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Homology of Algebraic Theories

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by

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The dissertation of Christopher Leonard Reedy is approved, and it is acceptable in quality and form for pulication on microfilm:

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ABSTRACT OF THE DISSERTATION

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A model category structure on the category of simplicial algebras over a simplicial theory is constructed. Given an extension of theories, $\varphi: S \to T$, and an algebra over S, we construct the free extension of this algebra to T. We define the homotopy free extension to T by using the definition of Anderson (unpublished.) It is shown that the natural map from the homotopy free extension to the free extension is a weak homotopy equivalence. This is also shown to be the case for extensions from based spaces to the algebras over pros and props with free symmetric group action. A decomposition theorem for the homology of the free algebra for this second extension is proved. As an example the graded group structure of $H_*(\Omega^\infty S^\infty X; Z/2)$ for X a connected space is computed.

Chapter I. Introduction.

The notion of an algebraic theory was originally introduced by Lawvere(AT). An algebraic theory is basically a formal description of a system of operations, satisfying certain relations, that can act on a set. Familiar objects of algebra, like groups and rings, can be described as the algebras of an algebraic theory.

A topological algebraic theory is an algebraic theory where the sets of operations have a topology. Ordinary algebraic theories can be considered as being topologized by the discrete topology. A topological algebra over a topological theory has the obvious definition.

The interest in topological theories has not arisen from the study of objects like topological rings; but, rather from the study of iterated loop spaces. One of the early examples of a topological theory was Stasheff's(HV) A^{∞}-spaces, which can easily be seen to be the algebras of the appropriate theory. Stasheff's work provides a good description of the theory of one-fold loop spaces.

Important later results were obtained by Beck, and Boardman-Vogt and May. Beck(IH) showed that there is a theory resembling $\Omega^n S^n(-)$, which has the property that the algebras over this theory are n-fold loop spaces. Boardman-Vogt(HE,AS) and May(IL) showed that n-fold loop spaces can be approximated by the algebras over certain objects, known as props, out of which theories can be built, but which are, in some ways, more easily handled than theories.

An important element of all of this work has been the study of the free algebras over these theories, which exist by category theory. The free algebra on a space, for any theory whose algebras approximate n-fold loop spaces, must approximate (homotopically) the $\Omega^n S^n$ functor.

One problem with the standard constructions of free algebras has been the rigidity of the constructions for these objects. For this reason, the study of these objects, from a homotopy point of view, can be difficult. The main results of this thesis, Theorems III and IV, state that, for a certain sufficiently large class of algebras over a theory, the homotopy type of the free algebra can be approximated by the realization of a certain simplicial space, this realization being the homotopy free algebra [Anderson(HF)]. This allows the use of the standard techniques of simplicial theory in the study of these objects.

In chapter two, theories and algebras over theories are defined, and some of the equivalent formulations of these objects are exhibited. In chapter three, a model category structure on the category of algebras over a theory is constructed. The importance of this work is to show that the class of algebras for which the later results hold is large enough to approximate all the algebras.

Chapter four is the proof that the construction of the homotopy free algebra produces a homotopy algebra. Chapter five contains the proofs of the main theorems. In the last chapter some examples and applications of the previous results are produced. These include a theorem describing the homology of a free algebra over a prop, and a computation of the homology of $\Omega^{\infty}S^{\infty}(X)$, for connected spaces X, based on this work. Other similar results in this line, have been proved by Fred Cohen(HL).

In this paper I always work in simplicial sets, rather than topological spaces. Though the proofs seem to depend explicitly on the simplicial structure, they could, presumably, be translated into topological spaces. The problem with this translation seems to be the "cofibrancy" of the theories, and not that of the spaces. Lacking a good description of what a cofibrant theory is I have not made this translation. I should point out, however, that these results do apply to topological theories which are the realizations of simplicial theories.

At all times Sp (spaces) will refer to the category of simplicial sets, always considered to have the standard closed simplicial model category structure [Quillen(HA)]. Several times throughout the paper I will use standard constructions in category theory, which can be found in MacLane(CT). I will use the term simplicial category to mean a simplicial object in the category of categories which has the additional property that all of the face and degeneracy functors are isomorphisms on objects. Such an object should be thought of as a category in which the objects are endowed with a simplicial-set-valued hom. The underlying category of a simplicial category is the degree zero part of the simplicial category. I will sometimes use constructions such as Kan extension in simplicial categories, or in more general simplicial objects in the category of categories. When I do this I will mean that the construction is to be done in each degree, and then pieced together to form a simplicial object. Finally, I will follow MacLane's convention, and take the symbol $C(a,b)$ to mean the morphisms in the category C from the object a to the object b.

Let N be the opposite category to the category of finite sets, the empty set included. Following Lawvere(AT), define a simplicial algebraic theory as a functor $T:N \to \mathcal{T}$ where $\mathcal{T}$ is a simplicial category, T is an isomorphism on objects, and the collection of maps $T(p_i):T(n) \to T(1)$ represent $T(n)$ as the product of n copies of $T(1)$. (p_i in $\mathcal{T}(n,1)$ is the dual to the map $\underline{1} \to \underline{n}$ of finite sets taking $1\in\underline{1}$ to $i\in\underline{n}$. This map corresponds to the projection onto the i^{th} factor.) Note that the identity functor $Id_N: N \to N$ is an algebraic theory, since the finite set with n elements is the n-fold coproduct of the finite set with one element.

In the future the objects of T will be written as 0, 1, ..., n, ... The identity morphism on any object will in general be denoted as Id_N or id_n.

The condition for a theory implies that the map $\prod_{i=1}^{n} T(p_i): T(m,n) \to T(m,1)^n$ is an isomorphism. We will use this to formulate an equivalent, but more useful definition of a simplicial theory. If T is a simplicial theory, let $T(n) = T(n,1)$. We have seen that $T(m,n) \cong T(n)^m$. We see also that the functor T induces a map $\varkappa_n: N(n) \to T(n)$. Composition induces a map $T(m,n)\times T(k,m) \to T(k,n)$, and so a map $T(m,1)\times T(k,m) \to T(k,1)$, which is a map $T(m)\times T(k)^m \to T(k)$. I will call this map c_k^m. If we let $(c_k^m)^n: T(m)^n \times T(k)^m \to T(k)^n$ represent the composition in T, then it is easy to see that the equation $c_k^n \circ id_n \times (c_k^m)^n = c_k^m \circ c_m^n \times id_k^m$ holds, since this is just the associativity of composition in the category T. From this it is easy to see that a theory is just a collection of spaces $T(n)$, and a collection of maps $c_m^n: T(n)\times T(m)^n \to T(m)$, and maps $\varkappa_n: N(n) \to T(n)$, for all integers $m,n \geq 0$, and such that the following relations hold:

(i) the associativity of composition holds.

(ii) the map $c_m^n \circ \varkappa_n \times id | \{p_i\}\times T(m)^n : \{p_i\}\times T(m)^n \to T(m)$ is just projection onto the i^{th} factor.

(iii) the map $c_m^m \circ id \times \varkappa_m^m | T(m)\times\{\iota_m\}: T(m)\times\iota_m \to T(m)$ is the identity, where $\iota_m \in N(m)^m$ corresponds to the identity function $\underline{m} \to \underline{m}$.

Lemma 1. Any collection of spaces $\mathcal{T}(n)$, $n \geqq 0$, together with maps $c^n_m:\mathcal{T}(n)\times\mathcal{T}(m)^n \to \mathcal{T}(m)$, and $\varkappa_n:\mathcal{N}(n) \to \mathcal{T}(n)$, satisfying (i), (ii), and (iii), give rise to a category $\mathcal{T}$ and a functor $T:\mathcal{N} \to \mathcal{T}$, where the objects of $\mathcal{T}$ are 0,1, ..., and the morphisms $\mathcal{T}(m,n) \cong \mathcal{T}(m)^n$. The functor T is given by $T(\underline{n}) = n$, and $T:\mathcal{N}(m,n) \to \mathcal{T}(m,n)$ is given by $\varkappa^n_m$. Further the functor T is a simplicial theory.

Proof: The proof is straightforward. The associativity of composition is guaranteed by condition (i). Conditions (ii) and (iii) are used to verify that the identities behave properly. Condition (ii) also shows that T is a functor, and the fact that $\mathcal{T}$ is a theory follows from the definition of $\mathcal{T}(m,n)$.

Given two theories S and T, $S:\mathcal{N} \to \mathcal{S}$, $T:\mathcal{N} \to \mathcal{T}$, then a functor $\varphi:\mathcal{S} \to \mathcal{T}$ is a morphism of theories if $\varphi\circ S = T$. In terms of Lemma 1 this is the same as having a collection of maps $\varphi_n:\mathcal{S}(n) \to \mathcal{T}(n)$, such that $\varphi_n\circ\varkappa_n = \varkappa_n$, and $\varphi_m\circ c^n_m = c^n_m\circ\varphi_n\times(\varphi_m)^n$. We will write $\varphi:S \to T$ for such a φ. Given any object $X\in\mathcal{S}p$ we have a natural functor $\hat{X}:\mathcal{N} \to \mathcal{S}p$ by $\hat{X}(n) = X^n$, and if $\alpha^*\in\mathcal{N}(m,n)$, (i.e. $\alpha:\underline{n} \to \underline{m}$, a map of finite sets), then $\hat{X}(\alpha^*) = X^\alpha:X^m \to X^n$. We define an algebra over T as a functor $X:\mathcal{T} \to \mathcal{S}p$ such that the functor $X\circ T$ is naturally isomorphic to the functor $\widehat{X(1)}$ (we require that $\widehat{X(1)}(1) = X(1) = X\circ T(1)$). We see that the functor X is an algebra over $\mathrm{Id}_{\mathcal{N}}$. We call the space X(1) the underlying space of X.

Given two T-algebras X and Y, then a T-algebra morphism is a natural transformation $\eta:X \to Y$. We define the category $\mathcal{T}\text{-}\mathcal{A}\ell g$ to be the full subcategory of the functor category $\mathcal{F}_*(\mathcal{T},\mathcal{S}p)$ of functors from $\mathcal{T}$ to $\mathcal{S}p$, whose objects are the T-algebras. We see that the underlying space defines a functor $U:T\text{-}\mathcal{A}\ell g \to \mathcal{S}p$.

Lemma 2. The category $Id_N\text{-}\mathcal{A}\ell g$ is equivalent to the category $\mathcal{S}p$.
Proof: The functor U and "^", exhibited above, provide the equivalence.

In the future I shall tend to confuse $Id_N\text{-}\mathcal{A}\ell g$ and $\mathcal{S}p$. The importance of the lemma above is that it tells us that a homomorphism of algebras is determined by the map on the underlying spaces. I.e. the forgetful functor $U:T\text{-}\mathcal{A}\ell g \to \mathcal{S}p$ is faithful, or one-to-one on morphisms.

If we let $X \in \mathcal{S}p$ and define $\mathcal{E}nd_X$ to be the full subcategory of $\mathcal{S}p$ whose objects are $*=X^0$, X^1, ... , then $\hat{X}$ can be considered as a functor $\hat{X}:N \to \mathcal{E}nd_X$. (if $X=*$, then we must make $*,*^1$, etc. into different objects.) The condition for a theory is easily verified, so that $\hat{X}$ is a theory. Further; we see that a transformation $\varphi:T \to \hat{X}$ of theories is just a functor $\varphi:T \to \mathcal{E}nd_X$ with $\varphi\circ T = \hat{X}$, i.e. φ is a T-algebra. On the other hand, any T-algebra comes from a functor $T \to \mathcal{E}nd_X$ and therefore, we get the familiar result that the T-algebra structures on X correspond to morphisms of theories $T \to \hat{X}$. [See May(IL).]

Combining the above observations and Lemma 1, we get the following result:

Proposition 3. A T-algebra structure on a space X is given by a collection of maps $\theta(n):T(n)\times X^n \to X$, such that:

(i) $\theta(n)(p_i,x_1, \ldots ,x_n) = x_i$ (i.e. $\theta(n)|\{p_i\}\times X^n$ is projection on the i^{th} factor.)

(ii) $\theta(m)\circ c^n_m = \theta(n)\circ id\times\theta(m)^n : T(n)\times T(m)^n\times X^m \to X$.

Proof: The proof is immediate. Condition (i) is a normalization condition, while condition (ii) is the associativity of composition.

Given two theories S and T, and a transformation $\varphi: S \to T$, there is a functor $\varphi^*: T\text{-}\mathcal{A}\ell g \to S\text{-}\mathcal{A}\ell g$, given by $\varphi^*(X) = X\circ\varphi$. Clearly $X\circ\varphi$ is an algebra since composition with φ does not change the Id_N structure, which determines whether a functor is an algebra. Consider the diagram:

$$\begin{array}{ccc} T\text{-}\mathcal{A}\ell g & \xrightarrow{\varphi^*} & S\text{-}\mathcal{A}\ell g \\ \cap & & \cap \\ F_*(T,Sp) & \xrightarrow{\varphi^*} & F_*(S,Sp) \end{array}$$

There is a standard construction for a left adjoint to the φ^* on the bottom line of the diagram, given by Lan^{φ} [MacLane(CT)].

Proposition 4. If $X \in S\text{-}\mathcal{A}\ell g$ then $Lan^{\varphi}(X) \in T\text{-}\mathcal{A}\ell g$.

I will defer the proof of this proposition until Chapter IV when it can be produced as a corollary to a more general theorem. This proposition does allow us to assert the following corollary which states the existence of free algebras.

Corollary: $T\text{-}\mathcal{A}\ell g(Lan^{\varphi}(X),Y) \cong S\text{-}\mathcal{A}\ell g(X, Y\circ\varphi)$.

Proof: This is the definition of the left adjointness of Lan^{φ} in the functor categories above. Since $T\text{-}\mathcal{A}\ell g$ and $S\text{-}\mathcal{A}\ell g$ are full subcategories of the functor categories, then we only need to know that $Lan^{\varphi}(X)$ is in $T\text{-}\mathcal{A}\ell g$, but this is the proposition.

If $S = Id_N$ then $\varphi = T$ above, and we write $Lan^T(X) = TX$, which is the free T-algebra on the space X. For reference I state the following standard result [Beck(IH)].

Proposition 5. If $X \in Sp$, and T is a simplicial theory, then TX is given as $\sum_n X^n \times T(n)/\sim$, where $\sim$ is the equivalence relation generated by $(\alpha(x),\vartheta) \sim (x,\vartheta\circ\alpha)$, where $x \in X^m$, and $\vartheta \in T(n)$, and $\alpha : m \to n$ is a morphism in $N(m,n)$.

Note: If T is a **theory,** then $T(n)$ is a T-algebra, given by the functor $Fn: T \to Sp$ by $Fn(m) = T(n,m)$. The condition for Fn to be an algebra is the same as that for T to be a theory. The maps c_n^m are the structure maps for the algebra Fn. An obvious result is the fact that Fn is the free T-algebra on the set $\{p_1, \dots , p_n\} \subset T(n)$. This follows easily from the fact that the element $(p_1, \dots , p_n) \in T(n)^n$ corresponds to the identity morphism in $T(n,n)$.

There are two other types of algebraic structures which will be studied in this paper, which are similar to theories. These two objects are known as Pros and Props [Beck(HS), or Boardman-Vogt(AS)]. Let P be the category of finite ordered sets (the empty set included), where the morphisms are the monomorphisms. Let P_σ be the category of monomorphisms of unordered sets. Note that $P \subset P_\sigma$ in the obvious way. P_σ differs from P in that Σ_n (the n^{th} symmetric group) acts on $\underline{n} \in P_\sigma$.

Definition: A Pro (Prop) A is a contravariant functor $A: P(P_\sigma) \to Sp$, together with maps $c_I^n : A(n) \times A(i_1) \times \dots \times A(i_n) \to A(i_1 + \dots + i_n)$, and a point $\iota \in A(1)$, such that

(i) $A(0) = *$,

(ii) the obvious associativity for c holds,

(iii) ι acts as a two sided identity for c, i.e. $c^1_n:\{\iota\}\times A(n) \to A(n)$, and $c^n_{1,\dots,1}:A(n)\times\{(\iota,\dots,\iota)\} \to A(n)$, are the identity maps,

((iv) and if A is a prop, c satisfies the obvious equivariance relations, with respect to the actions of the symmetric groups on A.)

Let N_b be the theory of based spaces. This theory is given by $N_b(n) = \underline{n}^+$, the set with n elements and a base point, and the compositions given in the obvious way. It is easily verified that an N_b-algebra is just a base pointed space. If (X,x) is a based space, then X defines a covariant functor $X:P_\sigma \to Sp$, given by $X(n) = X^n$, and if $\alpha:\underline{m} \to \underline{n}$ is a map in P_σ, then $X(\alpha)$ puts the i^{th} coordinate of X^m in the $\alpha(i)^{th}$ coordinate of X^n, and makes the other coordinates the base point.

Definition: If A is a pro (prop) then (X,x) is an A-algebra if there are based maps $\theta(n):A(n)\times X^n \to X$, such that the relation $\theta(n)\circ(id\times X(\alpha)) = \theta(m)\circ(A(\alpha)\times id):A(n)\times X^m \to X$ holds for all $\alpha:\underline{m} \to \underline{n}$ in P (P_σ), and such that θ is compatible with the composition c.

It is easy to show that there is a category of A-algebras, defined in the obvious way [cf. May(IL)], and a free A-algebra AX on any based space (X,x), which is the left adjoint to the forgetful functor $A\text{-}\mathcal{A}\ell g \to Sp$. In fact, AX is given in an analogous way to the construction of TX in Proposition 5. The following result of Beck(HS), is quoted to show that the category of A-algebras is the same as a category of algebras over a theory. This allows us to use the results of the next chapter when we are considering a pro or a prop, as well as when we are considering a theory.

Proposition 6. If we let A be the category with objects 0, 1, ... and $A(m,n) = (A\underline{m})^n$, and if we let $\hat{A}:N_b \to A$ be the obvious functor, then the functor $\hat{A}\circ N_b$ is a theory, and the functor $\hat{A}$ is a morphism of theories. Further; the forgetful functor $\hat{A}$-$\mathcal{Alg} \to$ A-$\mathcal{Alg}$ is an equivalence of categories.

Note from the proposition that $\hat{A}n = A(n,1) = An$. In fact it is clear that in order for the forgetful functor to be an equivalence of categories that it must be true that AX is isomorphic to $\mathrm{Lan}^{\hat{A}}X$ for all based spaces X.

One other concept I need to mention, is that of a monad (otherwise known as a triple.) If C is a category, then a monad in C is a functor $T:C \to C$ together with natural transformations $\eta:\mathrm{id}_C \to T$, and $\mu:T^2 \to T$, such that $\mu\circ T\eta = \mu\circ\eta T = \mathrm{id}_C$, and $\mu\circ T\mu = \mu\circ\mu T:T^3 \to T$. An algebra over T is an object $X \in C$, together with a morphism $\xi:TX \to X$, such that $\xi\circ\eta X = \mathrm{id}_X$, and $\xi\circ T\xi = \xi\circ\mu$. It is a standard result that T-algebra X in C has a simplicial resolution, where X is the augmentation of a simplicial C-object, whose n^{th} degree is $T^{n+1}X$, and where the face maps are induced by μ and ξ, and the degeneracy maps are induced by η. (See Beck(IH), for more details.) The first degrees of this simplicial object look like:

$$X \underset{\xi}{\overset{\eta}{\rightleftarrows}} TX \;\; \begin{matrix} \xrightarrow{\eta T} \\ \xleftarrow{\mu} \\ \xrightarrow{T\eta} \\ \xleftarrow{\xi} \end{matrix} \;\; T^2X \;\ldots$$

The maps η and ηT are the contracting degeneracies, and are the only maps in the diagram which are not T-algebra morphisms.

Any pair of adjoint functors induce a monad, which is the composition of the right adjoint with the left adjoint. Thus, if $\varphi : S \to T$ is a morphism of theories, then the functor $\varphi^* \circ \mathrm{Lan}^{\varphi} : S\text{-}\mathcal{A}\ell g \to S\text{-}\mathcal{A}\ell g$ is a monad. It is interesting to note that the algebras over this monad are just the T-algebras in $S\text{-}\mathcal{A}\ell g$, and that in fact the functor φ^* carries the category $T\text{-}\mathcal{A}\ell g$ to the category of $\varphi^* \circ \mathrm{Lan}^{\varphi}$ algebras, and is an equivalence of categories.

Chapter III. The Model Category Structure on T-$\mathcal{A}\ell g$

The object of this chapter is to prove:

Theorem: **The category T-$\mathcal{A}\ell g$ is a closed simplicial model category.**

And, also, to understand the structure of T-$\mathcal{A}\ell g$ as a model category. It will be helpful to know the following proposition:

Proposition 1. If $F:\mathcal{S}p \to \mathcal{S}p$ is a functor with a natural transformation $\varepsilon:\mathrm{Id}_{\mathcal{S}p} \to F$, such that the natural map $F(X\times Y) \to F(X)\times F(Y)$ is an isomorphism, then if X is a T-algebra, so is F(X). Further given another such functor G, and a natural transformation $\varphi:F \to G$, such that $\varphi\circ\varepsilon = \varepsilon$, then $\varphi(X):F(X) \to G(X)$ is a map of T-algebras. In particular the map $\varepsilon(X):X \to F(X)$ is a map of T-algebras.

Proof: Let $\theta(n):\mathcal{T}(n)\times X^n \to X$ be the structure map for X. The structure map for F(X) is given by the composition

$$\mathcal{T}(n)\times F(X)^n \xrightarrow{\varepsilon\times\mathrm{id}} F(\mathcal{T}(n))\times F(X)^n \cong F(\mathcal{T}(n)\times X^n) \xrightarrow{F(\theta(n))} F(X)$$

It is easy to verify that the fact that X is a T-algebra implies the associativity of the T-algebra for F(X). The projections are correct because of the definition of the isomorphism $F(X)^n \cong F(X^n)$. To verify that $\varphi(X)$ is a T-algebra map, it is enough to notice that φ induces a map from the above diagram for the structure maps of F(X) to the corresponding ones of G(X).

It will be important to know some category theoretic results about the category T-$\mathcal{A}\ell g$.

Proposition 2. The category $T\text{-}\mathcal{A}\ell g$ has images, i.e. given any T-algebra map $f:X \to Y$ then $f(X) \subset Y$ is a sub T-algebra of Y.

Proof: Obvious.

Proposition 3. The functor $U:T\text{-}\mathcal{A}\ell g \to \mathcal{S}p$ creates limits.

Proof: This is the familiar result that the categorical product for groups, rings, etc. is gotten by taking the categorical product as sets and then giving the product the obvious structure as a group, ring, etc. This result holds for pullbacks, equalizers, and all other forms of inverse limit.

Corollary: $T\text{-}\mathcal{A}\ell g$ has all small limits.

Proof: Since U creates limits, and $\mathcal{S}p$ has all small limits, then $T\text{-}\mathcal{A}\ell g$ has all small limits.

Proposition 4. $T\text{-}\mathcal{A}\ell g$ has co-equalizers.

Proof: Let $\mathcal{D}$ be the category $\cdot \rightrightarrows \cdot$ with two objects and two maps from the one to the other. There is a natural functor $R:T\text{-}\mathcal{A}\ell g \to F_*(\mathcal{D},T\text{-}\mathcal{A}\ell g)$ which is the adjoint to the functor $\mathcal{D} \to *$. R preserves limits since they are computed degreewise in the functor category. Further it is easy to verify that the smallness condition necessary for the Freyd Adjoint Functor Theorem holds [MacLane(CT), page 117], since the category $T\text{-}\mathcal{A}\ell g$ has images. Thus by the Adjoint Functor Theorem there is an adjoint functor $L:F_*(\mathcal{D},T\text{-}\mathcal{A}\ell g) \to T\text{-}\mathcal{A}\ell g$ to the functor R. It is now easy to verify that L applied to any coequalizer diagram is the coequalizer of that diagram.

Proposition 5. T-$\mathcal{Alg}$ has all small coproducts.

Proof: Take a family $\{X_\alpha\}_{\alpha\in A}$ of T-algebras. Consider the diagram:

$$T(\sum_\alpha UTX_\alpha) \overset{\sum_\alpha \mu}{\underset{\sum_\alpha T\xi_\alpha}{\rightrightarrows}} T(\sum_\alpha UX_\alpha) \to Z$$

where μ is the product in the monad determined by T, ξ_α is the structure map for X_α as a T-algebra, and Z is the coequalizer of the above diagram. Since T is a left adjoint then $T(\sum_\alpha X_\alpha) = \bigoplus_\alpha T(X_\alpha)$, where $\oplus$ is the categorical coproduct in T-$\mathcal{Alg}$, and Σ is the disjoint union, which is the categorical coproduct in $\mathcal{Sp}$.

Given a collection of maps $g_\alpha : X_\alpha \to Y$, we get a map $\sum_\alpha g_\alpha : \sum_\alpha UX_\alpha \to UY$, which gives a map $T(\sum_\alpha UX_\alpha) \to Y$ by the adjointness of T and U. This map equalizes the two arrows in the above diagram (since it does on each factor of the direct sum), and, therefore, we get a map $g : Z \to Y$ extending each $g_\alpha : X_\alpha \to Y$, which is uniquely determined, since it is on each TX_α. Thus $Z = \bigoplus_\alpha X_\alpha$ as desired. Q.E.D.

Corollary: T-$\mathcal{Alg}$ has all small co-limits.

Proof: This follows as a standard result from the fact that T-$\mathcal{Alg}$ has coequalizers and all small coproducts.

We will say that an algebra T is discrete if the spaces $\mathcal{T}(n)$ are discrete simplicial sets (i.e. the only non-degenerate simplices are in degree zero) for all n. It is clear that discrete algebras are just algebras in the sense of Lawvere(AT). In particular monoids, groups, abelian groups, etc. are discrete algebraic theories.

If T is a simplicial theory, define $\mathcal{T}_n(m) = (\mathcal{T}(m))_n$ that is the n^{th} degree of the simplicial set $\mathcal{T}(m)$. The sets $\mathcal{T}_n(m)$ fit together to form a theory (discrete). This allows us to consider a simplicial theory as a simplicial object in the category of theories. Given a theory T, let T_n be the n^{th} degree of T, which is a discrete theory. If X is a T-algebra then X_n (the n^{th} degree of X) is a T_n algebra. Given a simplicial map $\alpha : m \to n$, we have a transformation of theories $T\alpha : T_m \to T_n$, which induces transformations $T\alpha^* : T_n\text{-}\mathcal{A}\ell g \to T_m\text{-}\mathcal{A}\ell g$, and $T\alpha_* : T_m\text{-}\mathcal{A}\ell g \to T_n\text{-}\mathcal{A}\ell g$, where $T\alpha_*$ is the left adjoint of $T\alpha^*$ which is just the restriction of theories. If X is a T-algebra then in fact we can consider $X\alpha : X_m \to T\alpha^*(X_n)$ and this will be a T_m-algebra map. Thus we can consider a simplicial algebra over a simplicial theory as a space X such that each X_n is a T_n-algebra, and such that the function $X\alpha : X_m \to T\alpha^*(X_n)$ is a T_m-algebra homomorphism. This gives us an alternative description of T-algebras which will be useful later. Anderson(TT) has used this description of simplicial theories to provide another proof that simplicial T-algebras form a model category.

Consider the category of discrete algebras over a discrete theory. We define a projective T-algebra as a T-algebra which lifts through any map of T-algebras which is onto as sets. A projective extension is a map $S \to P$, such that the dotted arrow exists (making the diagram commute) in any diagram of the form:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & A \arrow[d] \\ P \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

where the morphism $A \to B$ is onto as sets. It is easy to verify that any map of the form $S \to S \oplus F$, where F is a free algebra, that is $F \cong TX$, for some set X, is a projective extension. Any projective object is a retract of a free object, and any projective extension is a retract of a free extension (that is one of the form $S \to S \oplus F$.)

Following Quillen(HA) define $T\text{-}\mathcal{Alg}(X \otimes K, Y)$, where $X, Y \in T\text{-}\mathcal{Alg}$, and $K \in \mathcal{Sp}$, to be the set of collections of maps $\{f_\sigma\}_{\sigma \in K}$ such that if $\sigma \in K_m$ then $f_\sigma : X_m \to Y_m$ is a T_m-algebra map, and such that the relation $Y\alpha \circ f_\sigma = f_{\alpha\sigma} \circ X\alpha$, where $\alpha : m \to n$ is a simplicial map. It is easy to verify that $T\text{-}\mathcal{Alg}(-\otimes-,-)$ defines a functor $T\text{-}\mathcal{Alg}^{op} \times \mathcal{Sp}^{op} \times T\text{-}\mathcal{Alg} \to \mathcal{Sp}$.

Definition: $\underline{T\text{-}\mathcal{Alg}}(X,Y) \in \mathcal{Sp}$ for $X, Y \in T\text{-}\mathcal{Alg}$, is given by $\underline{T\text{-}\mathcal{Alg}}(X,Y)_n = T\text{-}\mathcal{Alg}(X \otimes \Delta^n, Y)$.

By the arguments above, we see that $\underline{T\text{-}\mathcal{Alg}}$ defines a functor $T\text{-}\mathcal{Alg}^{op} \times T\text{-}\mathcal{Alg} \to \mathcal{Sp}$. Given $f = \{f_\sigma\} \in T\text{-}\mathcal{Alg}(X \otimes K, Y)$, and $g = \{g_\sigma\} \in T\text{-}\mathcal{Alg}(Y \otimes K, Z)$, then we can define $g \circ f \in T\text{-}\mathcal{Alg}(X \otimes K, Z)$ by the formula $(g \circ f)_\sigma = g_\sigma \circ f_\sigma$. This product allows us to define a composition $\underline{T\text{-}\mathcal{Alg}}(X,Y) \times \underline{T\text{-}\mathcal{Alg}}(Y,Z) \to \underline{T\text{-}\mathcal{Alg}}(X,Z)$. It is clear that $\underline{T\text{-}\mathcal{Alg}}(X,Y)_0 \cong T\text{-}\mathcal{Alg}(X,Y)$. In fact, if $f \in T\text{-}\mathcal{Alg}(X,Y)$ and $g \in \underline{T\text{-}\mathcal{Alg}}(Y,Z)_n$ then the composition $g \circ s_0^n(f)$ from the above composition, is the same as that that arises from the fact that $T\text{-}\mathcal{Alg}(X \otimes K, Y)$ is a functor. This shows that the composition above defines a simplicial category structure on the category $T\text{-}\mathcal{Alg}$.

Given $X \in T\text{-}\mathcal{Alg}$ and $K \in \mathcal{Sp}$, define the object $X \otimes K \in T\text{-}\mathcal{Alg}$, by $(X \otimes K)_m = \underset{\sigma \in K_m}{\oplus} (X_m)$, where $\oplus$ is the coproduct as T_m-algebras. Given $\alpha : m \to n$, a simplicial map, let $i_\sigma : X_m \to \oplus(X_m)_\sigma$ be the inclusion, then define $(X\otimes K)\alpha$ to make the relation $(X\otimes K)\alpha\circ i_{\sigma} = T\alpha^*(i_{\alpha\sigma})\circ X\alpha$ hold. Since we have defined $(X\otimes K)\alpha$ to be a T_m-algebra map, then $X\otimes K \in T\text{-}\mathcal{A}\ell g$. Further, a map $f \in T\text{-}\mathcal{A}\ell g((X\otimes K),Y)$, is a collection of maps $f_\sigma: X_m \to Y_m$ for all $\sigma \in K_m$, such that the relation $Y\alpha\circ f_\sigma = f_{\alpha\sigma}\circ X\alpha$. This is just the definition of a map in $T\text{-}\mathcal{A}\ell g(X\otimes K,Y)$. In fact there is a map canonical map in $T\text{-}\mathcal{A}\ell g(X\otimes K,(X\otimes K))$ which induces this isomorphism. Thus we have shown:

Proposition 6. There is a natural isomorphism of $T\text{-}\mathcal{A}\ell g(X\otimes K,Y)$ to $T\text{-}\mathcal{A}\ell g((X\otimes K),Y)$.

From now on I will use the terminology $T\text{-}\mathcal{A}\ell g(X\otimes K,Y)$ to mean either of the two sets above.

Define $\eta: K \to \underline{T\text{-}\mathcal{A}\ell g}(X,X\otimes K)$ as follows: if $\sigma \in K_m$ then $\eta(\sigma)$ equals $X\otimes\sigma: X\otimes\Delta^m \to X\otimes K$, where $\sigma:\Delta^m \to K$ is the standard map. It is easy to verify that η is a simplicial map.

Proposition 7. The composition $K\times\underline{T\text{-}\mathcal{A}\ell g}(X\otimes K,Y) \to \underline{T\text{-}\mathcal{A}\ell g}(X,X\otimes K)\times\underline{T\text{-}\mathcal{A}\ell g}(X\otimes K,Y) \to \underline{T\text{-}\mathcal{A}\ell g}(X,Y)$ has an adjoint $\underline{T\text{-}\mathcal{A}\ell g}(X\otimes K,Y) \to \underline{\mathcal{S}p}(K,\underline{T\text{-}\mathcal{A}\ell g}(X,Y))$, which is an isomorphism.

Note: $\underline{\mathcal{S}p}(K,L)$ is the simplicial function space L^K.

Proof: By definition $T\text{-}\mathcal{A}\ell g(X\otimes\Delta^m,Y) \cong \mathcal{S}p(\Delta^m,\underline{T\text{-}\mathcal{A}\ell g}(X,Y))$. Since both of these functors commute with colimits in the Δ^m variable this is also true with Δ^m replaced by any simplicial set K. Finally, it is enough to see that from the definitions all that is required is that $X\otimes(K\times L) \cong (X\otimes K)\otimes L$, for any simplicial sets K and L, which is clearly true.

Given a simplicial set K, define $f \in T\text{-}\mathcal{A}\ell g^P(X\times K,Y)$, to be a simplicial map $f:X\times K \to Y$, such that $f_m|X_m\times\{\sigma\} \to Y_m$ is a T_m-algebra map for all $\sigma \in K_m$. It is easy to see that a map in $T\text{-}\mathcal{A}\ell g^P(X\times K,Y)$ is the same as a map in $T\text{-}\mathcal{A}\ell g(X\otimes K,Y)$, and that in fact there is a natural map in $T\text{-}\mathcal{A}\ell g^P(X\times K,X\otimes K)$ which induces this isomorphism.

Proposition 8. $f:X\times K \to Y$, a simplicial map, is in $T\text{-}\mathcal{A}\ell g^P(X\times K,Y)$ if and only if the diagram,

$$\begin{array}{ccccc} \mathcal{T}(n)\times X^n\times K & \xrightarrow{\text{id}\times\text{shuffle}} & \mathcal{T}(n)\times(X\times K)^n & \xrightarrow{\text{id}\times f^n} & \mathcal{T}(n)\times Y^n \\ \downarrow{\scriptstyle \theta(n)\times\text{id}} & & & & \downarrow{\scriptstyle \theta(n)} \\ X\times K & & \xrightarrow{\qquad\qquad f \qquad\qquad} & & Y \end{array}$$

commutes, where $\theta(n)$, is the structure map for X or Y as a T-algebra, and shuffle is the obvious shuffle and diagonal homomorphism.

Proof: Write down what it means for the above diagram to commute, and it is clear that this is the same condition as the one that is required for f to be a $T\text{-}\mathcal{A}\ell g^P$ map.

A map in $T\text{-}\mathcal{A}\ell g^P$ should be thought of as a collection of maps $X \to Y$ parametrized by the points of K. Thus a map in $T\text{-}\mathcal{A}\ell g^P(X\times I,Y)$ is the analogous concept to the idea of homotopy of T-algebra morphisms, going through T-algebra morphisms.

We know that the functor $(-)^K:\mathcal{S}p \to \mathcal{S}p$, has the properties of Proposition 1, where the natural transformation $X \to X^K$ is given as the adjoint to the map $K \to *$ of simplicial sets. Thus if X is a T-algebra then X^K is also a T-algebra; further, the map $X^L \to X^K$ induced by a simplicial map $K \to L$ is a T-algebra map.

Proposition 9. Given $X,Y \in T\text{-}\mathcal{A}\ell g$, and $K \in \mathcal{S}p$, and $f:X\times K \to Y$, with adjoint $\hat{f}:X \to Y^K$, then $f \in T\text{-}\mathcal{A}\ell g^P(X\times K,Y)$ if and only if

$\hat{f} \in T\text{-}\mathcal{A}\ell g(X,Y^K)$.

Proof: The diagram that is required for $\hat{f}$ to be a T-algebra morphism is the adjoint to the diagram of Proposition 8. Thus the one diagram commutes if and only if the other diagram commutes, thus f is a $T\text{-}\mathcal{A}\ell g^P$ morphism if and only if $\hat{f}$ is a T-algebra morphism.

This proposition shows us that $T\text{-}\mathcal{A}\ell g^P(X\times K,Y) \cong T\text{-}\mathcal{A}\ell g(X,Y^K)$. We also know that the relation $(X^K)^L \cong X^{(K\times L)}$ holds, since it holds for simplicial sets. Thus we can put these results together and get that $\underline{T\text{-}\mathcal{A}\ell g}(X,Y^K) \cong \underline{T\text{-}\mathcal{A}\ell g}^P(X\times K,Y) \cong \underline{Sp}(K,\underline{T\text{-}\mathcal{A}\ell g}(X,Y))$. Thus we have shown:

Proposition 10. $T\text{-}\mathcal{A}\ell g$ is a simplicial category with objects $X\otimes K$ and X^K for all $X \in T\text{-}\mathcal{A}\ell g$, and $K \in \mathcal{S}p$. [see Quillen(HA)]

We are now ready for the main result of this chapter:

Theorem I. The category $T\text{-}\mathcal{A}\ell g$ is a closed simplicial model category, where $f:X \to Y$ is a weak equivalence (resp. fibration), if it is on the underlying simplicial set, and is a cofibration if it has the lifting property with respect to all trivial fibrations.

Proof: We have verified that $T\text{-}\mathcal{A}\ell g$ is both small complete and cocomplete (i.e. that is has all small limits and colimits). From the properties of $\mathcal{S}p$ it is immediate that weak equivalences compose and cancel, and that a map which is a retract of a weak equivalence or a fibration is a weak equivalence or fibration, respectively. The definition implies that any retract of a cofibration is a cofibration, and that cofibrations lift against trivial fibrations. Next we consider the factorizations.

Proposition 11. Every T-algebra homomorphism f may be factored as $f = p \circ i$, where i is a cofibration and p is a trivial fibration.

Proof: Use the "small object argument" [Quillen(HA), Ch. II, page 3.3], with the models $T(\partial\Delta^n) \subset T\Delta^n$. This inclusion is a cofibration, since it lifts against any trivial fibration If K is a finite simplicial set, TK will be a sequentially small T-algebra. Solving all lifting problems of the form $T(\partial\Delta^n) \subset T\Delta^n$ is the same as solving lifting problems of the form $\partial\Delta^n \subset \Delta^n$ on the underlying simplicial sets; and, this lifting problem can be solved if and only if the map in question is a trival fibration. Thus, every map can be factored as a cofibration and a trivial fibration.

Proposition 12. Every T-algebra homomorphism can be factored as $f = p \circ i$, where i is a trivial cofibration, and p is a fibration.

Proof: Use the small object argument again, using the models $T(\Lambda^n_k) \subset T\Delta^n$. This will factor any map as a fibration, and a map which lifts against all fibrations (which is the property that describes a trivial cofibration in a closed model category.) Thus we only need to show that a map which lifts against all fibrations is a trivial cofibration.

We note that the Kan Ex^∞ functor is a functor satisfying the conditions of Proposition 1. Thus if X is a T-algebra, then there is a natural map $X \to Ex^\infty(X)$, which is a T-algebra homomorphism, and the algebra $Ex^\infty(X)$ is a Kan complex, and so a fibrant T-algebra. [See Kan(EX).] It is clear that any map which lifts against all fibrations, a priori lifts against trivial fibrations, and is therefore a cofibration. It is sufficient, therefore, to show that such a map is a weak equivalence. This is done by using the argument that appears in Quillen(HA), Ch. II, page 4.9. The important point of this argument is the existence of the space $Ex^{\infty}(X)$, which allows us to find a fibrant approximation to any T-algebra. Therefore, any map can be factored as a trivial cofibration and a fibration. Q.E.D.

The only thing left for a closed model category is to show that any trivial cofibration lifts against any fibration. However, the map which is constructed in Proposition 12 lifts against any fibration. Any cofibration which is also a weak equivalence may be retracted from such a map by liting in the diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[d, "p"] \\ Y \arrow[r, "="] & Y \end{tikzcd}$$

Where f is a trivial cofibration, and p and i are constructed from Proposition 12. Since f is a weak equivalence, so is p. The lifting makes the morphism f a retract of the morphism i, and since i lifts against all fibrations, then f will also lift against all fibrations.

We have now shown that T-$\mathcal{Alg}$ is a closed model category. All that is left to show is that the simplicial structure is compatible with the model category structure. To show this it is enough to show that the maps $X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$, and $X^I \to X^{\{e\}} \times_{Y^{\{e\}}} Y^I$, are (trivial) fibrations when $f: X \to Y$ is a (trivial) fibration. This is clear since these limits are computed on the underlying spaces, and these maps have the desired properties in $\mathcal{Sp}$. Therefore, T-$\mathcal{Alg}$ is a closed simplicial model category. Q.E.D.

Given a morphism of algebras $\varphi: S \to T$, we know that there are adjoint functors φ^* and Lan^{φ}, between the categories $S\text{-}\mathcal{A}\ell g$, and $T\text{-}\mathcal{A}\ell g$. As a result of the theorem, φ^* preserves fibrations and weak equivalences, since φ^* does not change the underlying spaces. Lan^{φ} must preserve cofibrations since it is adjoint to φ^*, and φ^* preserves trivial fibrations. It is also true that $\underline{S\text{-}\mathcal{A}\ell g}(X,\varphi^*(Y)) \cong \underline{T\text{-}\mathcal{A}\ell g}(\mathrm{Lan}^{\varphi}X,Y)$, since $\varphi^*(Y)^K \cong \varphi^*(Y^K)$, since these are also computed on the underlying spaces. Thus, in particular, it is true that $\mathrm{Lan}^{\varphi}(X \otimes K) \cong \mathrm{Lan}^{\varphi}(X) \otimes K$. By Reedy(HM), we can now conclude that Lan^{φ} preserves weak equivalences of cofibrant objects. Therefore, we have adjoints $\underline{\underline{R}}(\varphi^*)$, and $\underline{\underline{L}}(\mathrm{Lan}^{\varphi})$ between the homotopy categories Ho $S\text{-}\mathcal{A}\ell g$, and Ho $T\text{-}\mathcal{A}\ell g$. Therefore, Lan^{φ} can be restricted to the homotopy categories, and is computed by restricting consideration to the cofibrant objects.

It is also interesting to know when the above pair of adjoints provide an equivalence of categories. From Quillen(HA), this pair of adjoints is an equivalence of categories if whenever X is a cofibrant S-algebra, and Y is a fibrant T-algebra, then a map $X \to \varphi^*(Y)$ is a weak equivalence if and only if the map $\mathrm{Lan}^{\varphi}X \to Y$ is a weak equivalence. By considering the composition $X \to \varphi^*\mathrm{Lan}^{\varphi}X \to \varphi^*(Y)$, where the second map comes from a map $\mathrm{Lan}^{\varphi}X \to Y$, it is easy to see that the required condition holds if and only if the adjunction morphism $X \to \varphi^*(\mathrm{Lan}\ X)$ is a weak equivalence for all cofibrant T-algebras X.

The later work requires a better understanding of cofibrations and cofibrant objects in the category $T\text{-}\mathcal{A}\ell g$.

In order to discuss cofibrancy, it is necessary to discuss the skeleton and coskeleton functors, since these are the functors that are used to construct liftings of simplicial maps inductively by degree. Formally, the skeleton and coskeleton are the left and right adjoint, respectively, to the functor which takes a simplicial object and truncates it above some degree. In fact, the skeleton and coskeleton are left and right adjoints to each other. In the category T-$\mathcal{A\ell g}$ it is easy to construct the coskeleton, since it is constructed on the underlying spaces. This works since the standard coskeleton functor on simplicial sets satisfies the conditions of Proposition 1; and therefore, is constructed as it is on simplicial sets. It is somewhat harder to construct the skeleton in the case of algebraic theories; however, it can be verified that the k^{th} degree of the n^{th} skeleton $(\mathrm{skel}_n(X)_k)$ is given by the requirements $\mathrm{skel}_0(X)_k = T(s_0^k)_*(X_0)$ and the following diagram is a pushout:

$$\begin{array}{ccc} \bigoplus_{\alpha:n\to k} T\alpha^*(\mathrm{skel}_{n-1}(X)_n) & \to & \bigoplus_{\alpha:n\to k} T\alpha^*(X_n) \\ \downarrow & & \downarrow \\ \mathrm{skel}_{n-1}(X)_k & \longrightarrow & \mathrm{skel}_n(X)_k \end{array}$$

A further discussion of the skeleton and coskeleton, and their uses in constructing liftings can be found in Reedy(HM).

Proposition 13. In the category T-$\mathcal{A\ell g}$ a morphism $i:A \to B$ is a cofibration if and only if the function $\mathrm{skel}_{n-1}(B)_n \vee A_n \to B_n$ (where the wedge is a pushout over $\mathrm{skel}_{n-1}(A)_n$) is a projective extension of T_n-algebras, for all n.

Proof: only if: It is easy to see that if $K \subset L$ is an inclusion of simplicial sets, then $TK \to TL$ is a free extension. It is also true that the pushout of a map satisfying the property also satisfies the property. Therefore, the cofibration constructed in Proposition 11 satisfies the property in question. Since any cofibration can be retracted from such a map, and it is clear that the retract of any map satisfying the property also satisfies the property, then any cofibration satisfies this property.

if: Given a lifting problem:

$$\begin{array}{ccc} A & \to & X \\ \downarrow & & \downarrow \\ B & \to & Y \end{array}$$

where p is a trivial fibration, then we construct the lifting inductively by degrees. In degree zero, p_0 is onto, and i_0 is a projective extension, so that a lifting exists. Assuming that we have constructed a lifting through degree n-1, we then have a diagram:

$$\begin{array}{ccc} \mathrm{skel}_{n-1}(B)_n \vee_{\mathrm{skel}_{n-1}(A)_n} A_n & \to & X_n \\ \downarrow & & \downarrow \\ B_n & \to & \mathrm{cosk}_{n-1}(X)_n \times_{\mathrm{cosk}_{n-1}(Y)_n} Y_n \end{array}$$

A lifting in this diagram will give us an extension of the lifting to degree n [Reedy(HM)]. The left hand map above is a projective extension by assumption. The right hand map above is the same set map as degree zero of the map $X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$. Since the map $X \to Y$ is a trivial fibration of simplicial sets, then this map is also a trivial fibration of simplicial sets, and is therefore onto in degree zero. Thus the right hand map above is an onto map of T_n-algebras; and, therefore, the lifting exists, which extends the lifting we are constructing to degree n. Therefore, any map satisfying the property is a cofibration. Q.E.D.

Corollary: If X is a cofibrant T-algebra, then X_n is a projective T_n-algebra for all n.

Proof: A cofibrant T-algebra, by the proposition, is one in which the map $\mathrm{skel}_{n-1}(X)_n \to X_n$ is a projective extension for all n. It is obvious that a projective extension from a projective maps to a projective. Thus, we only need to show that the skeleta of X are degreewise projective. We do this inductively. $\mathrm{Skel}_0(X)$ is degreewise projective, since by X_0 is a projective T_0-algebra. From the pushout diagram for $\mathrm{skel}_n(X)$, we see that the fact that $\mathrm{skel}_{n-1}(X)_n \to X_n$ is a projective extension, and the fact that $\mathrm{skel}_{n-1}(X)$ is degreewise projective, we can conclude that $\mathrm{skel}_n(X)$ is degreewise projective. Thus in particular, X is also degreewise projective. Q.E.D.

It is this description of cofibrant T-algebras, together with the result that $\underline{\underline{L}}(\mathrm{Lan}^{\varphi})$ only needs to be computed on the cofibrant algebras, that will allow us to produce the results of Chapter V.

Chapter IV. Homotopy Algebras.

In this chapter the concept of a homotopy T-algebra is defined, and some of the elementary theorems about them are discussed. There are several ideas about what a homotopy T-algebra should be. For the purposes of this paper I will use a definition due to Segal(HE).

Definition: Let $T:\mathcal{N} \to \mathcal{T}$ be a simplicial theory. A homotopy T-algebra is a simplicial functor $X:\mathcal{T} \to \mathcal{S}p$ such that

$$\prod_{i=1}^{n} X(p_i):X(n) \to X(1)^n$$

is a weak homotopy equivalence. ($X(0) \to *$ is also a weak homotopy equivalence.)

A map of homotopy T-algebras is a homotopy natural transformation, in the sense of Anderson(HF). The space X(1) is the "underlying space" of X. X will be said to admit the structure of a homotopy T-algebra if it is the "underlying space" of some homotopy T-algebra. $(T)\text{-}\mathcal{A}\ell g$ will be the category of homotopy T-algebras. It is true, for Kan complexes, that admitting the structure of a T-algebra is an invariant of homotopy type. If X and Y are homotopy T-algebras, then we define $[X,Y]_{(T)}$ to be the homotopy classes of homotopy T-algebra maps from X to Y.

Given a natural transformation of theories $\varphi:S \to T$, then we know that there is a functor $(T)\text{-}\mathcal{A}\ell g \to (S)\text{-}\mathcal{A}\ell g$, given by restriction. From Anderson(HF), there is a functor $LAN^{\varphi}:(S)\text{-}\mathcal{A}\ell g \to (T)\text{-}\mathcal{A}\ell g$ (homotopy left Kan extension), which has the property that there is an isomorpism $[X,\varphi^*(Y)]_{(S)} \cong [LAN^{\varphi}(X),Y]_{(T)}$.

The functor $LAN^{\varphi}(X)$ is given as the realization of a collection of simplicial spaces. For $n \in S$ let $L_n^{\varphi}(X)$ be the simplicial space given by:

$$L_n^{\varphi}(X)_k = \sum_{(n_0,\ldots,n_k)} X(n_0)\times S(n_0,n_1)\times \ldots \times S(n_{k-1},n_k)\times T(n_k,n)$$

where $(n_0, \ldots ,n_k)$ is a k+1 tuple of objects of S. (Σ is the disjoint union of simplicial sets.) The i^{th} face map is gotten by eliminating the object n_i by composition. The i^{th} degeneracy map is gotten by doubling the object n_i, and inserting parallel to the ideniity in the factor $S(n_i,n_i)$. It is easy to verify that this is a simplicial space. We define $LAN^{\varphi}(X)(n) = \|L_n^{\varphi}(X)\|$, where $\|-\|$ is the realization of a bisimplicial set (i.e. its diagonal.) Examination of the complex $L_n^{\varphi}(X)$ shows that $Lan^{\varphi}(X)(n)$ is the coequalizer of d_0 and d_1 going from degree one to degree zero.

Each map $p_i:n \to 1$ in N induces a map $L_{p_i}^{\varphi}(X):L_n^{\varphi}(X) \to L_1^{\varphi}(X)$, by taking the term $T(n_k,n)$ to $T(n_k,1)$ by composition with p_i.

Theorem II. The map $\prod_{i=1}^{n} L_{p_i}^{\varphi}(X):L_n^{\varphi}(X) \to L_1^{\varphi}(X)^n$, is a simplicial homotopy equivalence, for all $n \geq 0$. [See May(SO)]

Corollary: $LAN^{\varphi}(X)$ is a homotopy T-algebra.

Proof: $\|-\|$ commutes with products, and preserves homotopy equivalences; thus, the map which is required to be a weak equivalence is, by the theorem.

Corollary: If X is an S-algebra, then $Lan^{\varphi}(X)$ is a T-algebra.

Proof: The simplicial homotopy equivalence above will induce an isomorphism on π_0, which is just the coequalizer of d_0 and d_1.

Therefore $\mathrm{Lan}^{\varphi}(X)$ will preserve products on the nose, which is the condition required for $\mathrm{Lan}^{\varphi}(X)$ to be a T-algebra.

The rest of this chapter will be devoted to the proof of Theorem II.

Let I and J be discrete categories. Let $R:I \to J$ and $L:J \to I$, be right and left adjoints respectively. Define $C_*(J;F)$ where $F:J \to \mathcal{Set}$ is a functor, by $C_k(J;F) = \sum_{\sigma} F(j_k)_{\sigma}$, where $\sigma = (j_k \to \dots \to j_0)$ is a k-path in J, and Σ is the disjoint union of sets. We give $C_*(J;F)$ the obvious face and degeneracy functions to make it a simplicial set. There are two maps we wish to consider. $R_*:C_*(I;F\circ R) \to C_*(J;F)$ is given by $F\circ R(i_k)_{\sigma} \to F(Ri_k)_{R\sigma}$. $L_*:C_*(J;F) \to C_*(I;F\circ R)$ is given by $F(\eta j_k):F(j_k)_{\sigma} \to F\circ R(Lj_k)_{L\sigma}$, where $\eta j_k : j_k \to RLj_k$ is the unit of the adjunction between R and L. It is easy to verify that R_* and L_* are simplicial maps.

We now construct two simplicial homotopies. (We use the definition of a simplicial homotopy found in May(SO).) Define $h^1: id \sim R_* \circ L_*$ by $h^1_m : C_k(J;F) \to C_{k+1}(J;F)$ by $F(j_k)_{\sigma} \to F(j_k)_{\sigma^m}$, where σ^m is the k+1-path $(j_k \to \dots \to j_m \to RLj_m \to \dots \to RLj_0)$. The map $j_m \to RLj_m$ is the unit of the adjunction (ηj_m). Define $h^2 : L_* \circ R_* \sim id$ by h^2_m takes $F\circ R(i_k)_{\sigma}$ to $F\circ R(LRi_k)_{({}^m\sigma)}$ by $F(\eta Ri_k)$, where ${}^m\sigma$ is the k+1-path given by $(LRi_k \to \dots \to LRi_m \to i_m \to \dots \to i_0)$. The function $LRi_m \to i_m$ is the counit of the adjunction (εi_m). It is easy to verify that the maps defined provide a simplicial homotopy between the maps claimed. The only observation that needs to be made is that the composition $R\varepsilon i \circ \eta Ri = id_i$ for all objects $i \in I$. (This follows from the fact that L and R are adjoint.)

Given $\varphi:S \to T$ a morphism of algebraic theories, then consider $\varphi_k:S_k \to T_k$. There are natural functors $p:(\varphi_k\downarrow n) \to (\varphi_k\downarrow 1)^n$, and $\rho:(\varphi_k\downarrow 1)^n \to (\varphi_k\downarrow n)$, where $(\varphi\downarrow m)$ is the comma category of morphisms in S over the object $m \in T$. The functor p is given on objects by $p(\alpha) = (p_1\circ\alpha, \ldots ,p_n\circ\alpha)$, where $\alpha:k \to n$ is a morphism in T, and $p_i:n \to 1$ is the i^{th} projection. The functor ρ is given on objects by $\rho(\alpha_1, \ldots ,\alpha_n) = (\alpha_1\times\ldots\times\alpha_n)$, where $\alpha_i:k_i \to 1$ are morphisms in T and $\alpha_1\times\ldots\times\alpha_n:k_1\times\ldots\times k_n \to n$ is their product. ρ and p are left and right adjoints. The unit of the adjunction is induced by the diagonal map $k \to k\times\ldots\times k$, where the product is taken k times. The co-unit of the adjunction is induced by the projections $k_1\times\ldots\times k_n \to k_i$.

If we let $\pi_n:(\varphi\downarrow n) \to S$ be the standard functor, then the preceeding work gives us a natural homotopy equivalence $C_*((\varphi_k\downarrow n);X\circ\pi_n) \sim C_*((\varphi_k\downarrow 1)^n;X\circ\pi_n\circ\rho)$. Since this homotopy equivalence is natural, we can piece it together to get a simplicial homotopy equivalence of simplicial spaces $C_*((\varphi\downarrow n);X\circ\pi_n) \sim C_*((\varphi\downarrow 1)^n;X\circ\pi_n\circ\rho)$.

It is now easy to verify that $C_*((\varphi\downarrow n);X\circ\pi_n) \cong L_n^\varphi(X)$, and that $C_*((\varphi\downarrow 1)^n;X\circ\pi_n\circ\rho) \cong C_*((\varphi\downarrow 1);X\circ\pi_1)^n \cong L_1^\varphi(X)^n$, and that p_* is the product of the projections. Since realization of bisimplicial sets commutes with products, and preserves homotopies, then realizing the above homotopy will provide the desired homotopy equivalence $\mathrm{LAN}^\varphi(X)(n) \sim \mathrm{LAN}^\varphi(X)(1)^n$. Q.E.D.

Chapter V. The Main Theorem

We know from chapter III that there is a natural transformation $\mathrm{LAN}^{\varphi}(X) \to \mathrm{Lan}^{\varphi}(X)$, when φ is a morphism of theories. It is natural to ask what the relationship between these two objects is.

Theorem III. Let $\varphi: S \to T$ be a morphism of theories, and let X be a cofibrant S-algebra, then the map $\mathrm{LAN}^{\varphi}(X) \to \mathrm{Lan}^{\varphi}(X)$ is a weak homotopy equivalence.

Proof: We proceed in stages. First assume that $\varphi: S \to T$ is a morphism, where S and T are discrete theories. Take $X = S\underline{n}$, where $\underline{n} = \{1,\ldots,n\}$, so that X is the free S-algebra on n points. X is a discrete S-algebra. Note that in this case $\mathrm{Lan}^{\varphi}(X)$ is a discrete algebra, and $\mathrm{LAN}^{\varphi}(X)$ is a simplicial set. $\mathrm{Lan}^{\varphi}(X)$ is isomorphic to $T\underline{n}$, since by the adjointness $\mathrm{Lan}^{\varphi}(X)$ must be the free T-algebra on n points. We identify $\mathrm{Lan}^{\varphi}(X)$ and $T\underline{n}$ by making the inclusions of the set $\underline{n}$ in each agree. There is a distinguished element $\varkappa \in (T\underline{n})^n$, given by $\varkappa = (1,\ldots,n)$. $\varkappa$ is distinguished by the following:

Lemma: Given any $x \in (T\underline{n})^k$, there is a unique $\vartheta \in \mathcal{T}(n,k)$ such that $x = \vartheta(\varkappa)$. ($\vartheta(\varkappa)$ is the element given by the action of the operation ϑ on the element $\varkappa$. This can be described in this way since we are taking everything to be discrete.)

Proof: This is easy once we identify $T\underline{n} \cong \mathcal{T}(n)$. Then we see that $(T\underline{n})^k \cong \mathcal{T}(n,k)$, and that $\varkappa \in (T\underline{n})^n$ corresponds to the identity element in $\mathcal{T}(n,n)$. The result of the lemma then reduces to the fact that the identity element is an identity element for the composition in $\mathcal{T}$.

Now consider the map $LAN^{\varphi}(X) \to Lan^{\varphi}(X)$, where we now identify $X = S\underline{n}$, and $Lan^{\varphi}(X) \cong T\underline{n}$. The complex $L_1^{\varphi}(S\underline{n})$ is, in this case, a simplicial set, since all of the objects involved are discrete. Since $T\underline{n}$ is the coequalizer of d_0 and d_1, then we can consider $T\underline{n}$ as augmenting the complex $L_1^{\varphi}(S\underline{n})$. Write $L_1^{\varphi}(S\underline{n})(-1) = T\underline{n}$, and the map $L_1^{\varphi}(S\underline{n})(0) \to T\underline{n}$ as d_0. $L_1^{\varphi}(S\underline{n})$ is now an augmented simplicial set. We want to find a map $s_{-1}:L_1^{\varphi}(S\underline{n})(s) \to L_1^{\varphi}(S\underline{n})(s+1)$ $s \geq -1$, which satisfies the simplicial identities. If we can do this, then this extra degeneracy provides a homotopy equivalence $||L_1^{\varphi}(S\underline{n})|| \sim T\underline{n}$ [May(SO)].

To construct s_{-1}, take a typical point $(x,\alpha_1,\ldots,\alpha_s,\vartheta)$ in $L_1^{\varphi}(S\underline{n})(s)$, $s \geq 0$, where $x \in (S\underline{n})^{k_0}$, $\alpha_i \in \mathcal{S}(k_{i-1},k_i)$, and $\vartheta \in \mathcal{T}(k_s)$, where the k_i are objects in $\mathcal{S}$. Write $x = \alpha_0(\varkappa)$, where $\alpha_0 \in \mathcal{S}(n,k_0)$ is the unique element provided by the lemma. Define $s_{-1}(x,\alpha_1,\ldots,\alpha_s,\vartheta) = (\varkappa,\alpha_0,\alpha_1,\ldots,\alpha_s,\vartheta)$. For the case $s = -1$, define $s_{-1}(x) = (\varkappa,\vartheta)$, where $x \in T\underline{n}$, and $\vartheta \in \mathcal{T}(n)$ is the unique element such that $x = \vartheta(\varkappa)$. It is easy to verify that this extra degeneracy satisfies the simplicial identities, and that it, therefore, provides a contraction of the complex $L_1^{\varphi}(S\underline{n})$ to $T\underline{n}$.

Having shown that the theorem is true in this special case we first extend to the case where $X = S(A)$, where A is any discrete set. Write $A = \operatorname{colim}_{F \subset A} F$, where F runs over the finite subsets of A.

Then $Lan^{\varphi}(X) \cong \operatorname{colim} Lan^{\varphi}(SF)$, and $LAN^{\varphi}(X) \cong \operatorname{colim} LAN^{\varphi}(SF)$, since both of these functors commute with filtered direct limits. We know that the map $LAN^{\varphi}(SF) \to Lan^{\varphi}(SF)$ is a weak homotopy equivalence, since $F \cong \underline{n}$ for some n, and we have proved the theorem in that case. Since this is a filtered colimit, and filtered colimits commute with homotopy groups, then we see that the map $LAN^{\varphi}(X) \to Lan^{\varphi}(X)$ **must be a weak homotopy equivalence in this case.**

Now let X be a discrete projective S-algebra. Since X is projective we can retract X from a free S-algebra, SA. We then see that the map $LAN^{\varphi}(X) \to Lan^{\varphi}(X)$ can be retracted from the map $LAN^{\varphi}(SA) \to Lan^{\varphi}(SA)$. Since the second map is a weak equivalence, and since the retract of a weak homotopy equivalence is a weak homotopy equivalence, then the map $LAN^{\varphi}(X) \to Lan^{\varphi}(X)$ is a weak homotopy equivalence.

We are now ready to consider the general case. $\varphi: S \to T$ is a morphism of simplicial theories, and X is a cofibrant S-algebra. We see that there are morphisms $\varphi_n : S_n \to T_n$ of discrete theories, and that each X_n is a discrete projective S_n-algebra. Thus by the previous work, the map $LAN^{\varphi_n}(X_n) \to Lan^{\varphi_n}(X_n)$ is a weak homotopy equivalence.

Consider the complex $L_1^{\varphi}(X)$, which is augmented by the space $Lan^{\varphi}(X)$. $L_1^{\varphi}(X)$ is a bisimplicial set, and its vertical realization and horizontal realization are the same as its diagonal. In particular, we see that each horizontal section of $L_1^{\varphi}(X)$ is $L_1^{\varphi_n}(X_n)$, which is augmented by the algebra $Lan^{\varphi_n}(X_n)$. So we see that the vertical realization is $LAN^{\varphi}(X) = \|L_1^{\varphi_n}(X_n)\| \to \|Lan^{\varphi_n}(X_n)\| = Lan^{\varphi}(X)$. Thus we need only know that the realization of a degreewise weak equivalence is again a weak equivalence; however, this is known [Reedy (HM)]. Thus $LAN^{\varphi}(X) \to Lan^{\varphi}(X)$ is a weak homotopy equivalence, when X is a cofibrant S-algebra. Q.E.D.

It will be useful to know the analog of the above theorem when we are working in the case of a pro or a prop, especially since many of the theories that are of interest arise in this manner.

When we are considering the case of a prop some restriction on the prop is necessary. We say that a prop is Σ-free if the n^{th} symmetric group Σ_n acts freely on the space $A(n)$. We further assume that if A is a Σ-free prop, then A comes equipped with a choice (not necessarily natural) of one representative of every orbit class in A. Given this condition we can now formulate a uniqueness property analogous to the one in the Lemma to Theorem III.

Proposition 1. If A is a pro or a Σ-free prop, then given any $y \in AX$, where X is a based space, there exists a unique (x,ϑ), where $x \in X^k$, $\vartheta \in A(k)$, none of the coordinates of x are at the basepoint, $\vartheta(x) = y$ (and if A is a prop, ϑ is the representative of its orbit class.)

Proof: $AX = \sum_n X^n \times A(n)/\sim$. Take k minimal such that $y = |x',\vartheta'|$, where $x' \in X^k$, $\vartheta' \in A(k)$, and $|x,\vartheta|$ is the equivalence class of (x,ϑ). Let ϑ be the representative of the orbit class of ϑ', then $\vartheta' = \vartheta\circ\pi$, where $\pi \in \Sigma_n$. Thus $|x',\vartheta'| = |x',\vartheta\circ\pi| = |\pi(x'),\vartheta|$. Let $x = \pi(x')$. If x has a component at the base point, then there is an $\alpha{:}m \to k$, such that $x = \alpha(z)$, where $z \in X^m$. But, then, we would have $|x,\vartheta| = |z,\vartheta\circ\alpha|$, which would contradict the minimality of k. Thus, the pair (x,ϑ) satisfies the condition of the proposition.

We now prove uniqueness.

Lemma. Let $\alpha(x) = \beta(x') \in X^n$, where $x \in X^k$, and $x' \in X^m$, and such that x has no components which are the base point. Then there exists a $\gamma: k \to m$ such that $x' = \gamma(x)$, and $\beta \circ \gamma = \alpha$.

Proof: take $j \in k$. Then we must have that $\alpha(j) = \beta(j')$, for some unique $j' \in m$, since the $\alpha(j)^{th}$ component of $\alpha(x)$ is not the base point, and β is one-to-one. Then $\beta \circ \gamma = \alpha$. Since $\beta : X^m \to X^n$ is a monomorphism, then $\beta(x') = \beta \circ \gamma(x)$ implies that $x' = \gamma(x)$.

Suppose, now, that $(\alpha(x), \vartheta') = (\beta(x'), \vartheta')$, so that $|x, \vartheta' \circ \alpha| = |x', \vartheta' \circ \beta|$; and that the element x has no base point components. Then the lemma above constructs a γ such that $x' = \gamma(x)$, and so that $\vartheta' \circ \alpha = (\vartheta' \circ \beta) \circ \gamma$. In particular, if $\vartheta' \circ \alpha = \vartheta$, then $\vartheta = (\vartheta' \circ \beta) \circ \gamma$. This shows that if $|x', \vartheta'| = |x, \vartheta|$, then there is a γ such that $x' = \gamma(x)$, and $\vartheta = \vartheta' \circ \gamma$. Thus, if we have two elements satisfying the conditions of the proposition, then they must be connected by a permutation. But, the group of permutations acts freely on $A(n)$, so that the condition that ϑ be the representative of its orbit class means that any two elements satisfying the conditions must be equal.

Corollary: Given $(x', \vartheta') \in X^n \times A(n)$, then let (x, ϑ) be the pair chosen by the proposition (i.e. $|x', \vartheta'| = |x, \vartheta|$.) Then there exists a unique $\alpha : k \to n$, such that $x' = \alpha(x)$, and $\vartheta = \vartheta' \circ \alpha$.

Proof: (see above).

Theorem IV. If A is a pro or a Σ-free prop, then $LAN^A(X) \to AX$ is a weak homotopy equivalence.

Proof: (Note that $\mathrm{LAN}^A(X) = \|L_1^A(X)\|$, where $L_1^A(X)$ is the complex constructed analogously to $L_1(X)$.) As before we need only prove this when A is discrete, and X is a discrete based set. (Note that discrete based sets are always the free based set on the set minus the base point.)

As before we consider AX as augmenting $L_1^A(X)$, and attempt to construct an s_{-1}. Given $y \in AX$, let (x_y, ϑ_y) be the unique pair constructed by the previous proposition. Define $s_{-1}(y) = (x_y, \vartheta_y)$. Given a typical element $(x, \alpha_1, \ldots, \alpha_s, \vartheta) \in L_1^A(X)$, then let $y \in AX$ be the element $|x, \vartheta\circ\alpha_s\circ\ldots\circ\alpha_1|$. By the corollary, there is a unique α_0 such that $x = \alpha_0(x_y)$, and such that $\vartheta\circ\alpha_s\circ\ldots\circ\alpha_1\circ\alpha_0 = \vartheta_y$. Define $s_{-1}(x, \alpha_1, \ldots, \alpha_s, \vartheta) = (x_y, \alpha_0, \alpha_1, \ldots, \alpha_s, \vartheta)$. The simplicial relations follow from the uniqueness of α_0. Thus the result is true for discrete pros and props and discrete sets, and thus, as in the previous theorem, it is also true for simplicial pros and props and spaces. Q.E.D.

Note: This result also follows from a result of Ron Williams which states that if $\mathcal{A}$ is the theory derived from A and $\varphi: \mathcal{N}_b \to \mathcal{A}$ is the standard morphism, then $\mathrm{LAN}^A(X) \to \mathrm{LAN}^\varphi(X)$ is a weak homotopy equivalence.

There is another filtration on AX (or TX) which is standardly used [May(IL)]. This filtration has F_pAX is the image of $\sum_{k\le p} X^k \times A(k)$ (respectively $\Sigma X^k \times \mathcal{T}(k)$) in AX. If we let $F_p(\mathrm{LAN}^A(X))$ be the subcomplex generated by sequences $(k_0, \ldots, k_s)$ such that $k_i \le p$ for all i, then we see that $F_p(\mathrm{LAN}^A(X)) \to F_pAX$.

Corollary: The map $F_p LAN^A(X) \to F_p AX$ is a weak homotopy equivalence.

Proof: Examination of the proof of the theorem shows that s_{-1} respects the filtration. Thus the same proof applies.

It is also the case that the corollary is true for TX. However, the proof given for Theorem III does not preserve filtrations. It is the case, however, that there is a two stage homotopy contracting the complex in Theorem III that does preserve the filtration.

Corollary: [Beck(HS)] If $\varphi:S \to T$ is a morphism of theories, such that $\varphi(n):\mathcal{S}(n) \to \mathcal{T}(n)$ is a weak equivalence for all n, then $\varphi X:SX \to TX$ is a weak homotopy equivalence for all X, and the categories HoS-$\mathcal{A}\ell g$ and HoT-$\mathcal{A}\ell g$ are equivalent.

Proof: It is immediate that φ induces a degreewise weak equivalence $L_1^S(X) \to L_1^T(X)$. Since the realization of a degreewise weak equivalence is a weak equivalence [Reedy(HM)], then theorem III shows that $\varphi X:SX \to TX$ is a weak homotopy equivalence.

As for the second part, let $id:S \to S$ be the identity morphism. It is immediate that $Lan^{id}(X) \cong X$ for any S-algebra X. For any cofibrant S-algebra, there is a natural transformation $L_1^{id}(X) \to L_1^{\varphi}(X)$, which is a degreewise weak equivalence. Thus, again $LAN^{id}(X) \to LAN^{\varphi}(X)$ is a weak equivalence, and so by Theorem III $X \to Lan^{\varphi}(X)$ is a weak equivalence. But this is the condition required for HoS-$\mathcal{A}\ell g$ $\to$ HoT-$\mathcal{A}\ell g$ to be an equivalence of categories. Q.E.D.

In this chapter I will discuss some examples of pros, props and theories, and show some of the reasons for pursuing the previous results.

Example 1. Let A and E be the props defined by $A(n) = *$, and $E(n) = W\Sigma_n$, the contractible space on which Σ_n acts freely. The structure of A as a prop is obvious. The structure of E as a prop is given by the wreath product. There is a morphism of props $\varphi : E \to A$. Note that E is a Σ-free prop, and A is not. If Theorem IV were true for A then we would necesarily have $EX \to AX$ a weak homotopy equivalence, since $\varphi(n) : E(n) \to A(n)$ is, and so would induce a degreewise weak equivalence $L_1^E(X) \to L_1^A(X)$. It is easy to see that AX is the infinite symmetric product (free abelian monoid) on X. However, EX is known [Barratt(FG), for example] to be homotopy equivalent to $\Omega^\infty S^\infty(X)$ for connected spaces X. So the map $EX \to AX$ cannot be a weak homotopy equivalence, since, for example, EX is not a product of Eilenberg-MacLane spaces, and AX is. This example shows that Theorem IV does require the freeness of the symmetric group actions for props.

The idea behind Theorems III and IV is to use these results to study the homology groups $H_*(\mathrm{Lan}^\varphi X) \cong H_*(\mathrm{LAN}^\varphi X)$ for interesting φ. Theorem III does show that $H_*(\mathrm{Lan}^\varphi X;R)$ (R a ring) should depend only on the structure of $C_*(X;R)$ as an infinite homotopy S-algebra, provided we understood exactly what that meant. However, this does show that if T is a theory, then $H_*(TX;Q)$ depends at most on

$H_*(X;Q)$ as a graded coalgebra, since by Quillen(RH) the infinite homotopy coalgebra structure on $C_*(X;Q)$ is determined by the coalgebra structure on $H_*(X;Q)$.

In example four I will show why very general results of this sort are difficult to obtain. However, in the cases covered by Theorem IV we have the following result:

Theorem V. If A is a pro or a Σ-free prop, then $H_*(AX;R)$ is isomorphic $\bigoplus_{n\geq 0}\tilde{H}_*((A(n)\times X^{[n]})/\Sigma_n;R)$, where R is a principal ideal domain, and $X^{[n]}$ is the n-fold smash product of X.

Proof: The idea of the proof is to use the fact that $C_*(X;R)$ is isomorphic to $\tilde{C}_*(X;R)\oplus R$, naturally, since X is a base pointed space, and to exploit the observation that $A(X^+) \cong \sum_n A(n)\times X^n$.

Consider the complex $C_*^v(L_1^A(X);R)$, which is the simplicial chain complex gotten by taking the chains vertically. $C_*^v(L_1^A(X);R)_k$ (the k^{th} vertical slice) is naturally chain equivalent to the chain complex $D_{*,k} = \bigoplus_\alpha C_*(X^{n_0})\otimes C_*(A(n_k))$, by the Eilenberg-Zilber theorem. We then get a double complex D_{**} which is equivalent to the original complex.

Consider $C_*(X^n)$. There is an equivariant collection of commuting projections on X^n given by replacing the coordinates with basepoints. Let $X_d^n \subset X^n$ be the subset complex where one or more coordinates are the base point. Then $C_*(X^n) \cong C_*(X^n,X_d^n)\oplus C_*(X_d^n)$ equivariantly, since $C_*(X^n,X_d^n)$ is the kernel of all the projections. In fact, we have the decomposition $C_*(X^n) \cong \bigoplus_\alpha C_*(X^\ell,X_d^\ell)$, where α runs over all the ordered monomorphisms $\alpha:\ell \to n$.

Define $D_{*,-1}$, an augmentation to $D_{*,*}$ by $D_{*,-1} = \bigoplus_n [C_*(X^n,X^n_d)\otimes C_*(A(n))]/\Sigma_n$, and using the decomposition of $C_*(X^n)$ given above to define $d_0: D_{*,0} \to D_{*,-1}$. By choosing a basis for $C_*(X^n,X^n_d)$ for each n, the proof of Theorem **IV** can now be repeated to show that $H^h(D_{**})$, the horizontal homology, collapses onto an edge, so that $H^h_{*,n} = 0$ if $n>0$, and $H^h_{*,0} \cong D_{*,-1}$, naturally. Thus if $\mathrm{tot}(D_{**})$ is the total complex, then we know that $H(\mathrm{tot}(D_{**}))$ is isomorphic to $H(D_{*,-1})$.

The only observation needed now is that $H_*(C_*(X^n,X^n_d)\otimes C_*(A(n))/\Sigma_n)$ is isomorphic to $\tilde{H}_*((A(n)\times X^{[n]})/\Sigma_n)$, which is clearly true by the Eilenberg-Zilber Theorem.

Corollary: If R is a field (or if $H_*(X;R)$ is a free R-module), then $\tilde{H}_*(A(n)\times X^{[n]}/\Sigma_n) \cong H^{\Sigma_n}_*(A(n);\tilde{H}_*(X)^n)$ (the equivariant homology.)

Proof: By an acyclic models argument, $C_*(A(n))\otimes\tilde{C}_*(X^{[n]})$ is chain equivalent (equivariantly) to $C_*(A(n))\otimes\tilde{C}_*(X)^n$. If $H_*(X;R)$ is free, then $\tilde{C}_*(X)$ is chain equivalent to $\tilde{H}_*(X)$, thus giving the result. [See Dyer-Lashof(HI), Theorem 2.1, and following, for details].

Thus we see that if $H_*(X;R)$ is free then $H_*(AX;R)$ is isomorphic to $\bigoplus_n H^{\Sigma_n}_*(A(n);\tilde{H}_*(X;R)^n)$. Note that this implies that, for example, $H_*(A(S^2\vee S^4);Z)$ is isomorphic to $H_*(A(CP^2);Z)$, at least as a graded group. Thus, even though $S^2\vee S^4$ and CP^2 are two different spaces, they generate the same homology for $H_*(A(-))$. Thus, for example, $H_*(\Omega^\infty S^\infty(CP^2)) \cong H_*(\Omega^\infty S^\infty(S^2\vee S^4))$, at least as graded groups.

Example 2. As an immediate consequence of the above result we can prove the known result that the homology of $\Omega^n S^n(X)$

with field coefficients, depends only on the homology of X with coefficients in the same field. This is an immediate consequence of Theorem V, and the fact that there is a prop C_n [May(IL)] such that the space $C_n(X)$ is homotopically equivalent to $\Omega^n S^n X$, when X is connected. Thus by the above results, since we have seen that the homology of $C_n(X)$ depends only on the homology of X, then the homology of $\Omega^n S^n X$, must depend only on the homology of X.

Example 3. As another example of the results above, I will compute $H_*(EX;Z/2)$, where E is the prop of example one.

Let $J=(j_1,\ldots,j_\ell)$, $\ell\geq 0$, be a sequence of positive integers. J is admissible if $j_k \geq 2j_{k+1}$, for $1\leq k<\ell$, and $j_1 > j_2+\ldots+j_\ell$. We set $\mathrm{rank}(J) = 2^\ell$. Let $\{x_i\}$ be a homogeneous basis for $\tilde{H}_*(X)$, then $H_*(EX)$ is isomorphic (as a graded group) to the polynomial algebra on the symbols $Q_J(x_i)$, of dimension $j_1+2j_2+\ldots+2^{\ell-1}j\ +2^\ell \dim(x_i)$, where J is admissible. Note that this must surely be true as an algebra over the Dyer-Lashof algebra; but, I have not yet verified this.

The proof is as follows: Let P be the polynomial algebra on the specified set of generators. Given a monomial $Q_{J_1}(x_1)\ldots Q_{J_k}(x_k)$ then the rank of the monomial is the sum of the ranks of the Q_{J_i}. Let $U_m(P) \subset P$ be the subgroup generated by the rank m monomials. It is clear that $P \cong \bigoplus_{m\geq 0} U_m(P)$, so we only need to show that $U_m(P) \cong H_*(\Sigma_m;\tilde{H}_*(X)^m)$, which is the term given by Theorem V. This is done by writing $\tilde{H}_*(X)^m$ as the direct sum of the $Z/2[\Sigma_m]$-modules generated by the elementary tensors of basis elements. Any such tensor is Σ_m equivalent to some tensor of the form $x_1^{e_1}\otimes\ldots\otimes x_s^{e_s}$, where $e_1+\ldots+e_s = m$. and $x_i\neq x_j$ for $i\neq j$. For this element we get a copy of $H_*(\Sigma_{e_1}\times\ldots\times\Sigma_{e_s})$ offset by the dimension of $x_1^{e_1}\otimes\ldots\otimes x_s^{e_s}$ appearing in $H_*(\Sigma_m;\tilde{H}_*(X)^m)$.

Let A be the polynomial algebra generated by the symbols $Q_J(x)$ for some single generator x. Let $U_m(A)$ be the rank m part of A. Then, since $H_*(\Sigma_{e_1}\times\ldots\times\Sigma_{e_s})$ is isomophic to $H_*(\Sigma_{e_1})\otimes\ldots\otimes H_*(\Sigma_{e_s})$, then we only need to know that $H_*(\Sigma_m)\cong U_m(A)$. But this follows immediately from the computation of $H_*(\Sigma_m)$ in Nakaoka(SG), Theorem 6.3. This show that $H_*(EX;Z/2)$ is as claimed.

We note that the class $Q_J(x)$ ought to be the class $Q_{j_1}\cdots Q_{j_\ell}(x)$ in $H_*(EX)$, where $Q_j(x)$ is the standard Dyer-Lashof operation $Q^{n+j}(x)$, where the dimension of x is n.

Example 4. This final example is an attempt to show that the results that have been proved can be applied to examples that are different from the ones above, which are the types of examples which I had in mind when I pursued this work.

Let G be a simplicial group, then there is a theory given by $\mathcal{G}(n) = G\times\mathcal{N}(n)$, with the proper composition. It is immediate that the $\mathcal{G}$-algebras are just the G-spaces, in the ordinary sense. Let H be another simplicial group, and $\varphi:G\to H$ a homomorphism of groups, then φ induces a morpism of theories $\varphi:\mathcal{G}\to\mathcal{H}$. It is easily verified that $\mathrm{Lan}^{\varphi}(X) = X\times_G H$, and that $\mathrm{LAN}^{\varphi}(X) = (X\times WG)\times_G H$, where WG is a contractible space on which G acts freely.

A cofibrant G-space is just a free G-space, since any equivariant retract of a space on which G acts freely, must also have free G action. If we let H = {e}, the trivial group, in the above example, then the content of Theorem III is the standard result that X/G is homotopically equivalent to $X \times_G WG$.

Homologically this example is more complicated than the previous ones. I believe it is the case that the groups $H_*(X \times_G WG)$ do not depend in any nice way on $H_*(X)$ and the action of $H_*(G)$ on $H_*(X)$. Thus, no result as nice as Theorem V is likely to exist in all of the cases covered by Theorem III. However, Theorem V does show that results about the homology in certain situations are available.

As concluding remarks I would like to discuss some immediate areas of pursuit on these problems. First, Theorem V should be expanded to describe the complete structure on $H_*(AX;k)$, where k is a field. This would presumably include results giving the coalgebra structure and the Steenrod Algebra structure in terms of these structures on $H_*(X;k)$. This should also include a description of the structure that the homology of any algebra over A has, and possibly describe a convenient category of algebraic structures on graded groups where the homology of an algebra lies.

A second area would be a description of the properties that a theory must have in order for a result like Theorem V to be true in that case. This would hopefully allow a direct discussion of objects like $H_*(\Omega^n S^n(X))$, for non-connected spaces X, which cannot directly arise from a prop, because of the inverses, but which do arise indirectly from props, e.g. by group completion, as is the case for $\Omega^n X^n(X)$.

A last possible area of research, which is tangential to the results of this paper, would be a description of what an infinite homotopy differential graded coalgebra structure is. ($C_*(X;Z/p)$ clearly should be one.) This would be designed to allow one to make the statement that $H_*(\mathrm{Lan}^{\varphi}(X))$ depends only on the infinite homotopy S-algebra structure on $C_*(X)$. This might even include a model category structure on the homotopy coalgebras (or homotopy S-algebras), and also a theorem that the chains on a space constitute a good functor between these model categories. However, it is unfortunate that no result as nice as Quillen's(RH) is likely to be available here, since non-trivial higher operations do exist in the mod p homology of a space.

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