Prof. Bjorn Poonen
May 14, 2004, 8:10-11:00am, in 60 Evans Hall

MATH 140 FINAL SOLUTIONS

(1) *(5 pts. each) For each of (a)-(e) below: If the proposition is true, write TRUE* and explain why it is true. *If the proposition is false, write FALSE* and give a counterexample. *(Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.)*

(a) If S is a regular surface in $\mathbb{R}^3$, and every $p \in S$ has a neighborhood in S that is orientable, then S is orientable.

FALSE. A coordinate neighborhood of a regular surface is automatically oriented, so the hypothesis that every $p \in S$ have an orientable neighborhood is vacuous. Thus any nonorientable surface, such as the Möbius band, is a counterexample.

(b) If $\phi\colon S \to \bar{S}$ is a conformal diffeomorphism between regular surfaces in $\mathbb{R}^3$, then ϕ is a local isometry.

FALSE. Let S be the unit sphere, and let ϕ be the restriction to S of the diffeomorphism $(x, y, z) \mapsto (2x, 2y, 2z)$ of $\mathbb{R}^3$. Then ϕ is a diffeomorphism from S to $\bar{S} := \phi(S)$ (a sphere of radius 2). For any $p \in S$ and $v, w \in T_p(S)$, we have

$$\langle d\phi_p(v), d\phi_p(w)\rangle = 4\langle v, w\rangle$$

so ϕ is conformal, but not a local isometry ($4 \neq 1$).

(c) If every point on a nonempty connected compact surface S in $\mathbb{R}^3$ is an elliptic point, then S is homeomorphic to a sphere.

TRUE. By assumption $K > 0$ on S, and K is continuous on S, so $\iint_S K\, d\sigma > 0$. On the other hand, Gauss-Bonnet says $\iint_S K\, d\sigma = 2\pi\chi(S)$, so $\chi(S)$. Up to homeomorphism, the only nonempty connected compact surface in $\mathbb{R}^3$ with $\chi(S) > 0$ is the sphere.

(d) Let $\pi\colon S^2 \to P^2$ be the usual map sending each p on the unit sphere S^2 to the pair $\{p, -p\}$, which represents a single point on the projective plane P^2. Let $f\colon P^2 \to \mathbb{R}$ be a function. Then f is differentiable if and only if the composition $f \circ \pi\colon S^2 \to \mathbb{R}$ is differentiable.

TRUE. The differentiable structure on P^2 is given by a collection of maps $\pi \circ \mathbf{x}_\alpha$, for a set of parametrizations $\mathbf{x}_\alpha\colon U_\alpha \to S^2$ covering S^2 such that $\mathbf{x}_\alpha(U_\alpha)$ is disjoint from its image under the antipodal map. Then by definition,

$$f \text{ is differentiable} \iff f \circ (\pi \circ \mathbf{x}_\alpha) \text{ is differentiable for all } \alpha,$$
$$f \circ \pi \text{ is differentiable} \iff (f \circ \pi) \circ \mathbf{x}_\alpha \text{ is differentiable for all } \alpha.$$

The right hand sides are obviously equivalent, so the left hand sides are equivalent too.

(e) If $\phi\colon S \to \bar{S}$ is an orientation-preserving isometry between oriented regular surfaces in $\mathbb{R}^3$, then the mean curvature of S at a point p equals the mean curvature of $\bar{S}$ at $\phi(p)$.

FALSE. Let $S = (0, 2\pi) \times \mathbb{R} \subseteq \mathbb{R}^2$ with the standard orientation. Then $\phi(u, v) = (\cos u, \sin u, v)$ gives a diffeomorphism from S to its image $\bar{S} := \phi(S)$, and induces an orientation on $\bar{S}$ making ϕ orientation-preserving. The partial derivatives

$$\phi_u = (-\sin u, \cos u, 0)$$
$$\phi_v = (0, 0, 1)$$

are orthonormal, so ϕ is an isometry. At any $p \in S$, the principal curvatures of S are 0 and 0, so the mean curvature is $(0 + 0)/2 = 0$. At any $q \in \bar{S}$, the normal sections in the principal directions are a line and a circle, so the principal curvatures are 0 and something nonzero (in fact, 1), so the mean curvature is nonzero.

(2) *(15 pts.) Let C be a regular curve in $\mathbb{R}^3$ whose curvature vanishes nowhere. Let λ be a positive real number, and let $\bar{C}$ be the image of C under the map $v \mapsto \lambda v$. Suppose $p \in C$. Compute the torsion $\bar{\tau}$ of $\bar{C}$ at λp in terms of λ and the torsion τ of C at p.*

Choose a parametrization $\alpha(s)$ of C by arc length with $\alpha(0) = p$. Then $\bar{\alpha}(s) := \lambda\alpha(s/\lambda)$ is a parametrization of $\bar{C}$, again by arc length, since $\bar{\alpha}'(s) = \alpha'(s/\lambda)$. We compute

$$\begin{aligned}
\bar{\alpha}''(s) &= \lambda^{-1}\alpha''(s/\lambda)\\
\bar{k}(s) &= |\bar{\alpha}''(s)| = \lambda^{-1}|\alpha''(s/\lambda)| = \lambda^{-1}k(s/\lambda)\\
\bar{n}(s) &= \frac{\bar{\alpha}''(s)}{\bar{k}(s)} = \frac{\lambda^{-1}\alpha''(s/\lambda)}{\lambda^{-1}k(s/\lambda)} = \frac{\alpha''(s/\lambda)}{k(s/\lambda)} = n(s/\lambda)\\
\bar{b}(s) &= \bar{\alpha}'(s) \wedge \bar{n}(s) = \alpha'(s/\lambda) \wedge n(s/\lambda) = b(s/\lambda)\\
\bar{b}'(s) &= \lambda^{-1}b'(s/\lambda) = \lambda^{-1}\tau(s/\lambda)n(s/\lambda) = \lambda^{-1}\tau(s/\lambda)\bar{n}(s)\\
\bar{\tau}(s) &= \lambda^{-1}\tau(s/\lambda).
\end{aligned}$$

Taking $s = 0$, we get $\bar{\tau} = \lambda^{-1}\tau$.

(3) *Define $\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $\mathbf{x}(u, v) = (u\cos v, u\sin v, v)$.*

(a) (15 pts.) Prove that the set $S = \mathbf{x}(\mathbb{R}^2)$ is a regular surface in $\mathbb{R}^3$.

The map $\mathbf{x}$ is differentiable since its coordinate functions are. The differentiable map

$$f(x, y, z) = (x\cos z + y\sin z, z)$$

is a one-sided inverse to $\mathbf{x}$; that is, $f(\mathbf{x}(u, v)) = (u, v)$. Therefore $\mathbf{x}$ is a homeomorphism onto its image S. Taking the derivative of $f \circ \mathbf{x} = \mathrm{id}$ shows that $d\mathbf{x}_p$ is injective at each $p \in \mathbb{R}^2$.

(b) (15 pts.) Calculate the Gaussian curvature of S at $\mathbf{x}(u, v)$, as a function of u and v.

We compute

$$\begin{aligned}
\mathbf{x}_u &= (\cos v, \sin v, 0)\\
\mathbf{x}_v &= (-u\sin v, u\cos v, 1)\\
E &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle = 1\\
F &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0\\
G &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle = 1 + u^2\\
\mathbf{x}_u \wedge \mathbf{x}_v &= (\sin v, -\cos v, u)\\
\mathcal{N} &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = (1 + u^2)^{-1/2}(\sin v, -\cos v, u)\\
\mathbf{x}_{uu} &= (0, 0, 0)\\
\mathbf{x}_{uv} &= (-\sin v, \cos v, 0)\\
\mathbf{x}_{vv} &= (-u\cos v, -u\sin v, 0)\\
e &= \langle \mathcal{N}, \mathbf{x}_{uu}\rangle = 0\\
f &= \langle \mathcal{N}, \mathbf{x}_{uv}\rangle = -(1 + u^2)^{-1/2}\\
g &= \langle \mathcal{N}, \mathbf{x}_{vv}\rangle = 0\\
K &= \frac{eg - f^2}{EG - F^2} = \frac{-(1 + u^2)^{-1}}{1 + u^2} = -(1 + u^2)^{-2}.
\end{aligned}$$

(4) *(15 pts) Let S and $\bar{S}$ be oriented regular surfaces in $\mathbb{R}^3$, and let $N\colon S \to \mathbb{R}^3$ and $\bar{N}\colon \bar{S} \to \mathbb{R}^3$ be the associated differentiable fields of unit normal vectors. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular parametrized curve whose trace is contained in $S \cap \bar{S}$. Suppose moreover that S and $\bar{S}$ are orthogonal along α (i.e., for each t, the vectors $N(\alpha(t))$ and $\bar{N}(\alpha(t))$ are orthogonal). Show that the geodesic curvature of α as a curve in S equals the normal curvature of α as a curve in $\bar{S}$, up to a sign.*

We may assume that α is parametrized by arc length. Let $\mathcal{N} = N \circ \alpha$ and $\bar{\mathcal{N}} = \bar{N} \circ \alpha$. At any $t \in I$, the vector α'' is orthogonal to α' and hence is a combination of $\mathcal{N}$ and $\mathcal{N} \wedge \alpha'$. By definition, the geodesic curvature $k_g(\alpha \text{ in } S)$ is the component of α'' in the $\mathcal{N} \wedge \alpha'$ direction. But $\bar{\mathcal{N}}$ is a unit vector orthogonal to both of the orthogonal unit vectors α' and $\mathcal{N}$, so $\bar{\mathcal{N}} = \pm\mathcal{N} \wedge \alpha'$. Thus $k_g(\alpha \text{ in } S)$ also equals (up to sign) the component of α'' in the $\bar{\mathcal{N}}$ direction, which is the definition of the normal curvature of α in $\bar{S}$.

(5) *(15 pts.) Let S be a surface of constant Gaussian curvature -1. Let R be a simple geodesic n-gon in S. (That is, R is a subset of S homeomorphic to a closed disk, and the boundary of R is the union of n geodesics.) Prove that the area of R is strictly less than $(n-2)\pi$.*

Since R is homeomorphic to a closed disk, it is a homeomorphic also to a triangle T, so $\chi(R) = \chi(T) = 3 - 3 + 1 = 1$. Let $\theta_1, \ldots, \theta_n$ be the exterior angles of ∂R (oriented positively). We have $\theta_i \le \pi$ for each i, but equality is impossible, since there is a unique geodesic in each direction out of a point. Thus $\theta_i < \pi$. Then Gauss-Bonnet says

$$\sum_{i=1}^{n} \theta_i + \iint_R K\, d\sigma = 2\pi\chi(R)$$

$$\sum_{i=1}^{n} \theta_i - \text{Area}(R) = 2\pi$$

$$\text{Area}(R) = \sum_{i=1}^{n} \theta_i - 2\pi < \sum_{i=1}^{n} \pi - 2\pi = (n-2)\pi.$$