Math 1b Sections 104/106 Quiz 1 Solutions

Some solutions contain a bit less detail than what you should show.

Problem 1 (7 points, from the homework) Evaluate $\int \cos^3 x \sin^2 x dx$.

We first note that

$$\int \cos^3 x \sin^2 x dx = \int \cos x (1 - \sin^2 x) \sin^2 x dx$$

because $\cos^2 x = 1 - \sin^2 x$. *Additivity of the integral then yields*

$$\int \cos x \sin^2 x dx - \int \cos x \sin^4 x dx$$

Substituting $u = \sin x$ *and solving the resulting integrals, we obtain*
$\sin^3 x/3 - \sin^5 x/5 + C$

Problem 2 (7 points, from the homework) Evaluate by integration by parts

$$\int_{-1}^{0} \frac{t^3 dt}{\sqrt{4+t^2}}$$

Solution 1: *Let* $u = t^2$ *and* $dv = tdt/\sqrt{4+t^2}$. *Then* $du = 2tdt$. *We calculate* $v = \sqrt{4+t^2}$ *(say by substituting* $w = 4 + t^2$ *into* dv*). So by integration by parts, our integral equals*

$$t^2\sqrt{4+t^2}\,\Big]_{-1}^{0} - \int_{-1}^{0} 2t\sqrt{4+t^2}\,dt \;=\; \sqrt{5} - \int_{-1}^{0} 2t\sqrt{4+t^2}\,dt$$

Again we have an integral which can be solved by substituting $w = 4 + t^2$, *and we get*

$$-\sqrt{5} - \left((2/3)(4+t^2)^{3/2}\,\Big]_{-1}^{0} \right) = (7/3)\sqrt{5} - 16/3$$

Solution 2: *Substituting* $y = t^2$ *we have* $dy = 2tdt$, *and our limits of integration become* $(-1)^2 = 1$ *and* $0^2 = 0$, *and so we obtain the integral*

$$-\int_0^1 \frac{ydy}{2\sqrt{4+y}}$$

Now do integration by parts, with $u = y$ *and* $dv = dy/2\sqrt{4+y}$.

Problem 3a. (5 points) Suppose $p(x)$ is a polynomial of degree n. Explain how you would evaluate the integral $\int p(x)e^x dx$ by repeated integration by parts.

- How would you choose your u and dv (i.e. your f and g') at each step?
- Why would that strategy eventually succeed in solving the integral?

For each iteration of integration by parts, we choose u *to be the polynomial and* $dv = e^x$. *Then* du *is a polynomial of degree one less than the degree of* u, *and* $v = e^x$. *Thus after* $n+1$ *applications of this process, our* u *will be 0 and we obtain an easily solvable integral of the form* $\int 0\,dx =$ *constant* C.

Problem 3b. (3 points) (Continuation of problem 4a) Write a formula for $\int p(x)e^x dx$ in terms of p and its n derivatives $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$

Every time we apply integration by parts, the sign changes because of the 'minus' in $\int udv = uv - \int vdu$. Since $p^{(n+1)} = 0$, we have $\int p^{(n+1)}e^x =$ constant, and

$$\int p(x)e^x dx \;=\; e^x(p - p^{(1)} + p^{(2)} - \ldots + (-1)^n p^{(n)}) + C$$

Problem 4 (3 points, review of integral concepts) Suppose g is a function which is positive and continuous. Compute the limit (if it exists)

$$\lim_{t\to 0} \frac{\int_0^t g(x)dx}{\int_0^{2t} g(x)dx}$$

This problem was not intended to be a L'Hospital's rule problem, although L'Hospital's rule gives a nice rigorous solution that several students found. There's also another way to solve the problem by thinking about what an integral "really means."

Solution 1: *If g is continuous and a and b are close enough together, then $\int_a^b g(x)dx$ will become very close to $(b-a)g(a)$. This is because all values of $g(x)$ will become very close to $g(a)$ for $a \le x \le b$ since g is continuous. Hence, for t small enough, $\int_0^t g(x)dx$ will be approximately $(t-0)\,g(0)$, and similarly $\int_0^{2t} g(x)dx$ will be approximately $(2t-0)\,g(0)$. Note $g(0) \neq 0$ because g is positive. Thus our limit is*

$$\lim_{t\to 0} \frac{t\,g(0)}{2t\,g(0)} = \lim_{t\to 0} \frac{1}{2} = 1/2$$

Solution 2: *Let $G(t) = \int_0^t g(x)dx$. Then our limit is*

$$\lim_{t\to 0} \frac{G(t)}{G(2t)}$$

We have $G(0) = 0$ by basic properties of the integral, so the limit looks like $0/0$ which gives us no information. But we can apply L'Hospital's rule and differentiate the numerator and denominator. Note that $G' = g$ by the fundamental theorem of calculus. Furthermore $g(0) \neq 0$ because g is positive. Thus our limit equals

$$\lim_{t\to 0} \frac{G'(t)}{(G(2t))'} = \lim_{t\to 0} \frac{g(t)}{2g(2t)} = \frac{g(0)}{2g(0)} = 1/2$$