Math 1B Challenge Problems

Problem 1, chapter 7

a) Show that for any continuous function f,

$$\int_0^{2\pi} f(\sin x)\,dx = \int_0^{2\pi} f(\cos x)\,dx$$

b) Use (a) to give a one-line proof that

$$\int_0^{2\pi} \sin^2 x\,dx \;=\; \pi$$

c) Show that

$$\int_0^{2\pi} \sin^m x \cos^n x\,dx \;=\; 0$$

whenever m, n are nonnegative integers and either m is odd or n is odd.

Problem 2, chapter 7

The gamma function Γ is defined as

$$\Gamma(z+1) = \int_0^{\infty} x^z e^{-x} dx$$

a) Express $\Gamma(z+1)$ in terms of $\Gamma(z)$ if $z \neq 0$.

b) Show that for positive integers n, $\Gamma(n+1) = n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

c) For what real values of z is the integral $\Gamma(z)$ convergent?

Problem 3, chapter 7

a) If f is continuously differentiable, show that

$$\lim_{n\to\infty} \int_a^b \sin(nx)\, f(x)\, dx \;=\; 0$$

b) Think about the graphs of f and $\sin nx$, and explain why your result in (a) makes sense.

c) Suppose m, n are different nonnegative integers. Prove that

$$\int_0^{2\pi} \sin(mx) \sin(nx)\, dx \;=\; 0$$

Problem 4, chapters 7,11

a) Show that

$$\lim_{n\to\infty} \int_0^1 x^n e^x dx = 0$$

b) Show that for any polynomial p of degree n,

$$\int p(x)\, e^x\, dx \;=\; e^x(p - p^{(1)} + p^{(2)} - \ldots + (-1)^n p^{(n)})$$

c) Use (a) and (b) to show that

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

Problem 5, chapter 11

a) Show that $e^x \geq 1 + x$ for any x.

b) If $p \geq 2$ show that

$$1/p \geq 1/p^2 + 1/p^3 + 1/p^4 + \dots$$

c) Let p_n be the nth prime number. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges. (Hint: Assume it converges, and consider the exponential

$$e^{\left(\sum_{n=1}^{\infty} \frac{2}{p_n}\right)} = e^{2/p_1} \cdot e^{2/p_2} \cdot e^{2/p_3} \cdot \dots$$

Use your previous results and the fact that every positive integer has a unique factorization as a product of powers of prime numbers.)

Note that part (c) also shows that there are infinitely many primes.