A cool proof of the Cauchy-Schwarz inequality

Peyam Ryan Tabrizian

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Here's a cool and slick proof of the Cauchy-Schwarz inequality. It starts out like the usual proof of C-S, but the end is very cute! This proof is taken from Pugh's *Intro to Real Analysis*-book.

NOTE: This proof works **ONLY** for $\mathbb{F} = \mathbb{R}$. Do **NOT** memorize it, it's purely for your math-enjoyment!

Note: In this document, (u, v) stands for 'inner product of u and v' (the author uses $< u, v >$ instead)

Theorem: [Cauchy-Schwarz inequality]

$$|(u, v)| \leq \|u\|^2 \|v\|^2$$

Proof: Consider the following function from $\mathbb{R}$ to $\mathbb{R}$:

$$f(t) = (u + tv, u + tv) = (u, u) + t(u, v) + t(v, u) + (v, v)t^2 = at^2 + bt + c$$

Where:

$$\begin{aligned} a &= (v, v) = \|v\|^2 \geq 0 \\ b &= (u, v) + (v, u) = 2(u, v) \qquad \text{Remember } \mathbb{F} = \mathbb{R} \\ c &= (u, u) = \|u\|^2 \geq 0 \end{aligned}$$

However, since $f(t) = (u + tv, u + tv) \geq 0$, we know that $at^2 + bt + c \geq 0$.

However, this implies that the discriminant $\Delta = b^2 - 4ac$ of f has to be ≤ 0[1]

But then, we get:

[1]Proof: If $\Delta > 0$, then f has two distinct roots $x_1 < x_2$ by the quadratic formula, but then either $f < 0$ on $[x_1, x_2]$ or $f < 0$ on $(-\infty, x_1) \cup (x_2, \infty)$

$$
\begin{aligned}
\Delta &\leq 0 \\
b^2 - 4ac &\leq 0 \\
b^2 &\leq 4ac \\
\sqrt{b^2} &\leq 2\sqrt{a}\sqrt{c} \\
|b| &\leq 2\sqrt{a}\sqrt{c} \\
|2(u,v)| &\leq 2\,\|u\|\,\|v\| \\
|(u,v)| &\leq \|u\|\,\|v\|
\end{aligned}
$$

And we're done! $\square$