Colloques Internationaux C.N.R.S.
274 – ALGÈBRES D'OPÉRATEURS ET LEURS APPLICATIONS EN PHYSIQUE MATHÉMATIQUE

Mathematical Aspects of Local Cohomology (*)

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* Presented at the Colloquium on Operator Algebras and their Applications to Mathematical Physics.
Marseille 20 - 24 June, 1977.

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Preface

In the written version of this talk. I have omitted the material illustrating the notions of non-Abelian cohomology with examples drawn from the theory of operator algebras as well as that describing the applications of local cohomology to quantum field theory. A recent account of this latter topic can be found in [1].

1. Local Cohomology.

Local Cohomology is a novel form of cohomology which is intimately related to a number of interesting structural problems in quantum field theory. My aim is not to explain the physical background to local cohomology [1] but rather to discuss the more mathematical aspects.

Let me begin by recalling the standard notions of singular cohomology. If X is a topological space then an n-simplex of X is a continuous map of the standard n-simplex $\Delta^n = \{(t^o,t^1,\dots,t^n) \in \mathbb{R}^{n+1} : t^i \geq o, \sum_{i=o}^{n} t^i = 1\}$ into X. Let $\Sigma_n(X)$ denote the set of n-simplexes of X. There are face maps ∂_i: $\Sigma_n(X) \to \Sigma_{n-1}(X)$, $i = 0,1,2,\dots,n$ defined by

$$(\partial_i c)(t^o,t^1,\dots,t^{n-1}) = c(t^o,t^1,\dots,t^{i-1},o,t^i,\dots,t^{n-1}). \tag{1.1}$$

An n-cochain of X with values in an Abelian group A is a mapping f: $\Sigma_n(X) \to A$. The set of n-cochains forms an Abelian group under addition denoted by $C^n(X,A)$. Using the face operators and the group structure of A, One defines boundary operators d: $C^n(X,A) \to C^{n+1}(X,A)$ by

$$(df)(c) = \sum_{i=o}^{n+1} (-1)^i f(\partial_i c). \tag{1.2}$$

One checks that $d^2 = o$ and this gives rise to a cochain complex

$$C^o(X,A) \xrightarrow{d} C^1(X,A) \xrightarrow{d} C^2(X,A) \xrightarrow{d} \dots\dots \tag{1.3}$$

The n-cocycles $Z^n(X,A)$ and the n-coboundaries are the subgroups of $C^n(X,A)$ defined by $Z^n(X,A) = \ker d$, $B^n(X,A) = \mathrm{im}\, d$. Conventionally one sets $B^o(X,A) = o$. The cohomology groups $H^n(X,A)$ are the quotient groups $Z^n(X,A)/B^n(X,A)$.

In fact, I am interested in the case that X is Minkowski space $\mathbb{R}^{s+1}$, where s is the number of space dimensions. This has a trivial cohomology; let me remind you why. One picks a fixed origin x_o and if c is an n-simplex, one lets h(c) denote the n+1-simplex which is a cone with vertex x_o and base c.

$$\begin{aligned} h(c)(t^o,t^1,\dots,t^{n+1}) &= t^o x_o + (1-t^o)c(\tau t^1,\tau t^2,\dots,\tau t^{n+1}),\ t^o \neq 1 \\ &= x_o,\ t^o = 1. \end{aligned} \tag{1.4}$$

Here we have written $\tau = (1-t^o)^{-1}$. We have

$$\partial_o h(c) = c, \quad \partial_i h(c) = h(\partial_{i-1}c), \quad i > o \tag{1.5}$$

except that if $c \in \Sigma_o(X)$ then $\partial_1 h(c) = x_o$. A mapping h with this property is called a contracting homotopy. Once we have a contracting homotopy we proceed as follows: if $z \in Z^n(X,A)$, $n \geq 1$, define $y \in C^{n-1}(X,A)$ by $y(b) = z(h(b))$, $b \in \Sigma_{n-1}(X)$. Then if $c \in \Sigma_n(X)$,

$o = (dz)(h(c)) = z(c) - \sum_{i=o}^{n} z(h\partial_i c) = z(c) - dy(c)$.

Hence $H^n(X,A) = o$ for $n \geq 1$.

From now on, we restrict ourselves to Minkowski space and omit the symbol $X = \mathbb{R}^{s+1}$; however, the cohomology will not be trivial because the coefficient objects have a local structure and the definition of cochain is modified by a locality condition. This could be taken to refer to sheaf cohomology however the coefficient objects are not sheaves of Abelian groups but rather nets of Abelian groups over C, the set of compact subsets of Minkowski space ordered under inclusion. The term <u>net</u> is taken to mean a strict inductive system so that if $F_1, F_2 \in C$ and $F_1 \subset F_2$ then $A(F_1)$ is a subgroup of $A(F_2)$.

To date, the nets which have arisen in the study of local cohomology are those which reflect the causal structure of Minkowski space. This causal structure is defined in terms of the quadratic form $(x,x) = x_o^2 - \sum_{i=1}^{s} x_i^2$. x and y are said to be timelike, lightlike and spacelike according as $(x-y)^2 > o$, $(x-y)^2 = o$, $(x-y)^2 < o$. Let $V_+ = \{x : x_o \geq o \text{ and } (x,x) \geq o\}$. If $x - y \in V_+$ and $x \neq y$ then $= (x-V_+) \cap (y+V_+)$ is said to be the double cone with vertices x and y. Let K denote the set of double cones ordered under inclusion and K_o the subset of double cones centred on the origin, i. e. with $y = -x$.

A local n-cochain with values in a net A is a function $f: \Sigma_n \to \bigcup_{F \in C} A(F)$ such that there exists an $\mathcal{O} \in K_o$ with

$$f(c) \in A(\mathcal{O}+c), \quad c \in \Sigma_n. \tag{1.6}$$

Here $\mathcal{O} + c$ denotes $\{x+y: x \in \mathcal{O}, y \in c(\Delta^n)\}$. Since $\mathcal{O} + \partial_i c \subset \mathcal{O} + c$ if f is local so is df. Thus the local cochains give rise to a subcomplex of (1.3) denoted by

$$C_1^o(A) \overset{d}{\to} C_1^1(A) \overset{d}{\to} C_1^2(A) \overset{d}{\to} \ldots \tag{1.7}$$

and there are the obvious definitions of local cocycles, local coboundaries and local cohomology groups $H_1^n(A)$.

A typical example of a net which reflects the causal structure on Minkowski space is the net W constructed from the real C^∞-solutions of the wave equation by defining $W(F)$ to consist of those solutions which vanish on F', the spacelike complement of F, $F' = \{x:(x-y)^2 < o, y \in F\}$. I present here a preview in tabular form of some results involving coefficients which are real C^∞-solutions of invariant partial differential equations with analogous support conditions.

	$\mathcal{W}$	$\mathring{\mathcal{W}}$	K	L	M
H_1^o	o	o	o	o	o
H_1^1, s > 1	o	$\mathbb{R}$	o	o	o
H_1^2, s > 2	o	o	o	$\mathbb{R}$	$\mathbb{R}\times\mathbb{R}$

Here $\mathring{\mathcal{W}}$ denotes the solutions ξ of the wave equation such that $\int\dot{\xi}(o,\underline{x})\ d^s\underline{x} = o$; L denotes the vector wave equation with Lorentz condition $\Box\xi^\mu = o$, $\partial_\mu\xi^\mu = o$; K is the Klein-Gordon equation $(\Box+m^2)\xi = o$ and M Maxwell's equations $\partial^{[\lambda}\xi^{\mu\nu]} = o$, $\partial_\mu\xi^{\mu\nu} = o$. The results for H_1^o and H_1^1 are trivial although the dimensionality restriction s > 1 is essential. If s = 1, one finds for example that $H_1^1(\mathcal{W})$ can be identified with the set of all real C^∞-solutions of the wave equation. By contrast, the results for H_1^2 are not trivial and hinge on the fact that the sheaf of Cauchy data for the wave equation on a spacelike hyperplane is a soft sheaf. These results have some indirect physical interest; $H_1^2(L)$ may be regarded as parametrized by an electric charge and $H_1^2(M)$ by an electric and magnetic charge.

This simple setting is just a testing ground for local cohomology. It shows that it can lead to interesting results and some of the techniques developed do help in the more complicated setting of quantum field theory. Nevertheless, the mathematical nature of this local cohomology is still something of a mystery. Like Čech cohomology it is defined as the inductive limit of cohomology with values in a system of coefficients. Unlike this cohomology it depends on the uniform structure of Minkowski space and not just on its topological structure and the coefficients are nets rather than sheaves.

2. Non-Abelian Cohomology.

Unfortunately, the local cohomology of direct interest in algebraic field theory involves nets of non-Abelian coefficients. Non-Abelian cohomology has been largely developed with a view to applications to sheaf theory and its ramifications [2,3,4]. This sophistication is at present irrelevant to the local cohomology of net systems. Instead what is needed is to understand the purely algebraic problems involved in formulating and manipulating the cocycle identities, an aspect which seems to have been largely neglected in the efforts to achieve a "geometric" interpretation of non-Abelian cohomology.

Suppose one tries to take a non-Abelian group G as coefficients for simplicial cohomology. A o-cocycle is a function $w: \Sigma_o \to G$ with

$$w(\partial_o b) = w(\partial_1 b), \qquad b \in \Sigma_1 \tag{2.1}$$

The composition law in the group is not needed here but it can be used to give the set of o-cocycles a group structure under pointwise multiplication. A 1-cocycle is a function $x: \Sigma_1 \to G$ with

$$x(\partial_o c)\, x(\partial_2 c) = x(\partial_1 c), \qquad c \in \Sigma_2 \tag{2.2}$$

This identity makes use of the composition law in the group and, in revenge, the set of 1-cocycles is no longer, in general, a group under pointwise multiplication. (2.2) has an obvious interpretation in terms of the composition of paths in the basic 2-simplex Δ^2. This suggests that the natural coefficient object for the 1-cohomology is a category $\mathcal{C}$. If $\mathcal{C}_o$ and $\mathcal{C}_1$ denote respectively the set of objects and arrows of $\mathcal{C}$ then a 1-cocycle with values in $\mathcal{C}$ is a pair of functions $w: \Sigma_o \to \mathcal{C}_o$ and $x: \Sigma_1 \to \mathcal{C}_1$ with

$$x(b): w(\partial_1 b) \to w(\partial_o b), \qquad b \in \Sigma_1 \tag{2.3}$$
$$x(\partial_o c)\ x(\partial_2 c) = x(\partial_1 c), \qquad c \in \Sigma_2$$

In what follows, a composition law on a set will always be understood to be associative and to have left and right units so that the set with this composition law becomes a category.

To express the 2-cocycle identity in non-Abelian cohomology, we need a further composition law. The natural coefficient object here is a 2-category (see for example [5]). This is a set C with two composition laws $\times, o$ such that

a) every $\times$-unit is a o-unit

b) the $\times$-composition of o-units, when defined, is again a o-unit,

c) $(a \circ b) \times (a' \circ b') = (a \times a') \circ (b \times b')$ whenever the left hand side is defined.

If two composition laws $\times, o$ on the same set satisfy a), b) and c) we write $\times \prec o$. There are three sets associated with a 2-category: C_o the set of $\times$-units or objects, C_1 the set of o-units or 1-arrows and C_2 the set of all elements or 2-arrows.

To economize on brackets in what follows, we adopt the convention that if $\times \prec o$, a $\times$-composition is to be evaluated before a o-composition, so that $a \circ b \times c$ means $a \circ (b \times c)$ and not $(a \circ b) \times c$.

A composition law $\times$ is said to be <u>Abelian</u> if $a \times b = b \times a$ whenever either side is defined and the following lemma provides some insight into why a 2-category is a natural generalization of a set with an Abelian composition law

<u>Lemma</u> $\times \prec \times$ if and only if $\times$ is Abelian.

A special case of this Lemma is familiar as one of the steps in showing that the higher homotopy groups are commutative.

A 2-cocycle with values in a 2-category C is a triple of functions $w: \Sigma_o \to C_o$, $x: \Sigma_1 \to C_1$, $y: \Sigma_2 \to C_2$

with

$$x(b) \times w(\partial_1 b) = w(\partial_o b) \times x(b), \qquad b \in \Sigma_2$$

$$y(c) \circ x(\partial_1 c) = x(\partial_o c) \times x(\partial_2 c) \circ y(c), \qquad c \in \Sigma_3 \tag{2.4}$$

$$y(\partial_o d) \times x(\partial_2\partial_3 d) \circ y(\partial_2 d) = x(\partial_o\partial_1 d) \times y(\partial_3 d) \circ y(\partial_1 d), \; d \in \Sigma_4$$

It is instructive to look at these identities in terms of piecing together 2-simplexes to form a 3-simplex.

In general, the coefficient object for n-cohomology is an n-category (see [6; p. 552]). This is a set C with an ordered set of n composition laws, say, $\underset{p}{\circ}$, with $p = 0,1,2,\ldots,n-1$ such that $\underset{p}{\circ} \prec \underset{q}{\circ}$ whenever $p < q$. The set of $\underset{p}{\circ}$-units will be denoted by C_p. It is not easy to write down a formula for an n-cocycle, although I believe that such formulae can, in principle, be constructed recursively. Here is the description of a 3-cocycle with values in a 3-category C where the composition laws are denoted by $\times, \circ$ and $\cdot$. We need $w\colon \Sigma_o \to C_o$, $x\colon \Sigma_1 \to C_1$, $y\colon \Sigma_2 \to C_2$ and $z\colon \Sigma_3 \to C_3 = C$ with

$$x \times w_1 = w_o \times x$$

$$y \circ x_1 = x_o \times x_2 \circ y$$

$$z \cdot x_{o1} \times y_3 \circ y_1 = y_o \times x_{23} \circ y_2 \cdot z \tag{2.5}$$

$$z_o \times x_{234} \circ y_{23} \cdot x_{o12} \times y_{o4} \times x_{234} \circ z_2 \cdot x_{o12} \times z_4 \circ y_{12} =$$
$$= y_{o1} \times x_{o34} \times x_{234} \circ z_3 \cdot x_{o12} \times x_{o14} \times y_{34} \circ z_1.$$

Here, for brevity, we have written for example x_{o1} in place of $x(\partial_o\partial_1 d)$, $d \in \Sigma_3$ and z_o for $z(\partial_o e)$, $e \in \Sigma_4$. Thus the first two equations of (2.5) coincide with those of (2.4).

Of course in non-Abelian cohomology too, it is the cohomology classes rather than the cocycles which are important. As far as the o-cohomology goes, the two concepts coincide. The way to look at

1-cocycles is to consider them as the objects of a category $Z^1(\Sigma_*,C)$. If (w,x) and (w',x') are 1-cocycles as in (2.3) above then an arrow in $Z^1(\Sigma_*,C)$ from (w',x') to (w,x) is given by a mapping $r\colon \Sigma_o \to C_1$ such that

$$\begin{aligned} r(a) \times w'(a) &= w(a) \times r(a), & a \in \Sigma_o \\ x(b) \times r(\partial_1 b) &= r(\partial_o b) \times x'(b), & b \in \Sigma_1 . \end{aligned} \tag{2.6}$$

The composition law in $Z^1(\Sigma_*,C)$ is defined pointwise

$$(r\times r')(a) = r(a) \times r'(a) \tag{2.7}$$

Cohomologous 1-cocycles correspond to isomorphic objects in $Z^1(\Sigma_*,C)$. A 1-cocycle (w,x) is said to be trivial if $x(b)$ is a unit for each $b \in \Sigma_1$ and to be trivializable or a 1-coboundary if it is cohomologous to a trivial 1-cocycle.

The set of 2-cocycles with values in a 2-category C should likewise be considered as the objects of a 2-category $Z^2(\Sigma_*,C)$. Thus if (w,x,y) and (w',x',y') are 2-cocycles as in (2.4), a 1-arrow (r,s) from (w',x',y') to (w,x,y) in $Z^2(\Sigma_*,C)$ is given by mappings $r\colon \Sigma_o \to C_1$ and $s\colon \Sigma_1 \to C_2$ with

$$\begin{aligned} r \times w' &= w \times r \\ s \circ x \times r_1 &= r_o \times x' \circ s \\ r_{o1} \times y' \circ s_1 &= s_o \times x_2' \circ x_o \times s_2 \circ y \times r_{12} \end{aligned} \tag{2.8}$$

where we have again adopted the concise notation of (2.5).

The composition of 1-arrows is defined by

$$(r,s) \times (\hat{r},\hat{s}) = (r \times \hat{r},\ r_o \times \hat{s} \circ s \times \hat{r}_1) \tag{2.9}$$

If (r',s') is another 1-arrow from $(w',x'y')$ to (w,x,y) then a 2-arrow from (r',s') to (r,s) in $Z^2(\Sigma_*,C)$ is given by a mapping

$j\colon \Sigma_o \to C_2$ with

$$j \circ r' = r \circ j$$
$$s \circ x \times j_1 = j_o \times x' \circ s' \qquad (2.10)$$

whilst composition of 2-arrows is defined pointwise

$$(j \circ j')(a) = j(a) \circ j'(a); \; (j\times\hat{j})(a) = j(a) \times \hat{j}(a), \; a \in \Sigma_o. \qquad (2.11)$$

Two 2-cocycles are cohomologous if they are joined by an invertible 1-arrow. The role of 2-arrows can be illustrated by reference to Abelian cohomology. If y and y' are cohomologous 2-cocycles with values in an Abelian group A, there is a 1-cochain s such that $y - y' = ds$. If s and s' are two such 1-cochains $d(s-s') = o$ and one may ask if $s - s'$ is even a 1-coboundary, i. e. if $s - s' = dj$ for some o-cohain j. In the non-Abelian theory, j appears as a 2-arrow mapping from the 1-arrow s' to the 1-arrow s.

In general, it seems that one should regard the n-cocycles with values in an n-category $\mathcal{C}$ as the objects of an n-category $Z^n(\Sigma_*, C)$, although I have only verified this for $n \le 3$. One point which should be made here is that there are 2^n different conventions for the n-cocycle identity. One can, however, pass from one convention to another by dualizing with respect to some subset of the composition laws.

To see now non-Abelian cohomology works in practice, I recommend looking at the theory of (non-Abelian) group extensions of a group K by a group G from this point of view. Here Σ_* is a simplicial set constructed from the group G rather than from a topological space whereas the coefficient objects are constructed from K. Thus the coefficient object for the obstruction is a 3-category $\mathcal{Z}$ defined on the set $Z \times \mathrm{InK} \times \mathrm{AutK}$, where Z is the centre of K, AutK the group of automorphisms of K and InK the subgroup of inner automorphisms. The composition laws $\times, o, \cdot$ are defined by

$(z,\sigma,\alpha) \times (z',\sigma',\alpha') = (z\alpha(z'),\ \sigma\alpha\sigma'\alpha^{-1}, \alpha\alpha'),$

$(z,\sigma,\alpha)\ o\ (z',\sigma',\alpha')$ is defined if $\sigma'\alpha' = \alpha$ and equals $(zz',\sigma c$

$(z,\sigma,\alpha) \cdot (z',\sigma',\alpha')$ is defined if $\sigma = \sigma'$, $\alpha = \alpha'$ and equals (zz',σ,α).

The above discussion refers to the basic algebraic structure of non-Abelian cohomology. In practice, there can be variations on this basic structure. In applications to local cohomology, for example, C may have more algebraic structure, in particular more composition laws, than the minimum needed to define the cohomology. This is then reflected in additional structure on $Z^n(\Sigma_*,C)$. Furthermore, one may want to impose further restrictions on the nature of cocycles and require, for example, that x(b) in (2.3) is invertible for each $b \in \Sigma_1$, or, as in applications to local cohomology, that it is unitary. Nevertheless, these are minor details and, in its applications to algebraic field theory, local cohomology may be described mathematically by saying that the locality condition of section 1 is combined with the non-Abelian cohomology of section 2 in an operator-algebraic context.

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