MATH 274 HOMEWORK #6, DUE MARCH 31, 2003

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2.12) Let X be a nice genus 0 curve over a global field k. Use the description of $\operatorname{Br} k$ to prove the following:

(a) The curve X has a k-point if and only if X has a k_v-point for every place v of k.
(b) The number of places v for which $X(k_v) = \emptyset$ is finite and even.

Solution.

(a) Let x be the element of $\operatorname{Br} k$ corresponding to the Severi-Brauer variety X. Then X has a k-point if and only if $X \simeq \mathbb{P}^1_k$, which holds if and only if $x = 0$. The base extension X_{k_v} corresponds to the image of x in $\operatorname{Br} k_v$. So the result follows from the injectivity of $\operatorname{Br} k \to \bigoplus_v \operatorname{Br} k_v$.
(b) Equivalently, we must show that the number of places v for which the image of x in $\operatorname{Br} k_v$ is nonzero is finite and even. Since $2x = 0$, this image, if nonzero, satisfies $\operatorname{inv}_v(x) = 1/2$. Thus the result follows from the exactness of

$$0 \longrightarrow \operatorname{Br} k \longrightarrow \bigoplus_v \operatorname{Br} k_v \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

□

2.13) Let k be an imperfect field of characteristic p, and suppose $t \in k - k^p$. Let G be the k-subvariety of $\mathbb{G}_a^2$ defined by the equation $y^p = tx^p + x$. Prove that

(a) G is a subgroup scheme of $\mathbb{G}_a^2$.
(b) $G_{\overline{k}} \simeq (\mathbb{G}_a)_{\overline{k}}$ as $\overline{k}$-group schemes.
(c) G is not isomorphic to the k-group scheme $\mathbb{G}_a$.
(d) G as a k-variety (without group structure) is not isomorphic to $\mathbb{A}^1$.

Solution.

(a) The morphism $\phi : \mathbb{G}_a^2 \to \mathbb{G}_a$ given by (x, y) to $y^p - tx^p - x$ is clearly additive in characteristic p, so it is a homomorphism. Now $G = \ker \phi$, so G is a subgroup scheme of $\mathbb{G}_a^2$.
(b) Rewrite the equation as $(y - t^{1/p}x)^p = x$. We claim that the morphism $G_{\overline{k}} \to (\mathbb{G}_a)_{\overline{k}}$ defined by $(x, y) \mapsto y - t^{1/p}x$ is an isomorphism. Solving the system $y^p = tx^p + x$, $y - t^{1/p}x = z$ for (x, y) leads to the formula for an inverse: $z \mapsto (z^p, z + t^{1/p}z^p)$. Composing the two maps in either order, we find that they are indeed inverse isomorphisms between $G_{\overline{k}}$ and $(\mathbb{G}_a)_{\overline{k}}$.
(c) This will follow from the next part.
(d) Every isomorphism of $\overline{k}$-varieties $(\mathbb{G}_a)_{\overline{k}} \to \mathbb{A}^1_{\overline{k}}$ has the form $z \mapsto az + b$ for some $a \in \overline{k}^\times$, $b \in \overline{k}$. Any isomorphism $G \to \mathbb{A}^1$, when base extended to $\overline{k}$, must be a composition of the isomorphism $G_{\overline{k}} \to (\mathbb{G}_a)_{\overline{k}}$ in part (b) with an isomorphism

of $\overline{k}$-varieties $(\mathbb{G}_a)_{\overline{k}} \to \mathbb{A}^1_{\overline{k}}$, and hence must have the form

$$(x, y) \mapsto a(y - t^{1/p}x) + b.$$

But there is no choice of $a \in \overline{k}^\times$ and $b \in \overline{k}$ for which this has coefficients in k, because the ratio of the coefficients of x and y equals $t^{1/p}$, which is not in k.

□

2.14) Let k be a field. (Assume that the characteristic is not 2, if you want.)
(a) Find explicit equations for all 1-dimensional tori T over k.
(b) For each T, find explicit equations for all k-torsors under T.

Solution.
(a) Let L be a 2-dimensional étale k-algebra. Thus $L = k \times k$ or L is a separable quadratic extension of k. Let $f_L(x, y)$ denote a norm form for L/k. Let T^L be the affine k-variety $f_L(x, y) = 1$. The isomorphism type of T^L does not depend on the basis of L over k used to define f_L. If $L = L_0 := k \times k$, then the norm form relative to the standard basis $\{(1, 0), (0, 1)\}$ is xy, so $T^{L_0} = \mathbb{G}_m$.
For any $L \neq L_0$, we have $L \otimes k_s \simeq L_0 \otimes k_s$, and any choice of isomorphism ϕ induces an isomorphism $\phi' : (T^L)_{k^s} \simeq (\mathbb{G}_m)_{k^s}$; in other words, T^L is a 1-dimensional torus. Each $\sigma \in G_k$ transforms ϕ to ϕ or ϕ composed with the nontrivial k^s-algebra automorphism of $L_0 \otimes k_s = k_s \times k_s$ (interchanging the coordinates) according to whether $\sigma \in G_L \subseteq G_k$, and σ transforms ϕ' to ϕ' composed with the identity or the inverse on $(\mathbb{G}_m)_{k^s}$, accordingly. Thus T^L corresponds to a 1-cocycle representing the element of $\mathrm{H}^1(G_k, \pm 1) = \mathrm{Hom}(G_k, \pm 1)$ with kernel G_L. Thus the T_L (including the one with $L = L_0$) exhaust all possible cohomology classes, so we have obtained all twists of $\mathbb{G}_m$.
Explicitly, if k is of characteristic not 2, then Kummer theory describes the possibilities for L: for each $a \in k^\times$ (modulo squares), we have the torus $x^2 - ay^2 = 1$ corresponding to the basis $\{1, \sqrt{a}\}$ of $L = k[t]/(t^2 - a)$. Similarly if k is of characteristic 2, we use Artin-Schreier theory, and obtain the torus $x^2 + xy + ay^2 = 1$ for each $a \in k$ modulo the additive subgroup $\{\, b - b^2 : b \in k \,\}$.
(b) We need to compute $\mathrm{H}^1(G_k, T(k^s))$. If $T = \mathbb{G}_m$, then $\mathrm{H}^1(G_k, T(k^s)) = \mathrm{H}^1(G_k, (k^s)^\times) = 0$ by Hilbert's Theorem 90, so there are no nontrivial torsors under $\mathbb{G}_m$. Otherwise, we fix a quadratic extension L and norm form f_L such that $T \simeq T^L$. In the inflation-restriction sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(L/k), T(L)) \to \mathrm{H}^1(G_k, T(k^s)) \to \mathrm{H}^1(G_L, T(k^s)),$$

the term on the right is zero by Hilbert's Theorem 90, since $T_L \simeq (\mathbb{G}_m)_L$. Thus $\mathrm{H}^1(G_k, T(k^s)) = \mathrm{H}^1(\mathrm{Gal}(L/k), T(L))$. Under the isomorphism $T_L \simeq (\mathbb{G}_m)_L$, we have $T(L) = L^\times$ but the action of the nontrivial element $\sigma \in \mathrm{Gal}(L/k)$ is given by $\ell \mapsto 1/{}^\sigma\ell$. Thus

$$\mathrm{H}^1(\mathrm{Gal}(L/k), T(L)) = \frac{\ker(\text{norm})}{\mathrm{im}(\sigma - 1)} = \frac{\ker(\ell \mapsto \ell \cdot 1/{}^\sigma\ell)}{\mathrm{im}(\ell \mapsto (1/{}^\sigma\ell)/\ell)} = \frac{k^\times}{N_{L/k}(L^\times)}.$$

On the other hand, given $b \in k^\times$, the k-variety $f_L(x,y) = b$ is a k-torsor under T, and a straightforward calculation shows that the corresponding cohomology class in $\mathrm{H}^1(G_k, T(k^s))$ is identified with the image of b in $k^\times/N_{L/k}(L^\times)$ under the isomorphisms above.

Explicitly, if k is of characteristic not 2 (resp. characteristic 2), all the torsors of $x^2 - ay^2 = 1$ (resp. $x^2 + xy + ay^2 = 1$) are of the form $x^2 - ay^2 = b$ (resp. $x^2 + xy + ay^2 = b$) for some $b \in k^\times$ modulo the norm group $N_{L/k}(L^\times)$ where $L = \mathbb{Q}(\sqrt{a})$ (resp. L is obtained by adjoining a zero of $x^2 + x + a$ to k).

□

2.15) Show that the general definition of "S-torsor under G" is equivalent, in the case where $S = \operatorname{Spec} k$ and G is an algebraic group over k, to the definition of "k-torsor under G" given earlier. (Hint: use the fact that G is smooth over k to show that X is smooth over k.)

Solution. If X is a k-torsor under G, then explicit equations for the isomorphism $X_{k^s} \to G_{k^s}$ involve only finitely many elements of k^s, so the isomorphism comes from an isomorphism $X_L \to G_L$ for some *finite* Galois extension L/k. Let $S' = SpecL$. Since $S' \to S$ is fppf, X is an S-torsor under G.

Conversely, suppose that X is an S-torsor under G. Thus $X_{S'} \simeq G_{S'}$ for some fppf morphism $S' \to S$. By definition of algebraic group, G is smooth over k, so $G_{S'}$ is smooth over S', and $X_{S'}$ is smooth over S'. By fpqc descent, it follows that X is smooth over k. Thus X is geometrically reduced, and $X \neq \emptyset$, so $X(k^s) \neq \emptyset$ by Proposition **??**. Thus X_{k^s} is a trivial torsor under G_{k^s}. This means that X is a k-torsor under G. □

2.16) Let $L = \mathbb{F}_p(t)$ and $k = \mathbb{F}_p(t^p)$. Let X be the L-scheme $\operatorname{Spec} L[x]/(x^p - t)$. Compute $\operatorname{Res}_{L/k} X$.

Solution. The answer is the empty scheme! We use the basis $\{1, t, \dots, t^{p-1}\}$ for L/k. To find the restriction of scalars, we substitute

$$x = y_0 + ty_1 + \cdots + t^{p-1} y_{p-1}$$

into the equation $x^p - t = 0$ defining L, and rewrite the result as

$$F_0 + F_1 t + \cdots + F_{p-1} = 0$$

with $F_i \in k[y_0, \dots, y_{p-1}]$. We get

$$\begin{aligned} F_0 &= y_0^p + t^p y_1^p + \cdots + t^{p(p-1)} y_{p-1}^p, \\ F_1 &= -1, \\ F_i &= 0 \quad \text{for } i \geq 2. \end{aligned}$$

Then $\operatorname{Res}_{L/k} X$ is $\operatorname{Spec} k[y_0, \dots, y_{p-1}]/(F_0, \dots, F_{p-1})$. This is the empty scheme, since $F_1 = -1$ generates the unit ideal. □