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MATHEMATICS
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On the Equivariant Chern Homomorphism

by

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Summary. A split coefficient system for the equivariant Bredon cohomology is defined. Its properties are used to show that $K_G(X)\otimes Q$ is isomorphic to the Bredon cohomology of X with appropriate coefficients, provided G is a finite group and X is a compact G-CW complex. As a corollary we obtain that $K_G\otimes Q$ can be expressed in terms of the ordinary K-theory.

1. Split coefficient system. Let G be a finite group and $\mathfrak{O}_G$ the category of canonical G-orbits, i.e. G-sets of the form G/H, where H is a subgroup of G, and G-morphisms. Two orbits G/H and G/H' are identified in $\mathfrak{O}_G$ iff H and H' are conjugate in G. The category of contravariant functors from $\mathfrak{O}_G$ to the category Ab of abelian groups is denoted by $\mathfrak{C}_G$. Objects of $\mathfrak{C}_G$ are called G-coefficient systems. If H is a subgroup of G then there exists a functor $(\cdot)_H\colon \mathfrak{C}_G\to\mathfrak{C}_H$ such that for any G-coefficient system M

$$M_H(H/H')=M(G/H')$$

whenever H/H' is an object of $\mathfrak{O}_H$.

Let $\bar{\mathfrak{O}}_G$ be the full subcategory of $\mathfrak{O}_G$, consisting of all orbits different from G/G. If M is as above then for any canonical orbit $^G/H$ we will denote $\lim\limits_{\bar{\mathfrak{O}}_H} M_H$ by $\bar{M}(G/H)$ and the structural morphisms of this limit

$$\bar{M}(G/H)\to M_H(H/H')=M(G/H')$$

by $p(H/H')$. $\bar{M}(G/H)$ possess a natural structure of a $WH=NH/H$-module.

If $n\in NH$ then the composition

$$M(G/H)\xrightarrow{nH} M(G/H)\xrightarrow{p(H/H')} M(G/H')=(M\,^G/n^{-1}H'\,n)$$

is equal to $p(^H/n^{-1}H'\,n)$. Let

$$m(G/H)\colon M(G/H)\to\bar{M}(G/H)$$

be a WH-module morphism such that $p(H/H')\,m(G/H)$ is the morphism $M_H(^H/H\to\to M_H(H/H')$ induced by the map $H/H'\to H/H$, whenever H/H' is a canonical H-orbit. Ker $m(G/H)$ is denoted by $M(G/H)$.

1.1. DEFINITION. A G-coefficient system M is called split iff for any canonical orbit G/H there exists a WH-module morphism $t(G/H): \bar{M}(G/H) \to M(G/H)$ satisfying

$$m(G/H)\, t(G/H) = \mathrm{id}.$$

1.2. Examples. Let R_G be a coefficient system defined for objects as $R_G(G/H) = R(H) \otimes Q$, where $R(H)$ is a unitary representation ring, and for G-maps $G/H \to G/H'$ as the composition of a restriction homomorphism and conjugation by elements of G. This follows from the proof of the Artin theorem (see Serre [4]) that R_G is a split coefficient system. Furthermore, one can check that if M is an arbitrary Mackey functor (see Dress [2]) over $Z\left[\frac{1}{|G|}\right]$ then M is a split coefficient system.

1.3. LEMMA. *If M is a split G-coefficient system then*

$$\mathrm{Hom}_{\mathfrak{O}_G}(N, M) = \prod_{G/H \in \mathfrak{O}_G} \mathrm{Hom}_{WH}(N(G/H), \underline{M}(G/H))$$

whenever N is an object of $\mathfrak{C}_G$.

Proof. Let $\{\{e\}\} = \mathfrak{O}_1 \subset \mathfrak{O}_2 \subset \ldots \subset \mathfrak{O}_n = \mathfrak{O}_G$ be a certain filtration of $\mathfrak{O}_G$, such that any $\mathfrak{O}_k$ is a full subcategory of $\mathfrak{O}_G$ with k objects and if ${}^G/H$ is in $\mathfrak{O}_k$ and there exists a morphism in $\mathfrak{O}_G$ from G/H' to G/H then ${}^G/H'$ is in $\mathfrak{O}_k$, too. Let $\mathfrak{O}_k \setminus \mathfrak{O}_{k-1} = \{G/H_k\}$. For any G-map $f: G/H_l \to G/H_k$ one can find a subgroup $\bar{H}_l$ of H_k conjugate to H_l in G and an element w of WH such that the morphism $M(f)$ is equal to the composition $M(w)\, p(H_k/\bar{H}_l)$. This yields a group isomorphism:

$$\mathrm{Hom}_{\mathfrak{O}_k}(N, M) = \mathrm{Hom}_{\mathfrak{O}_{k-1}}(N, M) \oplus \mathrm{Hom}_{WH_k}\big(N(G/H_k), \ker m(G/H_k)\big),$$

where $\mathrm{Hom}_{\mathfrak{O}_k}(N, M)$ denotes the group of all natural transformations from $N|_{\mathfrak{O}_k}$ to $M|_{\mathfrak{O}_k}$.

The statement of the lemma follows from the above formula by induction.

2. Bredon cohomology with the coefficient system R_G and the $K_G \otimes Q$-theory. If X is a G-CW complex then the equivariant Bredon cohomology of X with a coefficient system M will be denoted by $H^*(X, M)$ (see [1]).

2.1. PROPOSITION. *There exists a natural transformation of equivariant cohomology theories*

$$ch_G: K_G \to \bigoplus_{k=0}^{\infty} H^{2k}(\ , R_G),$$

such that for any compact G-CW complex X

$$(ch_G^{\otimes}\ \mathrm{id})(X): K_G(X) \otimes Q \to H^{ev}(X, R_G)$$

is an isomorphism.

Proof. For any G-orbit G/H $\underline{R_G}(G/H)$ is a divisible group and hence a WH-injective one. Because R_G is a split coefficient system then from 1.3. it follows that

R_G is an injective object of $\mathfrak{C}_G$. Let $h_n(X)$ denote the object of $\mathfrak{C}_G$ determined by $h_n(X)(G/H)=H_n(X^H)$. The injectivity of R_G yields formulas (see [1], p. I-22).

$$H^n(X, R_G)=\operatorname{Hom}_{\mathfrak{D}_G}(h_n(X), RG)\subset \prod_{G/H\in\mathfrak{D}_G} \operatorname{Hom}(H_n(X^H), R(H)\otimes Q)= \\ = \prod_{G/H\in\mathfrak{D}_G} H^n(X^H, Q)\otimes R(H).$$

We define L_1 and L_2 as the maps

$$\prod_{G/H\in\mathfrak{D}_G} H^n(X^H, Q)\otimes R(H)\to \prod_{\substack{f:\,G/H_1\to GH_2\\ \text{in }\mathfrak{D}_G}} H^n(X^{H_2}, Q)\otimes R(H_1)$$

satisfying the conditions

$$S_f L_1=(H^n(f)\otimes\mathrm{id})\, S_{G/H_1}, \qquad S_f L_2=(\mathrm{id}\otimes R(f))\, S_{G/H_2},$$

where S_f and $S_{G/H}$ are structural morphisms of products.

It is easy to check that $\operatorname{Hom}_{\mathfrak{D}_G}(h_n(X), R_G)$ is isomorphic to $\ker(L_1-L_2)$. Let $q_H(X)$ be the restriction

$$K_G(X)\to K_H(X)\to K_H(X^H)$$

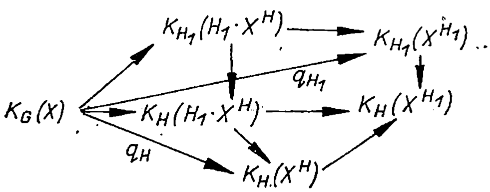
and let

$$q(X)=\prod_{G/H\in\mathfrak{D}_G} q_H(X)\colon K_G(X)\to \prod_{G/H\in\mathfrak{D}_G} K_H(X^H)=\prod_{G/H\in\mathfrak{D}_G} K(X^H)\otimes R(H).$$

If we let ch denote the ordinary Chern homomorphism then we can consider the composition

$$ch_G(X)=\prod_{G/H\in\mathfrak{D}_G} (ch(X^H)\otimes\mathrm{id})\, q(X)\colon K_G X\to \prod_{G/H\in\mathfrak{D}_G} H^{ev}(X^H, Q)\otimes R(H).$$

The inclusion $\operatorname{im} \widetilde{ch}_G X\subset\ker(L_1-L_2)$ follows from the commutativity of the diagrams



and

$$\begin{array}{ccccc}
K_G(X) & \xrightarrow{q_H} & K_H(X^H) & = & K(X^H)\otimes R(H) \\
\downarrow{\scriptstyle q_H} & & & & \downarrow{\scriptstyle \mathrm{id}\otimes R(w)} \\
K_H(X^H) & \xrightarrow{K_H(w)} & K_H(X^H) & & \\
\| & & & \searrow\!\!\searrow & \\
K(X^H)\otimes R(H) & \xrightarrow{K(w)\otimes\mathrm{id}} & & & K(X^H)\otimes R(H)
\end{array}$$

whenever $H \subset H_1$ are subgroups of G and w is an element of WH. This yields that we have well-defined the equivariant Chern homomorphism ch_G. One can easily verify that for any orbit G/H

$$ch_G(G/H) \otimes \mathrm{id}: K_G({}^G/H) \otimes Q \to H^{ev}(G/H, R_G)$$

is an isomorphism. Using the spectral sequence of the Atiyah—Hirzebruch type (see Matumoto [3]) we obtain the statement of the proposition.

Now, let $\mathfrak{O}_G^c$ be the full subcategory of $\mathfrak{O}_G$ consisting of all orbits G/H, such that H is a cyclic group.

2.2. Corollary. *If X is a compact G-CW complex then $K_G(X) \otimes Q$ is isomorphic to the direct sum*

$$\bigoplus_{G/H \in \mathfrak{O}_G^c} K(X^H) \underset{Z(WH)}{\otimes} \underline{R}_G(G/H).$$

Furthermore, if G is an abelian group then $K_G(X) \otimes Q$ is isomorphic to

$$\bigoplus_{G/H \in \mathfrak{O}_G^c} \bigoplus_{\varphi(|H|)} K(X^H/G) \otimes Q,$$

where φ denotes the Euler function.

Proof. If P is a G-module then $H_G^*(X, P)$ denotes the cohomology of the cochain complex $\mathrm{Hom}_G(C_*(X), P)$. From Lemma 1.3 it follows that

$$H^n(X, M) = \bigoplus_{G/H \in \mathfrak{O}_G} H^n_{WH}(X^H, \underline{M}(G/H)),$$

whenever M is a split coefficient system (see formulas 9.3 and 9.4, p. I-21 in [1]). If H is a noncyclic subgroup of G then $\underline{R}_G(G/H)$ is a trivial group. From proposition XII.2.5. in [5] it follows that

$$H^n_{WH}(X^H, \underline{R}_G(G/H))$$

is isomorphic to

$$H^n(X^H, Q) \underset{Q(WH)}{\otimes} \underline{R}_G(G/H)$$

since $\boldsymbol{R}_G(G/H)$ is a $Q(WH)$ projective module and

$$\begin{aligned}\mathrm{Hom}_{Z(WH)}\big(Z(WH/F),\quad & \boldsymbol{R}_G(G/H)\big) = \underline{R}_G(G/H)^F = \\ & = Z \underset{Z(F)}{\otimes} \underline{R}_G(G/H) = Z(WH/F) \underset{Z(WH)}{\otimes} \underline{R}_G(G/H)\end{aligned}$$

whenever F is a subgroup of WH. Now it is sufficient to use Proposition 2.1.

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Я. Сломиньски, **Об эквивариантности отображений Чженя**

Содержание. В представленной работе определено расщепление системы коэффициентов для эквивариантных гомологий Бредона. Пользуясь их свойствами доказывается, что для конечной группы G $K_G(X) \otimes Q$ изоморфно когомологиям Бредона, с соответствующей системой коэффициентов, компактного G-CW коплекса X. Из этого следует, что $K_G \otimes Q$ можно выразить через обыкновенную $K \otimes Q$-теорию.