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MATH 185 MIDTERM 2 SOLUTIONS

(1) (5 pts. each) For each of (a)-(c) below: If the statement is true, write TRUE. If the statement is false, write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistinguishable.) No explanations are required in this problem.

(a) If a function $f(z)$ is holomorphic on an open subset U of \mathbb{C} , and γ is a closed piecewise C^1 curve contained in U , then $\int_{\gamma} f(z) dz = 0$.

FALSE. Consider $f(z) = 1/z$, $U = \mathbb{C}^*$, and γ the unit circle.

(b) If f is an entire function, and $\gamma: [0, 1] \rightarrow \mathbb{C}$ and $\Gamma: [0, 1] \rightarrow \mathbb{C}$ are piecewise C^1 curves such that $\gamma(0) = \Gamma(0)$ and $\gamma(1) = \Gamma(1)$, then $\int_{\gamma} f = \int_{\Gamma} f$.

TRUE. The join of γ and $-\Gamma$ goes from $\gamma(0)$ to $\gamma(1)$ and then back to $\gamma(0)$, so it is a closed curve. By Cauchy's Theorem, the integral of f over this closed curve is 0. Thus $\int_{\gamma} f - \int_{\Gamma} f = 0$.

(c) Let \mathbb{C}^* be the set of nonzero complex numbers, and let f be a holomorphic function on \mathbb{C}^* . If f is bounded on \mathbb{C}^* , then f is constant.

TRUE. Since f is bounded on a punctured disk centered at 0, f has a removable singularity at 0. Thus f can be extended to a holomorphic function (which we will still call f) on all of \mathbb{C} . If M is an upper bound for the values of $|f(z)|$ for $z \in \mathbb{C}^*$, then $\max(M, |f(0)|)$ is an upper bound for $|f(z)|$ for all $z \in \mathbb{C}$, so f is bounded. By Liouville's theorem, f is constant on all of \mathbb{C} .

(2) (10 pts. each) Evaluate the following integrals:

(a) $\int_{|z|=3} \bar{z} dz$

Use the definition of the integral. Parametrize the circle by $z = \gamma(t) := 3e^{it}$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned} \int_{|z|=3} \bar{z} dz &= \int_0^{2\pi} 3e^{-it} 3ie^{it} dt \\ &= \int_0^{2\pi} 9i dt \\ &= 18\pi i. \end{aligned}$$

(b) $\int_{|z|=1} \frac{2z-1}{z^2-8z+15} dz$

The numerator and denominator are holomorphic on all of \mathbb{C} . The denominator $z^2 - 8z + 15 = (z - 3)(z - 5)$ is nonzero only at 3 and 5. Therefore the integrand is holomorphic everywhere on the open disk of radius 3. By Cauchy's Theorem for a convex region, the integral is 0.

(3) Let $f(z) = \frac{5z + 3}{e^z - 1 - z}$, defined on the open subset of \mathbb{C} where the denominator is nonzero.

(a) (15 pts.) What kind of singularity does $f(z)$ have at $z = 0$?

We have $\text{ord}_{z=0}(5z + 3) = 0$, since $5z + 3$ is holomorphic everywhere, and non-vanishing at 0. On the other hand,

$$e^z - 1 - z = \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

so $\text{ord}_{z=0}(e^z - 1 - z) = 2$. Thus $\text{ord}_{z=0} f(z)$ exists and equals $0 - 2 = -2$, which means that f has a pole of order 2 at $z = 0$.

(b) (25 pts.) Compute the residue of $f(z)$ at $z = 0$.

We need to compute the beginning of the Laurent series of f at 0. More precisely, it will be enough to compute the first two terms (the z^{-2} and z^{-1} terms). Therefore we need to use the first two terms of the power series for the numerator and denominator: for z in some punctured disk centered at 0, we have

$$\begin{aligned} f(z) &= \frac{3 + 5z}{\frac{z^2}{2} + \frac{z^3}{6} + \dots} \\ &= \frac{3 + 5z}{\frac{z^2}{2} \left(1 + \frac{z}{3} + \dots\right)} \\ &= \frac{3 + 5z}{\frac{z^2}{2} \left(1 + \frac{z}{3} + \dots\right)} \\ &= 2z^{-2}(3 + 5z) \frac{1}{1 - \left(-\frac{z}{3} + \dots\right)} \\ &= 2z^{-2}(3 + 5z) \left(1 + \left(-\frac{z}{3} + \dots\right) + \left(-\frac{z}{3} + \dots\right)^2 + \dots\right) \\ &= 2z^{-2}(3 + 5z) \left(1 - \frac{z}{3} + \dots\right) \\ &= 2z^{-2}(3 + 4z + \dots) \\ &= 6z^{-2} + 8z^{-1} + \dots, \end{aligned}$$

so the residue is 8.

(4) (25 pts.) For $R > 0$, let γ_R be the upper half of the circle $|z| = R$, taken counterclockwise. (Thus γ_R is a semicircular path from R to $-R$.) Prove that

$$\lim_{R \rightarrow +\infty} \left| \int_{\gamma_R} \frac{e^{iz}}{z^2} dz \right| = 0.$$

If $z = x + iy$, then $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = e^{-y}$. For z on γ_R , the imaginary part y of z is always ≥ 0 , so $|e^{iz}| \leq 1$. Thus for z on γ_R , we have

$$\left| \frac{e^{iz}}{z^2} \right| = \frac{|e^{iz}|}{R^2} \leq \frac{1}{R^2}.$$

The length of γ_R is $\frac{1}{2}(2\pi R) = \pi R$, so

$$\left| \int_{\gamma_R} \frac{e^{iz}}{z^2} dz \right| \leq \frac{1}{R^2} \pi R = \frac{\pi}{R},$$

which goes to 0 as $R \rightarrow \infty$. Thus $\left| \int_{\gamma_R} \frac{e^{iz}}{z^2} dz \right|$ is sandwiched between 0 and π/R , so as $R \rightarrow \infty$, it will tend to 0.