If you think something's supposed to hurt, you're less likely to notice if you're doing it wrong.

Name and section: ______________________________

1. (3 points) Solve $y'' - 4y' + 3y = e^{2x}$ using undetermined coefficients.

Solution: The homogeneous (complementary) equation is $y'' - 4y' + 3y = 0$, with characteristic equation $r^2 - 4r + 3 = (r-1)(r-3)$, so the homogeneous solution is

$$y_h = Ae^x + Be^{3x}. \tag{1}$$

e^{2x} is not a homogeneous solution, so for undetermined coefficients we can take $y_p = Ce^{2x}$ as a trial solution. This gives $y_p' = 2Ce^{2x}, y_p'' = 4Ce^{2x}$, so

$$y_p'' - 4y_p' + 3y_p = -Ce^{2x} = e^{2x} \tag{2}$$

so $C = -1$, giving our final answer

$$y = y_p + y_h = -e^{2x} + Ae^x + Be^{3x}. \tag{3}$$

2. (3 points) Solve $y'' + 9y = \cos 3x$ using undetermined coefficients.

Solution: The homogeneous equation is $y'' + 9y = 0$, with characteristic equation $r^2 + 9 = (r + 3i)(r - 3i)$, so the homogeneous solution is

$$y_h = A\cos 3x + B\sin 3x. \tag{4}$$

$\cos 3x$ is a homogeneous solution, so for undetermined coefficients we need to multiply by x; we can take $y_p = Cx\cos 3x + Dx\sin 3x$ as a trial solution. This gives

$$\begin{aligned} y_p' &= C\cos 3x - 3Cx\sin 3x + D\sin 3x + 3Dx\cos 3x && (5)\\ y_p'' &= -3C\sin 3x - 3C\sin 3x - 9Cx\cos 3x + 3D\cos 3x + 3D\cos 3x - 9Dx\sin 3x && (6)\\ y_p'' + 9y_p &= -6C\sin 3x + 6D\cos 3x = \cos 3x && (7) \end{aligned}$$

so $C = 0, D = \frac{1}{6}$, giving our final answer

$$y = y_p + y_h = \frac{1}{6}x\sin 3x + A\cos 3x + B\sin 3x. \tag{8}$$

3. (4 points) Solve $y'' + y = \sec x$ using variation of parameters.

Solution: The homogeneous equation is $y'' + y = 0$, with characteristic equation $r^2 + 1 = (r+i)(r-i) = 0$, so for two linearly independent homogeneous solutions we can take $y_1 = \cos x, y_2 = \sin x$. Variation of parameters tells us to write the general solution as

$$y = f(x)\cos x + g(x)\sin x \tag{9}$$

and then tells us that we get a system of equations

$$\begin{aligned} f'(x)\cos x + g'(x)\sin x &= 0 & (10)\\ -f'(x)\sin x + g'(x)\cos x &= \sec x. & (11) \end{aligned}$$

To solve the system, we can first multiply the first equation by $\sin x$ and the first equation by $\cos x$, getting

$$\begin{aligned} f'(x)\sin x\cos x + g'(x)\sin^2 x &= 0 & (12)\\ -f'(x)\sin x\cos x + g'(x)\cos^2 x &= 1. & (13) \end{aligned}$$

Adding these gives

$$g'(x) = 1 \Rightarrow g(x) = x + C_2. \tag{14}$$

Next, we can multiply the first equation by $\cos x$ and the second equation by $\sin x$, getting

$$\begin{aligned} f'(x)\cos^2 x\cos x + g'(x)\sin x\cos x &= 0 & (15)\\ -f'(x)\sin^2 x + g'(x)\sin x\cos x &= \tan x. & (16) \end{aligned}$$

Subtracting these gives

$$f'(x) = -\tan x \Rightarrow f(x) = -\ln\sec x + C_1. \tag{17}$$

Altogether, we get

$$y = (-\ln\sec x + C_1)\cos x + (x + C_2)\sin x. \tag{18}$$

We could also write $-\ln\sec x = \ln\cos x$.