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MATHEMATICS
(DIFFERENTIAL GEOMETRY)

The Graded Lie Algebra of Multivector Fields and the Generalized Lie Derivative of Forms

by

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Summary. The definition of a bracket operation for multivector fields and an extension of the notion of the Lie derivative are given. With the new bracket operation the exterior algebra of multivector fields acquires the structure of a graded Lie algebra.

1. Introduction. The object of this paper is to define for multivector fields a bracket operation which is a natural extension of the Lie bracket of vector fields. With this bracket operation the exterior algebra of multivector fields acquires the structure of a graded Lie algebra. The bracket is a special case of a bilinear differential concomitant for arbitrary contravariant tensor fields defined by Schouten [1]. A discussion of this special case of Schouten's concomitant is included in a paper by Nijenhuis [2]. Most properties of the bracket are derived there by coordinate methods of classical differential geometry.

Section **2** is a summary of generally known definitions and results stated without proofs. The main results of the paper are formulated in Sec. **3**. Sec. **4** is devoted to a discussion of the possibility of further generalizations. It is shown that the construction of the bracket operation for multivector fields is formally the same as that used by Frölicher and Nijenhuis [3] for the bracket of vector-valued forms. It is also shown that the construction does not extend to multivector-valued forms. The generalized Lie derivative of forms is defined in the last section.

2. The graded algebras of multivector fields and forms. Let M be a paracompact C^∞ manifold and F the ring of differentiable functions on M. Constant functions on M form a subring of F identified with the field $\boldsymbol{R}$ of real numbers.

2.1 DEFINITION. A *derivation* D of a commutative ring A relative to a subring K is a map $D\colon A\to A$ which satisfies

(a) $Dk=0$ if $k\in K$,

(b) $D(a+b)=Da+Db$,

(c) $D(ab)=(Da)\,b+aDb$.

Derivations of F relative to R form the F-module of *vector fields* on M, denoted by V.

Let $A^p = \wedge^p V$ denote the p-fold exterior power of V. Also let A^0 denote F and let $A^p = 0$ for $p < 0$. Elements of A^p are called *multivector fields of degree p*. The non-negative graded F-module $A = \{A^p\}$ is called the *exterior algebra* of V. It is a commutative, associative graded algebra with a product $\wedge$, called the *exterior product*, satisfying

(a) *degree* $(X \wedge Y) =$ *degree* $X +$ *degree* Y,
(b) $X \wedge Y = (-1)^{pq}\, Y \wedge X$ if *degree* $X = p$ and *degree* $Y = q$,
(c) $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$,
(d) $X \wedge (Y + Z) = X \wedge Y + X \wedge Z$ if *degree* $Y =$ *degree* Z.

Let A_p denote the F-module Hom (A^p, F) dual to A^p. Elements of A_p are called *forms* of *degree p*. The non-negative graded F-module $A^* = \{A_p\}$, dual to $\wedge$, is isomorphic to the exterior algebra of $V^* = \mathrm{Hom}\,(V, F)$. Formal properties of the exterior product $\wedge$ of forms are the same as those of the exterior product of multivector fields.

Duality between A and A^* implies the existence of the *evaluation map* $A \times A^* \to F\colon (X, \mu) \to \langle X \mid \mu \rangle$, where $\langle X \mid \mu \rangle = \mu(X)$ if *degree* $X =$ *degree* μ and $\langle X \mid \mu \rangle = 0$ if *degree* $X \neq$ *degree* μ.

The *interior products* $\lrcorner$ and $\llcorner$ are operations: $A \times A^* \to A^*\colon (X, \mu) \mapsto X \lrcorner \mu$ such that *degree* $(X \lrcorner \mu) =$ *degree* $\mu -$ *degree* X and $\langle Y \mid X \lrcorner \mu \rangle = \langle X \wedge Y \mid \mu \rangle$ for each multivector field Y; and $A \times A^* \to A\colon (X, \mu) \mapsto X \llcorner \mu$ such that *degree* $(X \llcorner \mu) =$ *degree* $X -$ *degree* μ and $\langle X \llcorner \mu \mid \nu \rangle = \langle X \mid \mu \wedge \nu \rangle$ for each form ν.

2.2 Proposition. *Relation* $(X \wedge Y) \llcorner \mu = (X \llcorner \mu) \wedge Y + (-1)^p X \wedge (Y \llcorner \mu)$ *holds if* μ *is a form of degree* 1, X *a multivector field of degree* p *and* Y *any multivector field. Also*

$$X \lrcorner (\mu \wedge \nu) = (X \llcorner \mu) \lrcorner \nu + (-1)^p \mu \wedge (X \lrcorner \nu)$$

holds if degree $X = p$, *degree* $\mu = 1$ *and* ν *is any form.*

2.3 Proposition. *There is a unique differential* d *in* A^* *satisfying* $\langle X \mid df \rangle = Xf$ *for each vector field* $X \in V$ *and each function* $f \in F$, *and*

(a) *degree* $d\mu =$ *degree* $\mu + 1$,
(b) $dd\mu = 0$,
(c) $d(\mu + \nu) = d\mu + d\nu$ *if degree* $\mu =$ *degree* ν,
(d) $d(\mu \wedge \nu) = d\mu \wedge \nu + (-1)^p \mu \wedge d\nu$ *if degree* $\mu = p$.

3. The graded Lie algebra of multivector fields. Let X and Y be multivector fields of degree p and q, respectively.

3.1. Proposition. *There is a unique multivector field* $[X, Y]$ *of degree* $p+q-1$ *such that*

$$\langle [X, Y] \mid d\mu \rangle = \langle X \mid d(Y \lrcorner d\mu) \rangle + (-1)^{pq+p+q} \langle Y \mid d(X \lrcorner d\mu) \rangle .$$

Proof. To define a multivector field it is sufficient to give its evaluation with forms of the type $f\,d\mu$, where f is a function. Let f, f' be functions and μ, μ' forms such that $f\,d\mu = f'\,d\mu'$. Then

$$
\begin{aligned}
&f\langle X \mid d(Y \lrcorner d\mu)\rangle + (-1)^{pq+p+q} f\langle Y \mid d(X \lrcorner d\mu)\rangle \\
&= \langle X \mid d(Y \lrcorner f d\mu)\rangle + (-1)^{pq+p+q}\langle Y \mid d(X \lrcorner f d\mu)\rangle - (-1)^{pq+q}\langle X\wedge Y \mid d(f\,d\mu)\rangle \\
&= \langle X \mid d(Y \lrcorner f' d\mu')\rangle + (-1)^{pq+p+q}\langle Y \mid d(X \lrcorner f' d\mu')\rangle - (-1)^{pq+q}\langle X\wedge Y \mid d(f'\,d\mu')\rangle \\
&= f'\langle X \mid d(Y \lrcorner d\mu')\rangle + (-1)^{pq+p+q} f'\langle Y \mid d(X \lrcorner d\mu')\rangle .
\end{aligned}
$$

Thus a multivector field $[X, Y]$ of degree $p+q-1$ is correctly defined by

$$\langle [X, Y] \mid f\,d\mu\rangle = f\langle X \mid d(Y \lrcorner d\mu)\rangle + (-1)^{pq+p+q} f\langle Y \mid d(X \lrcorner d\mu)\rangle .$$

Uniqueness of $[X, Y]$ is obvious.

3.2 Proposition. *The multivector field* $[X, Y]$ *satisfies*

$$[X, Y] \llcorner df = [X, Y \llcorner df] - (-1)^q [X \llcorner df, Y]$$

for each function f.

Proof.

$$
\begin{aligned}
&\langle [X, Y] \llcorner df \mid d\mu\rangle = \langle [X, Y] \mid d(f\,d\mu)\rangle \\
&= \langle X \mid d(Y \lrcorner (df\wedge d\mu))\rangle + (-1)^{pq+p+q}\langle Y \mid d(X \lrcorner (df\wedge d\mu))\rangle \\
&= \langle X \mid d((Y \llcorner df) \lrcorner d\mu)\rangle - (-1)^q \langle X \llcorner df \mid d(Y \lrcorner d\mu)\rangle \\
&= (-1)^{pq+p+q}\langle Y \mid d(X \lrcorner df) \lrcorner d\mu\rangle - (-1)^{pq+q}\langle Y \llcorner df \mid d(X \lrcorner d\mu)\rangle \\
&= \langle [X, Y \llcorner df] \mid d\mu\rangle - (-1)^q \langle [X, Y \llcorner df] \mid d\mu\rangle
\end{aligned}
$$

for each function f and each form μ. Hence $[X, Y] \llcorner df = [X, Y \llcorner df] - (-1)^q [X \llcorner df, Y]$.

3.3. Proposition. *The multivector field* $[X, Y]$ *satisfies*

$$
\begin{aligned}
&[X, Y] \lrcorner \mu = X \lrcorner d(Y \lrcorner \mu) + (-1)^{pq+p+q}\, Y \lrcorner d(X \lrcorner \mu) \\
&\quad - (-1)^{pq+p}\, d((X\wedge Y) \lrcorner \mu) - (-1)^{pq+q} (X\wedge Y) \lrcorner d\mu
\end{aligned}
$$

for each form μ.

Proof. The proposition is true for *degree* $\mu = 0$, and if the proposition is true for *degree* $\mu = p$ then

$$
\begin{aligned}
&[X, Y] \lrcorner (df\wedge \mu) = ([X, Y] \llcorner df) \lrcorner \mu + (-1)^{p+q-1}\, df \wedge ([X, Y] \lrcorner \mu) \\
&= [X, Y \llcorner df] \lrcorner \mu - (-1)^q [X \llcorner df, Y] \lrcorner \mu - (-1)^{p+q}\, df\wedge ([X, Y] \lrcorner \mu) \\
&= X \lrcorner d((Y \llcorner df) \lrcorner \mu) - (-1)^{pq+q} (Y \llcorner df) \llcorner d(X \lrcorner \mu) - (-1)^{pq}\, d((X\wedge (Y \llcorner df)) \lrcorner \mu) \\
&+ (-1)^{pq+p+q} (X \wedge (Y \llcorner df)) \lrcorner d\mu - (-1)^q (X \llcorner df) \lrcorner d(Y \lrcorner \mu) + (-1)^{pq+p+q} \\
&Y \lrcorner d((X \llcorner df) \lrcorner \mu) - (-1)^{pq+p} d(((X \llcorner df)\wedge Y) \lrcorner d\mu) + (-1)^{pq+q} ((X \llcorner df)\wedge Y) \lrcorner d\mu \\
&- (-1)^{p+q}\, df\wedge (X \lrcorner d(Y \lrcorner \mu)) - (-1)^{pq}\, df \wedge (Y \lrcorner d(X \lrcorner \mu)) \\
&+ (-1)^{pq+q}\, df\wedge d((X\wedge Y) \lrcorner \mu) + (-1)^{pq+p}\, df\wedge ((X\wedge Y) \lrcorner d\mu) \\
&= X \lrcorner d(Y \lrcorner (df\wedge \mu)) - (-1)^{pq+p+q}\, Y \lrcorner d(X \lrcorner (df\wedge\mu)) \\
&- (-1)^{pq+p}\, d((X\wedge Y) \lrcorner (df\wedge \mu)) - (-1)^{pq+q} (X\wedge Y) \lrcorner d(df\wedge\mu) .
\end{aligned}
$$

Hence proof by induction.

3.4 Theorem. *There is a unique operation* $A\times A \to A\colon (X, Y) \mapsto [X, Y]$ *which satisfies* $[X, f] = Xf$ *for each vector field* $X \in V$ *and each function* $f \in F$, *and*

(a) *degree* $[X, Y]=\mathit{degree} X+\mathit{degree}\ Y-1$,

(b) $[X, Y]+(-1)^{(p-1)(q-1)}[Y, X]=0$ *if degree* $X=p$ *and degree* $Y=q$,

(c) $(-1)^{(p-1)(r-1)}[X,[Y, Z]]+(-1)^{(q-1)(p-1)}[Y,[Z, X]]$
$+(-1)^{(r-1)(q-1)}[Z,[X, Y]]=0$ *if degree* $x=p$, *degree* $Y=q$ *and degree* $Z=r$,

(d) $[X, Y+Z]=[X, Y]+[X, Z]$ *if degree* $Y=$ *degree* Z,

(e) $[X, Y\wedge Z]=Y\wedge[X, Z]+(-1)^{r(p-1)}[X, Y]\wedge Z$ *if degree* $X=p$ *and degree* $Z=r$.

Proof. It follows from elementary computation that $(X, Y)\mapsto[X, Y]$, where $[X, Y]$ is defined in Proposition 3.1, has all the required properties. To prove uniqueness note that $[X,f]=Xf=X\llcorner df$ for *degree* $X=1$, and $[Y,f]=Y\llcorner df$ imply $[X\wedge Y,f]=[X,f]\wedge Y-X\wedge[Y,f]=(X\llcorner df)\wedge Y-X\wedge(Y\llcorner df)=(X\wedge Y)\llcorner df$ by the use of (b) and (e). Hence $[X,f]=X\llcorner df$ by induction on *degree* X. This result, together with (b) and (c), implies $[X, Y]\llcorner df=[[X, Y],f]=[X,[Y,f]]-(-1)^q[[X,f], Y]=$ $=[X, Y\llcorner df]-(-1)^q[X\llcorner df, Y]$. An inductive argument similar to that used to prove Proposition 3.3 leads to $\langle[X, Y]|\mu\rangle=\langle X|d(Y\lrcorner\mu)\rangle+(-1)^{pq+p+q}\langle Y|d(X\lrcorner\mu)\rangle$ $-(-1)^{pq+q}\langle X\wedge Y|\,d\mu\rangle$. Hence $[X, Y]$ is unique.

3.5 Definition. The graded algebra of multivector fields with the operation $(X, Y)\mapsto[X, Y]$ is called the *graded Lie algebra of multivector fields.*

4. Multivector forms. Let Λ_p^q denote the F-module $\mathrm{Hom}(\Lambda^p, \Lambda^q)$. Elements of Λ_p^q are called *multivector forms* of *degree* (p, q). The module Λ_0^q is identified with Λ^q and $\Lambda_p^0=\Lambda_p$. Elements of Λ_p^1 are the vector forms discussed by Frölicher and Nijenhuis [3].

For each multivector form K of degree (p, q) and each form μ of degree r let $K\lrcorner\mu$ be a form of degree $p+r-q$ such that $K\lrcorner\mu=0$ for $r-q<0$ and for $r-q\geqslant 0$

$$(K\lrcorner\mu)(X_1\wedge\ldots\wedge X_{p+r-q})=$$
$$=\frac{1}{p!(r-q)!}\sum_\alpha \operatorname{sgn}\alpha\,\mu\left(K(X_{\alpha_1}\wedge\ldots\wedge X_{\alpha_p})\wedge X_{\alpha_{p+1}}\wedge\ldots\wedge X_{\alpha_{p+r-q}}\right),$$

where $X_1, \ldots, X_{p+r-q}$ are vector fields and α ranges over the symmetric group S_{p+r-q}. Also let $K\llcorner\mu$ be a multivector form of degree $(p, q-r)$ such that $(K\llcorner\mu)(X)=$ $=K(X)\llcorner\mu$ for each multivector field X of degree p.

The above definitions extend in a natural way the applicability of interior products. Relation

$$K\lrcorner(\mu\wedge\nu)=(K\llcorner\mu)\lrcorner\nu+(-1)^{p-q}\mu\wedge(K\lrcorner\nu)$$

for *degree* $K=(p, q)$ and *degree* $\mu=1$, is an immediate consequence of the definitions.

To each multivector form K of degree (p, q) there correspond operators $i_K\colon\Lambda^*\to$ $\to\Lambda^*\colon\mu\mapsto K\lrcorner\mu$, and $d_K=i_K d-(-1)^{p-q}d i_K\colon\Lambda^*\to\Lambda^*\colon\mu\mapsto K\lrcorner d\mu-(-1)^{p-q}$ $d(K\lrcorner\mu)$. The operator i_K is an operator of degree $p-q$ since *degree* $(i_K\mu)=$ $=$ *degree* $\mu+p-q$, and d_K is an operator of degree $p-q+1$ since *degree* $(d_K\mu)=$

degree $\mu+p-q+1$. For two operators a and b of degrees r and s respectively let $[a, b]$ denote the commutator $ab-(-1)^{rs}ba$. Then $d_K=[i_K, d]$. Also $[i_K, i_\mu]=i_{K\llcorner\mu}$ if μ is a form of degree 1 and $[d_K, i_f]=[i_K, i_{df}]=i_{K\llcorner df}$ if f is a function.

4.1 Proposition. *Let $X\in A^p$ ond $Y\in A^q$ be multivector fields. There is a unique multivector field $[X, Y]\in A^{p+q-1}$ such that*

$$[d_X, i_Y]=i_{[X,Y]}.$$

Also

$$[d_X, d_Y]=d_{[X,Y]}.$$

The proof of this proposition follows directly from Propositions 3.1 and 3.3. The following proposition is due to Frölicher and Nijenhuis.

4.2 Proposition. *Let $K\in A^1_p$ and $L\in A^1_p$ be vector forms. There is a unique vector form $[K, L]\in A^1_{p+q}$ such that $[d_K, d_L]=d_{[K,L]}$.*

Proposition 4.2 implies the existence of a bracket operation for vector forms. It has been shown by Frölicher and Nijenhuis that this operation makes the module of vector forms a graded Lie algebra over R. The intersection of this algebra with the graded Lie algebra of multivector fields is the Lie algebra of vector fields.

Let $R\in A^r_p$, $S\in A^s_q$ and $K\in A^m_n$ be multivector forms such that $n-m=p+q-r-s+1$. Then $[[d_K, i_f], i_g]=[i_{K\llcorner df}, i_g]=0$ for any functions f and g. However $[[[d_R, d_S], i_f], i_g]=[i_{R\llcorner df}, i_{S\llcorner dg}]+[i_{R\llcorner dg}, i_{S\llcorner df}]$ is in general different from zero. Hence re'ations of the type

$$[d_R, d_S]=d_K$$

do not exist in general and Propositions 4.1 and 4.2 do not generalize to the case of arbitrary multivector forms.

5. The generalized Lie derivative

5.1 Definition. Let X be a multivector field of degree p and μ a form. Then $\mathcal{L}_X\mu=X\lrcorner d\mu-(-1)^p d(X\lrcorner\mu)$ is called the *generalized Lie derivative of μ with respect to X.*

5.2 Proposition. *The following relations hold for all multivector fields X and Y, all forms μ and ν and each function f:*

(a) *degree $\mathcal{L}_X\mu=$ degree $-$ degree $X+1$*

(b) $\mathcal{L}_X(\mu+\nu)=\mathcal{L}_X\mu+\mathcal{L}_X\nu$, *if degree $\mu=$ degree ν*

(c) $\mathcal{L}_X(f\mu)=f\mathcal{L}_X\mu+(X\llcorner df)\lrcorner\mu$

(d) $\mathcal{L}_X d\mu=d\mathcal{L}_X\mu$

(e) $\mathcal{L}_X(Y\lrcorner\mu)=[X, Y]\lrcorner\mu+(-1)^{q(p-1)}Y\lrcorner\mathcal{L}_X\mu$, *if degree $X=p$ and degree $Y=q$*

(f) $\mathcal{L}_{X+Y}\mu=\mathcal{L}_X\mu+\mathcal{L}_Y\mu$, *if degree $X=$ degree Y*

(g) $\mathcal{L}_{X\wedge Y}\mu=f\mathcal{L}_X\mu-(-1)^p df\wedge(X\lrcorner\mu)$, *if degree $X=p$*

(h) $\mathcal{L}_{X\wedge Y}\mu=Y\lrcorner\mathcal{L}_X\mu+(-1)^{pq}X\lrcorner\mathcal{L}_Y\mu+(-1)^p[X, Y]\lrcorner\mu$, *if degree $X=p$ and degree $Y=q$*

(i) $\mathcal{L}_{[X,Y]}\mu=\mathcal{L}_X\mathcal{L}_Y\mu-(-1)^{(p-1)(q-1)}\mathcal{L}_Y\mathcal{L}_X\mu$, *if degree $X=p$ and degree $Y=q$.*

The proof of this proposition follows easily from Definition 5.1 and the results of the preceding sections.

In agreement with a theorem by Frölicher and Nijenhuis [3], the generalized Lie derivative with respect to a multivector field X is not a derivation of the graded algebra of forms unless *degree* $X=1$.

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В. М. Тульчиев, Алгебра Ли с градацией мультивекторных полей и обобщенная производная форм Ли

Содержание. Дается определение скобки Ли мультивекторных полей, и обобщение понятия производной Ли. Новые скобки придают наружной алгебре мультивекторных полей структуру алгебры Ли сградацией.