# MATH 274 HOMEWORK #5, DUE MARCH 10, 2003

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2.8) Let $k$ be a field, and let $A = k[x_1, x_2, \ldots]$ be the polynomial ring in countably many indeterminates. Let $I$ be the ideal $(x_1, x_2, \ldots)$. Show that the finitely generated $A$-algebra $A/I$ is not finitely presented.

*Solution.* Suppose $A/I \simeq A[t_1, \ldots, t_n]/(g_1, \ldots, g_m)$ for some $n \geq 0$ and $g_1, \ldots, g_m \in A[t_1, \ldots, t_n]$. The $g_i$ involve only finitely many of the $x$'s, say $x_1, \ldots, x_N$. Then

$$\frac{A[t_1, \ldots, t_n]}{(g_1, \ldots, g_m)} \simeq B[x_{N+1}, x_{N+2}, \ldots]$$

where $B = k[x_1, \ldots, x_N, t_1, \ldots, t_n]/(g_1, \ldots, g_m)$. If $B = 0$, then so is $B[x_{N+1}, x_{N+2}, \ldots]$. If $B \neq 0$, then $\dim_k B[x_{N+1}, x_{N+2}, \ldots]$ is infinite. Both cases contradict $\dim_k A/I = 1$. $\square$

2.9) Let $R$ be a Dedekind domain, let $K = \operatorname{Frac}(R)$, and let $X$ be a smooth proper $K$-scheme. Suppose that for each nonzero prime $\mathfrak{p}$ of $R$, the scheme $X$ has good reduction at $\mathfrak{p}$. Prove that there is a smooth proper $R$-model of $X$.

*Solution.* By Theorem **??**, we can spread $X$ out to a smooth proper model $X_U$ over somedense open subscheme $U \subseteq \operatorname{Spec} R$. Then $(\operatorname{Spec} R) - U$ is 0-dimensional and noetherian, hence finite. It will suffice to show that given $\mathfrak{p} \in (\operatorname{Spec} R) - U$, the $U$-scheme $X_U$ extends to a smooth proper model over the open subscheme $U \cup \{\mathfrak{p}\}$ of $\operatorname{Spec} R$, since then we can repeat the argument to enlarge the base one point at a time until we obtain a smooth proper model over $R$.

By assumption, there exists a smooth proper $R_{\mathfrak{p}}$-model of $X$. By Remark **??**, we can spread this out to a smooth proper $V$-model $X_V$ of $X$ for some open neighborhood $V$ of $\mathfrak{p}$ in $\operatorname{Spec} R$. By Theorem **??**, there exists a dense open subscheme $W \subseteq U \cap V$ such that the restrictions of $X_U$ and $X_V$ to $W$ are isomorphic. Then we can glue $X_V|_{W \cup \{\mathfrak{p}\}}$ to $X_U$ along their common restrictions over $W$ to get a scheme $X'$ over $U \cup \{\mathfrak{p}\}$. Since smoothness and properness are local on the base, $X'$ is smooth and proper over $U \cup \{\mathfrak{p}\}$. $\square$

2.10) Let $k$ be a global field, and let $\mathbf{A}_k$ denote its ring of adèles. Let $X$ be a proper $k$-variety. Prove that the natural map $X(\mathbf{A}_k) \to \prod_v X(k_v)$ is a bijection, where the product is over all places $v$ of $k$, and $k_v$ is the completion of $k$ at $v$. (Warning: if you really want to do all the details, such as proving that $X(\prod_v k_v) = \prod_v X(k_v)$, then this is not an easy problem.)

*Solution.* Apply Theorem **??** to spread $X$ out to a proper $\mathcal{O}_{k,S}$-scheme $\mathcal{X}$ where $S$ is a finite nonempty set of places of $k$ including all archimedean places, and $\mathcal{O}_{k,S}$ is the ring of $S$-integers in $k$, that is, the set of elements of $k$ that are integral at all $v \notin S$. For $v \notin S$, let $R_v$ be the valuation ring of $k_v$. For $v \in S$, let $R_v = k_v$. Then we have inclusions of $\mathcal{O}_{k,S}$-algebras $\prod R_v \hookrightarrow \mathbf{A}_k \hookrightarrow \prod k_v$. Each is a localization of the previous, so we get inclusions $\coprod \operatorname{Spec} k_v \hookrightarrow \operatorname{Spec} \prod k_v \hookrightarrow \operatorname{Spec} \mathbf{A}_k \hookrightarrow \operatorname{Spec} \prod R_v$. Moreover, $\coprod \operatorname{Spec} k_v$ is dense in $\operatorname{Spec} \prod R_v$, since if $f \in \prod R_v$ has zero image in each $k_v$, then $f = 0$. Since $\mathcal{X}$ is separated, and the rings $\prod R_v$, $\mathbf{A}_k$, and $\prod k_v$ are reduced, it follows (by the same argument used for Exercise II.4.2 in [**?**]) that the two top horizontal maps and the map following them from $\mathcal{X}(\prod k_v) \to \mathcal{X}(\coprod \operatorname{Spec} k_v)$ in the commutative diagram

$$\begin{tikzcd}
\mathcal{X}(\prod R_v) \arrow[r] \arrow[d] & \mathcal{X}(\mathbf{A}_k) \arrow[r] & \mathcal{X}(\prod k_v) \arrow[d] \\
\mathcal{X}(\coprod \operatorname{Spec} R_v) \arrow[d, equal] & & \mathcal{X}(\coprod \operatorname{Spec} k_v) \arrow[d, equal] \\
\prod \mathcal{X}(R_v) \arrow[rr, "\sim"] & & \prod \mathcal{X}(k_v)
\end{tikzcd}$$

are all injective. Also, the bottom horizontal map is an isomorphism, by the valuative criterion for properness.

To show that all the maps in the diagram are bijective, we need only show that the left vertical map $\mathcal{X}(\prod R_v) \to \prod \mathcal{X}(R_v)$ is surjective. Suppose we are given $(f_v) \in \prod \mathcal{X}(R_v)$. Pick an affine open covering $\{U_1, \ldots, U_m\}$ of $\mathcal{X}$. Since $R_v$ is local, the image of each $f_v : \operatorname{Spec} R_v \to \mathcal{X}$ is contained in some $U_i$, depending on $v$; in other words $f_v$ comes from an element of $U_i(R_v)$. Partition the set of places into $S_1, \ldots, S_m$ such that the image of $f_v$ is contained in $U_i$ whenever $v \in S_i$. Write $U_i = \operatorname{Spec} B$. Then

$$U_i(\prod_{v \in S_i} R_v) = \operatorname{Hom}(B, \prod_{v \in S_i} R_v) \to \prod_{v \in S_i} \operatorname{Hom}_k(B, R_v) = \prod_{v \in S_i} U_i(R_v)$$

is bijective. Thus $(f_v)_{v \in S_i}$ comes from a point in $U_i(\prod_{v \in S_i} R_v)$, which can be mapped to a point in $\mathcal{X}(\prod_{v \in S_i} R_v)$. Taking the *finite* product over $i$ yields a point in $\mathcal{X}(\prod R_v)$ mapping to $(f_v) \in \prod \mathcal{X}(R_v)$.

Finally observe that in the bijection $\mathcal{X}(\mathbf{A}_k) \to \prod \mathcal{X}(k_v)$, we may replace $\mathcal{X}$ by $X$ in both places, since $\mathbf{A}_k$ and each $k_v$ are $k$-algebras. □

2.11) Show that the Lang-Nishimura Proposition can fail if either of the following changes is made:

(a) The assumption that $Y$ is proper is dropped.

(b) The given $k$-point on $X$ is not assumed to be smooth.

*Solution.*

(a) Let $k$ be a finite field. Let $X = \mathbb{A}^1_k$, which is irreducible. Let $Y$ be the open subvariety $X - X(k)$ of $X$. The identity $Y \to Y$ represents a rational map

$X \dashrightarrow Y$. Moreover $X$ has a smooth $k$-point (in fact, all of $X$ is smooth over $k$), but $Y(k) = \emptyset$.

(b) Let $k = \mathbb{R}$. Let $X$ be the affine curve $x^2 + y^2 = 0$ in $\mathbb{A}^2_{\mathbb{R}}$, which has exactly one $\mathbb{R}$-point, namely $P := (0, 0)$. Note that $X$ is not smooth at $P$. Since the polynomial $x^2 + y^2$ is irreducible over $\mathbb{R}$, the variety $X$ is irreducible. Let $Y$ be the subscheme of $\mathbb{A}^1_{\mathbb{R}}$ defined by $t^2 + 1 = 0$. Since $Y$ is finite over $\mathbb{R}$, it is proper. Then $(x, y) \mapsto y/x$ defines a rational map $X \dashrightarrow Y$ defined everywhere except $P$. But $Y(\mathbb{R}) = \emptyset$.

One can also find a counterexample in which $X$ and $Y$ are geometrically integral. For example, let $X$ be the projective closure in $\mathbb{P}^2_{\mathbb{R}}$ of the affine curve $y^2 = -(x^4 + 1)$ over $\mathbb{R}$, and let $Y$ be its smooth projective model. One can show that $X$ has a nonsmooth $\mathbb{R}$-point at infinity, and that $Y(\mathbb{R})$ is empty.

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