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A Non-Commutative Theory of Integration for Operators

By

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The so-called "non-commutative theory" of integration for rings of operators on Hilbert spaces has been much developed by Segal [**26**] and Dixmier [**9**], independently. The former's theory is a theory of integrals (or traces) for certain (unbounded) "measurable operators", analogous to measurable functions in the classical theory of integrations over abstract measure spaces. His idea of the "measurable operators" originates from the works of Murray and v. Neumann ([**18**], Chap. 16) for factors of type II, and of Dye [**11**] for finite rings. The latter's theory is a theory of integrals as linear forms. For both theories the rings may be assumed to be semi-finite without loss of generality. A ring $\mathbb{M}$ of operators is called semi-finite [**15**] provided every non-zero projection $\in \mathbb{M}$ contains a non-zero finite projection $\in \mathbb{M}$. Let $\mathbb{M}$ and $\mathbb{N}$ be $*$-isomorphic rings of operators, and let m and μ be regular gages of $\mathbb{M}$ and $\mathbb{N}$ respectively such that m and μ correspond by means of the above $*$-isomorphism. If we stand on the view-point of Dixmer [**9**], the measurable integrable operators with respect to m and μ must correspond $*$-isomorphically. We show (Theorem 1) that if $\mathbb{M}$ is $*$-isomorphic with $\mathbb{N}$ by means of a mapping θ, then θ is uniquely extended to a $*$-isomorphic mapping between measurable operators with respect to $\mathbb{M}$ and $\mathbb{N}$. To develop the theory of Segal [**26**] for a given ring $\mathbb{M}$ it seems, therefore, preferable to take an appropriate ring $\mathbb{N}$ $*$-isomorphic with $\mathbb{M}$ and to develop the theory for $\mathbb{N}$ instead of $\mathbb{M}$ and then to transfer it to that for $\mathbb{M}$, if such a process is more suitable. It is known that every semi-finite ring $\mathbb{M}$ is $*$-isomorphic with the left ring $\mathbb{L}$ of an H-system $\mathbf{H}$, and the regular gage of $\mathbb{M}$ in question corresponds to the canonical gage μ of $\mathbf{H}$. Left multiplication operators L_x, $x \in \mathbf{H}$ form a Hilbert space when the inner product $<L_x, L_y>$ is defined by $<L_x, L_y> = <x, y>$. The set $\mathfrak{L}_2$ of all L_x is the set of square integrable measurable operators with respect to μ. Thus in $\mathbf{H}$ the square integrable measurable operators are given *a priori*. We define that $T = L_x \cdot L_y$ is integrable with respect to μ and define its integral $\mu(T)$ by $<L_x, L_{y^*}>$. Let $\mathfrak{L}_1$ be the set of all T. To prove that $\mathfrak{L}_1$ is the set of all measurable integrable operators is reduced to the proof of the following: in **H** (a) strong and ultrastrong (=strongest) topologies, (b) weak and ultraweak ($=\sigma$–weak [**15**]) topologies coincide respectively (Theorem 3). This is an easy consequence of a theorem of Griffin ([**15**], Theorem 12). But we shall prove it by an elementary way somewhat similar to Segal's method of proof of a certain theorem on a commutative ring [**25**]. As its consequence, the Radon-Nikodym theorem and Lebesgue monotone convergence theorem follow.

If $\mathbb{M}$ is commutative, then the above $\mathbb{L}$ is a masa (=maximal abelian self-adjoint) algebra which is $*$-isomorphic with $\mathbb{M}$. In this case the set $\mathbf{H}'$ of self-adjoint elements of **H** is a vector lattice in which the lattice order is the usual operator order. Finally we shall give a somewhat axiomatic definition of $\mathfrak{L}_1$ for a general ring $\mathbb{M}$ and compare it with the AL-space of a vector lattice developed previously by one of the present authors ([**19**] p. 86).

Some applications to the structure of $\mathbb{L}$ are given in **3**.

1. Measurable operators

1.1. Let $\mathbb{M}$ be a ring of operators on a Hilbert space $\mathfrak{H}$ of arbitrary dimensions. We shall always assume that $\mathbb{M}$ contains the identity operator I on $\mathfrak{H}$. $\mathbb{M}_P$ and $\mathbb{M}_U$, respectivly, stand for the set of projections and that of unitary operators in $\mathbb{M}$. Let $\mathfrak{m}$ be an ideal of $\mathbb{M}$ generated by a certain set of finite projections $\in \mathbb{M}$. Any projection $\in \mathfrak{m}$ is then finite since the ideal $\mathfrak{m}_0$ generated by all finite projections $\in \mathbb{M}$ contains only finite projections.

Definition 1.1. (cf. [**26**], Def. 2.1). A linear set $\mathfrak{D}$ in $\mathfrak{H}$ is said to be *strongly* $\mathfrak{m}$*-dense* provided (a) $U'\mathfrak{D} \subset \mathfrak{D}$ for every $U' \in \mathbb{M}'_U$; (b) there exists a sequence of projections $P_n \in \mathbb{M}$ such that $P_n\mathfrak{H} \subset \mathfrak{D}, P_n^{\perp} \downarrow 0$ and $P_n^{\perp} \in \mathfrak{m}$. An operator $T\eta\mathbb{M}$ is called *essentially* $\mathfrak{m}$*-restrictedly measurable* if T has a strongly $\mathfrak{m}$–dense domain and a closed extension. Moreover if T is closed, T is called $\mathfrak{m}$*-restrictedly measurable.* In case $\mathfrak{m}=\mathfrak{m}_0$, we shall say simply that $\mathfrak{D}$ is *strongly dense*, T is *essentially measurable* or T is *measurable* as the case may be.

Lemma 1.1. *Let T be a closed densely defined operator $\eta\mathbb{M}$, Then:*

(i) *T is $\mathfrak{m}$-restrictedly measurable if and only if so is $|T|$;*

(ii) *Let $T \geq 0$ and let $T=\int_0^{\infty} \lambda dE_\lambda$ be its spectral resolution. T is $\mathfrak{m}$-restrictedly measurable if only if $E_\lambda^{\perp}(=I-E_\lambda) \in \mathfrak{m}$ for a positive λ.*

Proof. (i) is evident since T and $|T|$ have the same domain. The "if" part of (ii) is clear. Let $T \geq 0$ be $\mathfrak{m}$-restrictedly measurable. Then there exists a projection $P \in \mathbf{M}$ such that TP is bounded and $P^{\perp} \in \mathfrak{m}$. Let $\|TP\| < \lambda_0$. We show that $P \frown E_{\lambda_0}^{\perp} = 0$. If the contrary holds, there exits a non-zero $x \in \mathfrak{H}$ with $P \frown E_{\lambda_0}^{\perp} x = x$. $\|Tx\| = \|TPx\| < \lambda_0 \|x\|$, while $\|Tx\| = \|TE_{\lambda_0}^{\perp} x\| \geq \lambda_0 \|x\|$. This is a contradiction. Since for every projection $Q, R \in \mathbf{M}$, $Q - Q \frown R \sim Q \smile R - R$ [**17**], we have $E_{\lambda_0}^{\perp} = E_{\lambda_0}^{\perp} - P \frown E_{\lambda_0}^{\perp} \sim P \smile E_{\lambda_0}^{\perp} - P \leq P^{\perp} \in \mathfrak{m}$, as desired.

Segal [**26**] proved that if S and T are essentially measurable and agree on a strongly dense domain, then they have identical closures. Next is its slight generalization.

LEMMA 1.2. *If two essentially* $\mathfrak{m}$*-restrictedly measurable operators* S *and* T *agree on a dense domain, then they have identical closures.*

PROOF. With no loss of generality, we may assume that S and T are $\mathfrak{m}$-restrictedly measurable. The set $\mathfrak{D} = \{x ; Tx = Sx\}$ is obviously invariant under every $U' \in \mathbf{M}'_U$, and is dense in $\mathfrak{H}$. Let T_0 be the restriction of S and T on $\mathfrak{D}$. $T \supset T_0$ implies $T^* \subset T_0^*$. As T^* is $\mathfrak{m}$-restrictedly measurable, as proved below, so is T_0^* by the very definition of measurability. It follows, from the result of Segal above mentioned, that $T^* = T_0^*$ and hence $T = T_0^{**}$. By symmetry $S = T_0^{**}$, and we have $T = S$; as desired.

From Lemma 1.1. if T is $\mathfrak{m}$-restrictedly measurable, then so are T^*T, $|T|^{\alpha}$ $(\alpha > 0)$. We show that T^* is $\mathfrak{m}$-restrictedly measurable if so is T. Let $T = W|T|$ be the polar decomposition of T, where W is a partially isometric operator $\in \mathbf{M}$ with the closure of the range of $|T|$ as the initial set and with the closure of the range of T as the final set. Let $WW^* = E$ and let $|T| = \int_0^{\infty} \lambda dE_{\lambda}$, $|T^*| = \int_0^{\infty} \lambda dF_{\lambda}$ be the spectral resolutions of $|T|$ and $|T^*|$ respectively. $|T^*| = W|T|W^*$ yields $F_{\lambda} = WE_{\lambda}W^* + E^{\perp}$ $(\lambda > 0)$. Hence $F_{\lambda}^{\perp} = WE_{\lambda}^{\perp}W^*$. This implies by Lemma 1.1 that $|T^*|$ is $\mathfrak{m}$-restrictedly measurable. It is clear that the intersection of a finite number of strongly $\mathfrak{m}$-dense domains is so also. After Segal we define the strong sum $S \dot{+} T$ and strong product $S \cdot T$ of two $\mathfrak{m}$-restrictedly measurable operators S and T. $S \dot{+} T$ and $S \cdot T$ are the closures of $S + T$ and ST respectively. (cf. [**26**], Def. 2.2). But in case of our $\mathfrak{m}$-restrictedly measurable operators, $S+T$ is seen to be essentially $\mathfrak{m}$-restrictedly measurable from the above. That ST is so also, follows from a modification of a proof given in [**26**], and details are omitted. Hence in our case $S \dot{+} T$ and $S \cdot T$ are $\mathfrak{m}$-restrictedly measurable. Thus we have the

LEMMA 1.3. *The set of all* $\mathfrak{m}$*-restrictedly measurable operators forms a *-algebra with respect to the strong sum* $S \dot{+} T$ *and product* $S \cdot T$, *the scalar multiplication (except that*

$0 \cdot T=0$) *and adjunction.*

We remark that the two measurable self-adjoint operators S, T are commutative ($S \cdot T=T \cdot S$) if and only if every two projections in their spectral resolutions commute (This is usually a definition of commutativity of two self-adjoint operators on $\mathfrak{H}$). The "if" part is well known. Let S and T commute, and if we put $V=S+iT$, then $V^*V=VV^*$ will follow and therefore V is normal. From this we obtain the statement of the "only if" part.

1.2. A projection $P \in \mathbf{M}_P$ is called *countably decomposable* if each set of mutually orthogonal non-zero projections in $P\mathbf{M}P$ is at most countable. In the sequel only three types of ideals $\mathfrak{m}$ are concerned: (a) $\mathfrak{m}_0$ is the ideal of $\mathbf{M}$ generated by all finite projections $\in \mathbf{M}$; (b) $\mathfrak{m}_1$ is the ideal of all finite countably decomposable projections $\in \mathbf{M}$; (c) $\mathfrak{m}_2$ is an ideal of $\mathbf{M}$ generated by the metrically finite projections with respect to a regular gage. In the last case we assume that $\mathbf{M}$ is semi-finite. A ring $\mathbf{M}$ is called *semi-finite* [**15**] if every non-zero projection $\in \mathbf{M}$ contains a non-zero finite projection $\in \mathbf{M}$. Clearly $\mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_2$. Let $d(P)$ be a dimension function on $\mathbf{M}_P$ in a certain sense of Segal [**26**]. He proved that if we let $\{P_{i,j};\ i, j=1, 2, 3, \cdots\}$ be an indexed family of projections $\in \mathfrak{m}_1$ such that for each i, $d(P_{i,j}) \downarrow 0$ (pointwise except for a non dense set) as $j \uparrow \infty$, then there exists a subsequence $\{j(i)\}$ of the integers such that $\sum_{i=1}^{\infty} d(P_{i,j(i)}) < \infty$ (pointwise except for a non-dense set). In particular, if $P_{i,j} \downarrow 0$ for each i as $j \uparrow \infty$, then there exists a subsequence $\{j(i)\}$ of the integers such that $\cup_{i=n}^{\infty} P_{i,j(i)} \in \mathfrak{m}_1$ and $\downarrow 0$ as $n \uparrow \infty$. For $\mathfrak{m}=\mathfrak{m}_2$, if we use a regular gage instead of a dimension function, we get a corresponding result. Segal's discussion is concerned with the case when the center $\mathbf{M}^\natural$ of $\mathbf{M}$ is countably decomposable, but it holds as well for the modified statement above mentioned, since a countable number of countably decomposable finite projections $\in \mathbf{M}$ is contained in a center which is countably decomposable in $\mathbf{M}^\natural$.

Segal exposed a convergence discussion by the following definition ([**26**], Def. 2.3). A sequence $\{T_n\}$ of measurable operators is said to converge *nearly everywhere* (n. e.) to a measurable operator T, if for every positive ε there exists a sequence $\{P_n\}$ of projections, such that $\mathfrak{m}_0 \ni P_n^{\perp} \downarrow 0$ as $n \uparrow \infty$ and $\|(T_n - T)P_n\| < \varepsilon$ ($n=1, 2, 3, \cdots$). In case $\mathbf{M}$ is a factor of Type III, a measurable operator is nothing but an element of $\mathbf{M}$, and n. e. convergence in this sense means $T_n = T$ ($n=1, 2, 3, \cdots$). This shows that the uniform convergence does not necessarily imply the n. e. convergence. On account of this unsuitableness, we shall give the following improved

Definition 1.2. Let $\{T_n\}$ be a sequence of $\mathfrak{m}$-restrictedly measurable operators

η M. $\{T_n\}$ is said to *converge* $\mathfrak{m}$*-nearly everywhere* ($\mathfrak{m}$-n. e.) to a measurable operator T if for every positive number ε, there exists a sequence of projections $P_n \in \mathrm{M}$ $(n \geq n_\varepsilon)$ such that $\|(T-T_n)P_n\| < \varepsilon$, $P_n^\perp \downarrow 0$ as $n \uparrow \infty$ and $P_n^\perp \in \mathfrak{m}$. If $\mathfrak{m} = \mathfrak{m}_0$, we shall omit "$\mathfrak{m}-$".

It will turn out from the discussion below that T is necessarily $\mathfrak{m}$-restrictedly measurable. Segal [**26**] proved that T is unique for an n. e. convergent sequence $\{T_n\}$.

Lemma 1.4. *Let $\{T_n\}$ be a sequence of $\mathfrak{m}$-restrictedly measurable operators ηM. A necessary and sufficient condition for $\{T_n\}$ to converge $\mathfrak{m}$-n. e. to a measurable operator ηM is that, for erery positive $\varepsilon > 0$, there exists a sequence of projections $P_n \in \mathrm{M}$ $(n \geq n_\varepsilon)$ such that $\|(T_m - T_n)P_n\| < \varepsilon$ for $m > n \geq n_\varepsilon$ and $P_n^\perp \downarrow 0$, $P_n^\perp \in \mathfrak{m}$.*

Proof. That the condition is necessary is evident. For the sufficiency proof we only consider the cases $\mathfrak{m} = \mathfrak{m}_0$ and $\mathfrak{m} = \mathfrak{m}_1$. For the case $\mathfrak{m} = \mathfrak{m}_2$, it is treated in much the same way as in the case $\mathfrak{m} = \mathfrak{m}_1$. First consider the case $\mathfrak{m} = \mathfrak{m}_1$.

Write $n_\varepsilon = n_k$ and $P_n = P_n^{(k)}$ when $\varepsilon = \frac{1}{k^2}$. We may assume that $n_k \uparrow \infty$ as $k \uparrow \infty$, and that

(1) $$Q^\perp = \cup_{k=1}^\infty P_{n_k}^{(k)\perp} \in \mathfrak{m}_1, \quad \cup_{k=n}^\infty P_{n_k}^{(k)\perp} \downarrow 0 \text{ as } n \uparrow \infty.$$

We use the symbol $\mathfrak{D}_T$ to denote the domain of operator T. The intersection $\mathfrak{D} = \cap \mathfrak{D}_{T_n}$ is strongly $\mathfrak{m}_1$-dense [**26**]. By definition there exists a sequence of projections $E_n \in \mathrm{M}$ such that $E_n \mathfrak{H} \subset \mathfrak{D}$, $E_n^\perp \downarrow 0$ and $E_n^\perp \in \mathfrak{m}_1$. Put

(2) $$Q_n = \cap_{k=1}^\infty P_{\max(n,\, n_k)}^{(k)} \cap E_n. \quad (Q_1 = Q).$$

Then $Q_n^\perp = \cup_{k=1}^\infty P_{\max(n,\, n_k)}^{(k)\perp} \cup E_n^\perp \downarrow 0$ as $n \uparrow \infty$, since $\cup_{k=1}^\infty P_{\max(n,\, n_k)}^{(k)\perp} \downarrow 0$ and $E_n^\perp \downarrow 0$ as $n \uparrow \infty$. Evidently $Q_n^\perp \in \mathfrak{m}_1$. We obtain

(3) $$\|(T_p - T_q)Q_n\| < \frac{1}{k^2} \text{ for every } p > q \geq \max(n, n_k).$$

Let $\mathfrak{D}_0$ be the set-theoretical union of $\{Q_n \mathfrak{H}\}$. Then $\mathfrak{D}_0$ is strongly $\mathfrak{m}_1$-dense. (3) shows that $\{T_p\}$ is a Cauchy sequence on each $Q_n \mathfrak{H}$ in the uniform topology. Hence we have an operator $T \eta$ M with domain $\mathfrak{D}_0$ such that $\|(T - T_q)Q_n\| \leq \frac{1}{k^2}$ for $q \geq \max(n, n_k)$. For any positive number $\varepsilon > 0$, we take $k = k(\varepsilon)$ so large that $\frac{1}{k^2} < \varepsilon$. Then we have

(4) $$\|(T - T_n)Q_n\| < \varepsilon \text{ for } n \geq n_{k(\varepsilon)},\ Q_n^\perp \downarrow 0, \text{ and } Q_n^\perp \in \mathfrak{m}_1.$$

If we can show that T has a closed extension $\tilde{T}$, then $\tilde{T}$ will be $\mathfrak{m}_1$-restrictedly measurable and $\{T_n\}$ converges $\mathfrak{m}_1$-n. e. to $\tilde{T}$. The proof will follow from the following lemma (we take $\mathfrak{m}=\mathfrak{m}_1$).

LEMMA 1.5. *If $\{T_n\}$ satisfies the condition of the preceding lemma, then every subsequence of $\{T_n^*\}$ has a subsequence satisfying a condition of the same type.*

PROOF. It suffices to prove that $\{T_n^*\}$ has a subsequence stated in this lemma. We use the notation in the proof of the preceding lemma. Let $\mathfrak{D}^*=\cap\mathfrak{D}_{T_n^*}$, which is strongly $\mathfrak{m}_1$-dense. There exists a sequence of projections $F_n\in\mathbf{M}$ such that $F_n\mathfrak{H}\subset\mathfrak{D}^*$, $F_n^{\perp}\downarrow 0$ and $F_n^{\perp}\in\mathfrak{m}_1$. Put

$$\mathfrak{L}_{l,k}=F_{nk}\mathfrak{H}\cap(T_{nl}^*-T_{nk}^*)^{-1}(Q_{nk}\mathfrak{H}) \text{ for } l>\text{k}.$$

Then $\mathfrak{L}_{l,k}^{\perp}=F_{nk}^{\perp}\mathfrak{H}\cup[(T_{nl}^*-T_{nk}^*)^{-1}(Q_{nk}\mathfrak{H})]^{\perp}$. By a result of Segal ([**26**], Lemma 3.1) we have $d(P\mathfrak{L}_{l,k}^{\perp})\leqq d(F_{nk}^{\perp})+2d(Q_{nk}^{\perp})$. We select a subsequence $\{n_{k_i}\}$ such that $\sum_i d(F_{nk_i}^{\perp})<\infty$, $\sum_i d(Q_{nk_i}^{\perp})<\infty$ except for a non-dense set. Let

$$\mathfrak{L}_n=\cap_{i=n}^{\infty}[\mathfrak{L}_{k_{i+1},k_i}] \text{ and } G_n=P_{\mathfrak{L}_n}. \tag{5}$$

Then $G_n^{\perp}\in\mathfrak{m}_1$, $G_n^{\perp}\downarrow 0$ as $n\uparrow\infty$. It follows from (3) that $\|(T_{nk_{i+1}}-T_{nk_i})Q_{ni}\|<\frac{1}{k_i^2}$. Then we obtain $\|(T_{nk_{i+1}}^*-T_{nk_i}^*)x\|\leqq\frac{1}{k_i^2}\|x\|$ for every $x\in G_n\mathfrak{H}$ for $i\geqq n$ ([**26**], the proof of Theorem 9). Hence

$$\|(T_{nk_j}^*-T_{nkl}^*)\ G_l\|\leqq\sum_{i=l}^{\infty}\frac{1}{k_i^2} \text{ for } j\geqq l, \tag{6}$$

which shows that $\{T_{nk_j}^*\}$ satisfies the condition of the preceding lemma. The proof is completed.

We return to the proof of Lemma 1.4. By making use of Lemma 1.5 and the result so far obtained in the proof of Lemma 1.4, we can infer that there may exist a subsequence $\{T_{pn}^*\}$ of $\{T_n^*\}$ converging pointwise to an operator T' in a strongly $\mathfrak{m}_1$-dense domain $\mathfrak{D}_0^*$. Let $x\in\mathfrak{D}_0$, $y\in\mathfrak{D}_0^*$ be chosen arbitrarily. Then $\langle T_{pn}x, y\rangle=\langle x, T_{pn}^*y\rangle$, which yields $\langle Tx, y\rangle=\langle x, T'y\rangle$. This implies that T^* has a dense domain, so that, T has a closed extension $\tilde{T}$, as desired. It is noted that $\tilde{T}$ is $\mathfrak{m}_1$-restrictedly measurable.

We show that $\|(\tilde{T}-T_n)P_n\|\leqq\varepsilon$. $\|(T_m-T_n)\ (P_n\cap Q_p)\|<\varepsilon$ for $m>n$. Let $m\uparrow\infty$ in this inequality, then we have $\|(\tilde{T}-T_n)(P_n\cap Q_p)\|\leqq\varepsilon$. Since $P_n-P_n\cap Q_p\sim P_n\cup Q_p-Q_p$ by [**17**] and $P_n\cup Q_p-Q_p\leqq Q_p^{\perp}\downarrow 0$ as $p\uparrow\infty$, we can easily obtain the desired inequality.

Next we turn to the case $\mathfrak{m}=\mathfrak{m}_0$. Let $\{Q_\iota\}$ be a maximal orthogonal family of projections $\in\mathbf{M}^\natural$, each of which is countably decomposable in $\mathbf{M}^\natural$. For each Q_ι, consider the sequence $\{T_{n,\iota}\}$, where $T_{n,\iota}=T_nQ_\iota$. Put $P_{n,\iota}=P_n\cup Q_\iota^{\perp}$. Then $(T_{m,\iota}-T_{n,\iota})$

$P_{n,\iota}=(T_m-T_n)Q_\iota P_n$ and therefore $\|(T_{m,\iota}-T_{n,\iota})P_{n,\iota}\|<\varepsilon$ for $m>n$, $P_{n,\iota}^{\perp}=P_n^{\perp}Q_\iota\downarrow 0$ as $n\uparrow\infty$ and $P_{n,\iota}^{\perp}\in\mathfrak{m}_1$. $T_{n,\iota}$ is evidently $\mathfrak{m}_1$–restrictedly measurable. We can apply the result for $\mathfrak{m}=\mathfrak{m}_1$ to $\{T_{n,\iota}\}$. Let T_ι be the limit of $\{T_{n,\iota}\}$ in the $\mathfrak{m}_1$–n. e. convergence. Then $\|(T_\iota-T_{n,\iota})\ Q_\iota P_n\|\leq\varepsilon$. Let T be the closed operator such that $TQ_\iota=T_\iota$. Evidently $T_\iota Q_\iota^{\perp}=0$, so that the existence of T is proved in a usual way. It is easy to see that $T\eta\mathrm{M}$ and $\|(T-T_n)P_n\|\leq\varepsilon$. Therefore T is the n. e. limit of $\{T_n\}$. The proof of Lemma 1.4 is completed.

From (3) in this proof we can incidentally read off the following

LEMMA 1.6. *Let $\{T_n\}$ be a sequence of $\mathfrak{m}$–restrictedly (where $\mathfrak{m}=\mathfrak{m}_1$ or $\mathfrak{m}_2$) measurable operators converging $\mathfrak{m}$–n. e., and $\{\varepsilon_k\}$ be a sequence of positive numbers decreasing to 0. Then there exists a sequence of projections $\{Q_k\}$, $Q_k^{\perp}\in\mathfrak{m}$, $Q_k\downarrow 0$ as $k\uparrow\infty$, and an increasing sequence of positive integers $\{n_k\}$, such that $\|(T_m-T_n)Q_k\|<\varepsilon_k$ for every $m>n\geq n_k$.*

REMARK. At this juncture we shall point out the following fact which will be used later. If $\{T_n\}$ be a sequence of uniformly bounded $\mathfrak{m}$–restrictedly (where $\mathfrak{m}=\mathfrak{m}_0$, $\mathfrak{m}_1$ or $\mathfrak{m}_2$) measurable operators converging $\mathfrak{m}$–n. e. in the star sense to an $\mathfrak{m}$–restrictedly measurable operator T, then T is bounded and $T_n\to T$ strongly. This follows easily from Lemma 1.6 if $\mathfrak{m}=\mathfrak{m}_1$ or $\mathfrak{m}_2$. As for the case $\mathfrak{m}=\mathfrak{m}_0$, we decompose M into direct summands by the family of projections $\{Q_\iota\}$ used in the last part of the proof of Lemma 1.4, and the problem can be reduced to the case $\mathfrak{m}=\mathfrak{m}_1$ on each direct summand $\mathrm{M}Q_\iota$.

Lemma 1.4 together with Lemma 1.5 shows that if a sequence $\{T_n\}$ of $\mathfrak{m}_1$–restrictedly measurable operators converges $\mathfrak{m}_1$–n. e. to a measurable operator T, then T is necessarily $\mathfrak{m}_1$–restrictedly measurable and $\{T_n^*\}$ converges $\mathfrak{m}_1$–n. e. to T^* in the star sense. This is also proved by Segal [26].

Let $\{T_n\}$ be a sequence of $\mathfrak{m}_1$–restrictedly measurable operators converging $\mathfrak{m}_1$–n. e. to 0, Then $\|T_nP_n\|<\frac{1}{k}$ for $n\geq n_k$, $P_n^{\perp}\downarrow 0$ and $P_n^{\perp}\in\mathfrak{m}_1$. Let $\mathfrak{M}_n=P_n\mathfrak{H}\cap T_n^{*-1}(P_n\mathfrak{H})$. Let E_n be a projection on the closure of $\mathfrak{M}_n$. Then $d(E_n^{\perp})\leq d(P_n^{\perp})+2d(P_n^{\perp})=3d(P_n^{\perp})$. And we can find a subsequence $\{p_n\}$ of the integers, such that $\cup_n E_{p_n}^{\perp}\in\mathfrak{m}_1$ and $\cup_{k=n}^{\infty}E_{p_k}^{\perp}\downarrow 0$ as $n\uparrow\infty$. It is easy to see $\|T_{p_n}^*E_{p_n}\|\leq\|T_{p_n}P_{p_n}\|$ and therefore $\|T_{p_n}T_{p_n}^*E_{p_n}\|<\|T_{p_n}P_{p_n}\|^2$. Thus $\{T_{p_n}T_{p_n}^*\}$ converges $\mathfrak{m}_1$–n. e. to 0. From this we see that $\{T_nT_n^*\}$ converges $\mathfrak{m}_1$–n. e. to 0 in the star sense. Similarly $\{T_n^*T_n\}$ converges $\mathfrak{m}_1$–n. e. to 0 in the star sense. Segal [26] proved that if $\{T_n\}$ is a sequence of $\mathfrak{m}_1$–restrictedly measurable operators $\eta\mathrm{M}$ converging $\mathfrak{m}_1$–n. e. to a measurable operator

$T\eta\mathbf{M}$ and S is an $\mathfrak{m}_1$-restrictedly measurable operator, then $\{S\cdot T_n\}$ and $\{T_n\cdot S\}$ converges $\mathfrak{m}_1$-n. e. to $S\cdot T$ and $T\cdot S$ in the star sense respectively.

LEMMA 1.7. *Let $\{S_n\}$ and $\{T_n\}$ be sequences of $\mathfrak{m}_1$-restrictedly measurable operators converging $\mathfrak{m}_1$-n. e. to S and T in the star sense respectively, then so does $\{S_n\cdot T_n\}$ to $S\cdot T$.*

PROOF. Since $S\cdot T=\frac{1}{4}\{(S^*+T)^*\cdot(S^*+T)-(S^*-T)^*\cdot(S^*-T)-i(S^*+iT)^*\cdot(S^*+iT)+i(S^*-iT)^*\cdot(S^*-iT)\}$, it is sufficient to prove the lemma under the assumption $S=T^*$. $T_n^*T_n-T^*T=(T^*-T_n^*)\cdot(T-T_n)+T_n^*\cdot T+T^*\cdot T_n-2T^*T$. This equation yields that $\{T_n^*T_n\}$ converges $\mathfrak{m}_1$-n. e. to T^*T in the star sense. But $(S_n-T_n^*)\cdot(S_n^*-T_n)=S_nS_n^*+T_n^*T_n-(S_n\cdot T_n+T_n^*\cdot S_n^*)$ and $(S_n+iT_n^*)\cdot(S_n^*-iT_n)=S_nS_n^*+T_n^*T_n-i(S_n\cdot T_n-T_n^*\cdot S_n^*)$. The first of these equations shows that $\{S_n\cdot T_n+T_n^*\cdot S_n^*\}$ converges $\mathfrak{m}_1$-n. e. to $2T^*T$ in the star sense and the second one shows that $\{S_n\cdot T_n-T_n^*\cdot S_n^*\}$ converges $\mathfrak{m}_1$-n. e. to 0 in the star sense. Therefore $\{S_n\cdot T_n\}$ converges $\mathfrak{m}_1$-*n. e.* to T^*T in the star sense. The proof is completed.

The discussions so far given hold also for $\mathfrak{m}=\mathfrak{m}_2$. Hence Lemma 1.7 is true for $\mathfrak{m}=\mathfrak{m}_2$. Therefore if $\mathfrak{m}=\mathfrak{m}_1$, or $\mathfrak{m}_2$, the algebra of $\mathfrak{m}$-restrictedly measurable operators is a topological algebra with respect to the star topology. Let $\mathfrak{m}=\mathfrak{m}_1$ or $\mathfrak{m}_2$. Then we have the following

LEMMA 1.8. *If a sequence $\{T_n\}$ of $\mathfrak{m}$-measurable operators $\eta\mathbf{M}$ converges $\mathfrak{m}$-n. e. to 0 in the star sense, then so does $\{|T_n|\}$.*

PROOF. From the above discussion $\{T_n^*T_n\}$ converges $\mathfrak{m}$-n. e. to zero in the star sence. Therefore any subsequence of $\{T_n^*T_n\}$ contains a subsequence converging $\mathfrak{m}$-n. e. to 0. Let it be denoted by $\{T_{pn}^*T_{pn}\}$. For any given positive ε, there exists a sequence of projections $P_n\in\mathbf{M}$ $(n\geq n_\varepsilon)$ such that $\|T_{pn}^*T_{pn}P_n\|<\varepsilon^2$, $P_n^\perp\in\mathfrak{m}$ and $P_n^\perp\downarrow 0$. Let x be an arbitrary element of $P_n\mathfrak{H}$. $\||T_{pn}|x\|^2=\langle T_{pn}^*T_{pn}x, x\rangle\leq\|T_{pn}^*T_{pn}x\|\,\|x\|\leq\varepsilon^2\|x\|^2$. Hence $\||T_{pn}|P_n\|\leq\varepsilon$. The proof is completed.

1.3. Let $\mathbf{N}$ be a ring of operators on a Hilbert space $\mathfrak{H}'$. Suppose that there exists a $*$-isomorphic mapping $\theta(A)$ from $\mathbf{M}$ onto $\mathbf{N}$ (θ is bi-continuous in the ultrastrong (= strongest = ultrafort [9]) and ultraweak (= σ-weak = ultrafaible [15], [9]) topologies [9]). Let $\mathfrak{n}$ be the ideal of $\mathbf{N}$ corresponding to $\mathfrak{m}$ under θ, that is, $\mathfrak{n}=\theta(\mathfrak{m})$. Let $\mathcal{M}$ and $\mathcal{N}$ be the $*$-algebras of all $\mathfrak{m}$-and $\mathfrak{n}$-restrictedly measurable operators respectively. We shall show that θ can be uniquely extended to a $*$-isomorphic mapping from $\mathcal{M}$ onto $\mathcal{N}$. When this is once done, we see that θ will preserve the convergence character, since the Definition 1.2 is concerned only with the algebraic property of $\mathbf{M}$ (note that $*$-isomorphism preserves norms). The fact that there is a unique extension of θ will be important for our theory of integration for operators, because our point of view is that the theory is first developed for a certain ring $\mathbf{N}$ $*$-isomorphic with $\mathbf{M}$ and then we transfer it to the theory for $\mathbf{M}$ through the extended $*$-isomorphism.

THEOREM 1. *Let $\mathbf{M}$ and $\mathbf{N}$ be $*$-isomorphic rings of operators by a mapping θ. θ can be uniquely extended to a $*$-isomorphic mapping from $\mathcal{M}$ onto $\mathcal{N}$.*

PROOF. We prove the theorem for $\mathfrak{m}=\mathfrak{m}_0$ and $\mathfrak{m}=\mathfrak{m}_1$. For, the case $\mathfrak{m}=\mathfrak{m}_2$ is treated along the same line as in the case $\mathfrak{m}=\mathfrak{m}_1$. First we shall consider the case $\mathfrak{m}=\mathfrak{m}_1$. Let T be any element of $\mathcal{M}$. There exists a sequence $\{A_n\}$ of operators $\in \mathbf{M}$ converging $\mathfrak{m}_1$–n. e. to T. For example, let $T=W|T|$ be the polar decomposition of T, let $|T|=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of $|T|$, and put $A_n=W\int_0^n \lambda dE_\lambda$, then it is clear that $\{A_n\}$ converges $\mathfrak{m}_1$–n. e. to T. A $*$-isomorphism θ preserves norms. The criterion for $\mathfrak{m}$–n. e. convergence given in Lemma 1.4 is concerned only with the operators $\in \mathbf{M}$. Therefore $\{\theta(A_n)\}$ converges $\theta(\mathfrak{m}_1)$–n. e. to a $\theta(\mathfrak{m}_1)$–restrictedly measurable operator which we shall denote by $\theta(T)$. $\theta(T)$ is independent of the particular sequence $\{A_n\}$, because if $\{A_n'\}$ is another sequence with the same property as $\{A_n\}$, then $\{A_n-A_n'\}$ converges $\mathfrak{m}_1$–n. e. to 0 and then $\{\theta(A_n)-\theta(A_n')\}$ converges $\theta(\mathfrak{m}_1)$–n. e. to 0. $\{A_n^*\}$ contains a subsequence converging $\mathfrak{m}_1$–n. e. to T^*, so that we obtain $\theta(T)^*=\theta(T^*)$. It is clear that the mapping θ is linear and one-to-one. From Lemma 1.7 we see that $\theta(S\cdot T)=\theta(S)\cdot\theta(T)$. Therefore θ is a $*$-isomorphism. Uniqueness is evident, and details are omitted.

Next consider the case $\mathfrak{m}=\mathfrak{m}_0$. Let the mapping $\theta(T)$ be defined in the same manner as before. Only points for us to make clear are the following: $\theta(T^*)=\theta(T)^*$ and $\theta(S\cdot T)=\theta(S)\cdot\theta(T)$. Let $\{Q_\iota\}$ be a maximal orthogonal system of central projections $\in \mathbf{M}^\natural$ each of which is countably decomposable in $\mathbf{M}^\natural$. From the proof of Lemma 1.4, $\{A_nQ_\iota\}$ converges $\mathfrak{m}_1$–n. e. to TQ_ι if $\{A_n\}$ converges n. e. to T. Hence $\theta(TQ_\iota)=\theta(T)\theta(Q_\iota)$. Since TQ_ι is $\mathfrak{m}_1$–restrictedly measurable, $\theta(T^*)\theta(Q_\iota)=\theta(T^*Q_\iota)=\theta(TQ_\iota)^*=\theta(T)^*\theta(Q_\iota)$. This equation holds for every Q_ι and $\{\theta(Q_\iota)\}$ is also a maximal orthogonal system of central projections. Hence $\theta(T^*)=\theta(T)^*$. In like manner it is easy to see that $\theta(ST)=\theta(S)\theta(T)$. The proof is completed.

COROLLARY 1.1. *$\theta(|T|^\alpha)=|\theta(T)|^\alpha$ $(\alpha>0)$ for every measurable $T\eta\mathbf{M}$.*

PROOF. First suppose that $T\geq 0$. Let $T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T. Put $F_\lambda=\theta(E_\lambda)$. Then $\{F_\lambda\}$ is a resolution of identity. Put $A_n=\int_\lambda^n \lambda dE_\lambda$. Then $\theta(A_n)=\int_0^n \lambda dF_\lambda$ and $\theta(A_n^\alpha)=\int_0^n \lambda^\alpha dF_\lambda$. It follows from the manner of extension of θ that $\theta(T)=\int_0^\infty \lambda dF_\lambda$ and $\theta(T^\alpha)=\int_0^\infty \lambda^\alpha dF_\lambda$. Therefore $\theta(T^\alpha)=\theta(T)^\alpha$. The general case follows easily from the above. $\theta(|T|^\alpha)=\theta(|T|^2)^{\frac{\alpha}{2}}=\theta(T^*T)^{\frac{\alpha}{2}}=\{\theta(T)^*\theta(T)\}^{\frac{\alpha}{2}}=|\theta(T)|^\alpha$. The proof is completed.

The theorem is wellknown when $\mathbf{M}$ is commutative ([**26**], Lemma 15.1).

1.4. DEFINITION 1.3. A linear set $\mathfrak{L}$ of measurable operators $\eta\mathbf{M}$ is called an *invariant linear system* of $\mathbf{M}$ if $T\in\mathfrak{L}$ implies UT, $TU\in\mathfrak{L}$ for every $U\in\mathbf{M}_U$.

Let $\mathfrak{L}$ be an invariant linear system. Let K be a self-adjoint operator $\in\mathbf{M}$ such that $0\leq K\leq T$. Then $U=K+i(I-K^2)^{\frac{1}{2}}\in\mathbf{M}_U$ and $2K=U+U^*$. Hence $K\cdot\mathfrak{L}$, $\mathfrak{L}K\subset\mathfrak{L}$. As every operator $A\in\mathbf{M}$ is expressed as a linear combination of such K, we see that $A\cdot\mathfrak{L}$, $\mathfrak{L}A\subset\mathfrak{L}$ for every $A\in\mathbf{M}$. Let T and S be measurable operators such that $0\leq S\leq T$ and $T\in\mathfrak{L}$. Following Dixmier [**9**], we show $S\in\mathfrak{L}$ as follows. It is easy to see that the domain $\mathfrak{D}_{T^{\frac{1}{2}}}$ of $T^{\frac{1}{2}}$ is contained in the domain $\mathfrak{D}_{S^{\frac{1}{2}}}$ of $S^{\frac{1}{2}}$ and $\|S^{\frac{1}{2}}x\|\leq\|T^{\frac{1}{2}}x\|$ for every $x\in\mathfrak{D}_{T^{\frac{1}{2}}}$. Let C be an operator such that $Cy=S^{\frac{1}{2}}x$ for $y=T^{\frac{1}{2}}x$ and zero for any $y\in[\text{range of } T^{\frac{1}{2}}]^\perp$. We denote by the same C the closed linear extension of C. Then $C\in\mathbf{M}$ and $S^{\frac{1}{2}}=C\cdot T^{\frac{1}{2}}$. And in turn $S=C\cdot TC^*\in\mathfrak{L}$. We can also show that $T\in\mathfrak{L}$ implies T^*, $|T|\in\mathfrak{L}$. For let $T=W|T|$ be the polar decomposition of T, then $|T|=W^*T\in\mathfrak{L}$ and $T^*=|T|W^*\in\mathfrak{L}$. Let $\mathfrak{L}^+$ stand for the set of positive operators $\in\mathfrak{L}$. Every operator $\in\mathfrak{L}$ is expressed as a linear combination of operators $\in\mathfrak{L}^+$.

It follows from the above discussion that the set $\mathfrak{L}^+$ has the following properties:

(a) if $T\in\mathfrak{L}^+$ and $U\in\mathbf{M}_U$, then $UTU^*\in\mathfrak{L}^+$;

(b) if $T\in\mathfrak{L}^+$ and $0\leq S\leq T$, then $S\in\mathfrak{L}^+$, S being a measurable operator;

(c) if S, $T\in\mathfrak{L}^+$, theu $S+T\in\mathfrak{L}^+$.

Conversely let $\mathfrak{L}^\times$ be any set of positive measurable operators satisfying the conditions (a), (b) and (c). Then $\mathfrak{L}^\times$ is an $\mathfrak{L}^+$ of an invariant linear system $\mathfrak{L}$ determined as the set of linear combinations of elements of $\mathfrak{L}^\times$. This is also shown by the method of proof due to Dixmier [**7**] for an ideal. The main idea of the proof is that we let $\mathfrak{L}$ denote the set of all $\sum_{i=1}^n T_i\cdot S_i^*$, where T_i and S_i are measurable and $T_i\cdot T_i^*, S_i\cdot S_i^*, \in\mathfrak{L}^\times$. The details are omitted.

DEFINITION 1.4. (cf. [**8**] Def. 2). Let $\mathfrak{L}$ be an invariant linear system of $\mathbf{M}$. The power $\mathfrak{L}^\alpha(\alpha>0)$ is defined as the invariant linear system generated by all T^α such that $T\in\mathfrak{L}^+$.

Let T and S be positive $\mathfrak{m}$-restricted measurable operators and $T=\int_0^\infty \lambda dE_\lambda$, $S=\int_0^\infty \lambda dF_\lambda$ be their spectral resolutions respectively. If we put $G_\lambda=E_\lambda \cap F_\lambda$, then $G_\lambda^\perp=E_\lambda^\perp \cup F_\lambda^\perp \in \mathfrak{m}$ for sufficiently large λ by Lemma 1.1 and $G_\lambda^\perp \downarrow 0$ as $\lambda \uparrow \infty$. We define $T\vee S=\int_0^\infty \lambda dG_\lambda$, which is also positive and $\mathfrak{m}$-restrictedly measurable. We note that if P and Q are projections $\in \mathbf{M}$, then $P\vee Q$ coincides with the usual one $(P\cup Q)$. We write $T\ll S$ [8] if $E_\lambda \geq F_\lambda$ for every positive $\lambda>0$. $T\ll S$ implies $T^\alpha \leq S^\alpha$ for every $\alpha>0$. Since we can write $T^\alpha=\int_0^\infty E^\perp_{\lambda^{\frac{1}{\alpha}}} d\lambda$ and $S^\alpha=\int_0^\infty F^\perp_{\lambda^{\frac{1}{\alpha}}} d\lambda$. It is clear from the definition of $T\vee S$ that T, $S\ll T\vee S$. Assume that $\mathfrak{L}$ satisfies the conditions:

$(\ll)_1$ If $T=\int_0^\infty \lambda dE_\lambda \in \mathfrak{L}^+$ and $S=\int_0^\infty \lambda dF_\lambda \geq 0$ are measurable opertors such that $F_\lambda^\perp < E_\lambda^\perp$ for every positive λ, then $S\in\mathfrak{L}^+$.

$(\ll)_2$ If T, $S\in\mathfrak{L}^+$, then $T\vee T\in \mathfrak{L}^+$.

These conditions are always satisfied if $\mathfrak{L}\subset\mathbf{M}$ ([8], Lemma 7 and 8). By using $(\ll)_1$ and $(\ll)_2$ we can show after Dixmier [8] that the set $\{T^\alpha; T\in\mathfrak{L}^+\}$ satisfies the conditions (a), (b) and (c). (a) is evident. Let T_1, $T_2\in\mathfrak{L}^+$ and $S^\alpha\leq T_1^\alpha+T_2^\alpha$, S being a positive measurable operator. Put $T=T_1\vee T_2\in\mathfrak{L}^+$. Then $T_1^\alpha+T_2^\alpha\leq 2T^\alpha$. Let $S=\int_0^\infty \lambda dF_\lambda$ and $2^{\frac{1}{\alpha}}T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolutions respectively. Then $E_\lambda\cap F_\lambda^\perp=0$ is easily verified. $F_\lambda^\perp=F_\lambda^\perp-F_\lambda^\perp\cap E_\lambda \sim F_\lambda^\perp\cup E_\lambda - E_\lambda \leq E_\lambda^\perp$ [17]. Therefore by $(\ll)_1$ we have $S\in\mathfrak{L}^+$. Hence (b) and (c) are satisfied.

We note that if $\mathfrak{L}$ satisfies $(\ll)_1$ and $(\ll)_2$ then so do all the other $\mathfrak{L}^\alpha(\alpha>0)$. For if $T\in\mathfrak{L}^{\alpha+}$ and S satisfy the hypothesis of $(\ll)_1$, then so do $T^{\frac{1}{\alpha}}\in\mathfrak{L}^+$ and $S^{\frac{1}{\alpha}}$. Therefore $S^{\frac{1}{\alpha}}\in\mathfrak{L}^+$, that is, $S\in\mathfrak{L}^\alpha$. $T^\alpha\vee S^\alpha=(T\vee S)^\alpha$ shows that $\mathfrak{L}^\alpha$ satisfies $(\ll)_2$.

We state the following theorem for the powers of $\mathfrak{L}$, corresponding to that of the powers of ideals due to Dixmier [8].

Theorem 2. *Let $\mathfrak{L}$ be an invariant linear system of $\mathbf{M}$ satisfying the conditions $(\ll)_1$ and $(\ll)_2$. Then,*

(i) $(\mathfrak{L}^\alpha)^\beta=\mathfrak{L}^{\alpha\beta}$, $\mathfrak{L}^\alpha\cdot\mathfrak{L}^\beta=\mathfrak{L}^{\alpha+\beta}$, $\alpha,\beta>0$;

(ii) *if an $\mathfrak{L}^\alpha$ is an algebra for some $\alpha>0$ then so are all the other $\mathfrak{L}^\beta$.*

Proof. (i): The proof is modelled after that given by Dixmier [8] for the case $\mathfrak{L}\subset\mathbf{M}$. Let T be any positive element of $(\mathfrak{L}^\alpha)^\beta$, then $T^{\frac{1}{\beta}}\in\mathfrak{L}^{\alpha+}$ and therefore

$T^{\frac{1}{\alpha\beta}} \in \mathfrak{L}$ that is, $T \in \mathfrak{L}^{\alpha\beta+}$. The converse is also true. Therefore $(\mathfrak{L}^{\alpha})^{\beta} = \mathfrak{L}^{\alpha\beta}$. Let $T_1^{\alpha} \in \mathfrak{L}^{\alpha+}$ and $T_2^{\beta} \in \mathfrak{L}^{\beta+}$. We show that $T_1 \cdot T_2 \in \mathfrak{L}^{\alpha+\beta}$ [**8**]. Let $T = T_1 \vee T_2$. Then $T_1^{2\alpha} \leq T^{2\alpha}$, $T_2^{2\beta} \leq T^{2\beta}$ therefore $T_1^{\alpha} = C_1 \cdot T^{\alpha}$ and $T_2^{\beta} = C_2 \cdot T^{\beta}$ for some $C_1, C_2 \in \mathbf{M}$. $T_1^{\alpha} \cdot T_2^{\beta} = T_1^{\alpha} \cdot (T_2^{\beta})^* = C_1 \cdot T^{\alpha+\beta} C_2^* \in \mathfrak{L}^{\alpha+\beta}$ Therefore $\mathfrak{L}^{\alpha} \cdot \mathfrak{L}^{\beta} \subset \mathfrak{L}^{\alpha+\beta}$. Conversely let $T^{\alpha+\beta} \in (\mathfrak{L}^{\alpha+\beta})^+$. Then $T \in \mathfrak{L}^+$ and $T^{\alpha+\beta} = T^{\alpha} \cdot T^{\beta}$, $T^{\alpha} \in \mathfrak{L}^{\alpha+}$, $T^{\beta} \in \mathfrak{L}^{\beta+}$, and therefore $T^{\alpha+\beta} \in \mathfrak{L}^{\alpha} \cdot \mathfrak{L}^{\beta}$. Hence $\mathfrak{L}^{\alpha} \cdot \mathfrak{L}^{\beta} = \mathfrak{L}^{\alpha+\beta}$.

(ii) That $\mathfrak{L}^{\alpha}$ is an algebra is the same as $\mathfrak{L}^{\alpha} \supset \mathfrak{L}^{2\alpha}$ and therefore $\mathfrak{L} \supset \mathfrak{L}^2$. From this we ohtain $\mathfrak{L}^{\beta} \supset \mathfrak{L}^{2\beta}$ for every $\beta > 0$. The proof is completed.

We note that if an $\mathfrak{L}^{\gamma}$ is composed of $\mathfrak{m}$--restrictedly measurable operators, then so are all the other $\mathfrak{L}^{\beta}$.

Lemma 1.9. *Let $\mathfrak{L}'$ stand for the linear space composed of the self-adjoint operators $\in$ an invariant linear system $\mathfrak{L}$. If $\mathfrak{L}'$ is a vector lattice by the ordering of operators, then $\mathfrak{L}$ is commutative.*

Proof. The lemma follows immediately from a result of Sherman [**27**] or of Kadison [**16**]. But it seems that the following direct proof has some interest. We have only to show that any two projections $E, F \in \mathfrak{L}$ are commutative. We show first that $E \cap F$ is a projection. Let $E \cap F = \int_0^1 \lambda dG_{\lambda}$ be the spectral resolution of $E \cap F$. It follows from $(E \cap F)^{\frac{1}{2}} \leq E$, F [**22**] that $(E \cup F)^{\frac{1}{2}} \leq E \cap F$, and therefore $\int_0^1 \lambda^{\frac{1}{2}} dG_{\lambda} \leq \int_0^1 \lambda dG_{\lambda}$. On the other hand $\int_0^1 \lambda^{\frac{1}{2}} dG_{\lambda} \geq \int_0^1 \lambda dG_{\lambda}$ since $\lambda^{\frac{1}{2}} \geq \lambda$ for $0 < \lambda \leq 1$. Hence $(E \cap F)^{\frac{1}{2}} = E \cap F$, that is, $E \cap F$ is a projection. As $\mathfrak{L}'$ is assumed to be a vector lattice, $E \cup F - F = E - E \cap F$. Since the right side of the equation and F are projections, so is $E \cup F$. Put $E' = E - E \cap F$ and $F' = F - E \cap F$, then $E' + F' = E \cup F - E \cap F = E' \cup F'$. This means that $E' + F'$ is a projection, and therefore $E'F' = 0$, that is, $(E - E \cap F)(F - E \cap F) = 0$. This yields $EF = E \cap F$. By symmetry we have $EF = FE$, as desired.

2. Integrals with respect to a canonical gage.

2.1. Segal [**26**] has developed a theory of non-commutative extension of integration for the measurable operators associated with a ring of operators on a Hilbert space. Theorem 1 shows that a $*$-isomorphism between two rings of operators has a unique $*$-isomorphic extension between measurable operators. Therefore in order to develop such a theory it does not matter how to choose any one of $*$-isomorphic rings. For his theory the singular part of a ring plays no essential rôle, and therefore we assume, otherwise stated, that rings are semi-finite [**15**]. A semi-finite ring is $*$-isomorphic with a left ring of a Hilbert system (H-system), which we shall take as a basic ring for our development of Segal's theory of integration for operators.

Let **A** be a unitary algebra [**14**]: **A** is a $*$-algebra and a pre-Hilbert space with the inner product $\langle a, b\rangle$ satisfying the following conditions:

(a) $\langle a, a\rangle = \langle a^*, a^*\rangle$ for every $a \in \mathbf{A}$;

(b) $\langle ab, c\rangle = \langle b, a^*c\rangle$ for every $a, b, c \in \mathbf{A}$;

(c) the mapping $b \to ab$ is continuous for every fixed $a \in \mathbf{A}$;

(d) $\mathbf{A}^2$ is dense in **A**.

Generally **A** is not a Hilbert space. If **A** is a Hilbert space, then **A** becomes an H^*-algebra of Ambrose [**1**], taking the norm multiplied by an appropriate positive number as its new norm. In this case we say that **A** is *essentially* an H^*-algebra.

The completion **H** of a unitary algebra **A** is equivalent to an H-system [**2**]. For any $x \in \mathbf{H}$, x^*, xa, ax are defined by continuity. Let L'_x denote the operator $a \to xa$ $(a \in \mathbf{A})$ and we define $L_x = (L'_{x^*})^*$. Likewise we define R_x. $L_x y$ is defined if and only if $R_y x$ is defined. Then $L_x y = R_y x$ will be denoted by xy. The left ring $\mathbb{L}$ of an H-system **H** is the ring of operators on **H** generated by $L_a (a \in \mathbf{A})$. Similarly the right ring $\mathbb{R}$ is defined. The operation $J : x \to x^*$ is a conjugation of **H** and $\mathbb{R} = J\mathbb{L}J$. $\mathbb{L}$ and $\mathbb{R}$ are commutants of each other [**10**], [**13**], [**14**], [**26**]. By making use of this fact R. Pallu de la Barrière [**24**] proved that $L_x^* = L_{x^*}$ and $R_x^* = R_{x^*}$. $x \in \mathbf{H}$ is called bounded if L_x (equivalently R_x) is bounded. The set **B** of bounded elements of **H** becomes a $*$-algebra called *bounded algebra of* **H**. We denote by $\mathbb{L}_\mathbf{B}$ the set $\{L_x ; x \in \mathbf{B}\}$. $\mathbb{L}_\mathbf{B}$ is an ideal of $\mathbb{L}$ and is dense in $\mathbb{L}$ in the strong topology. Any projection $P \in \mathbb{L}_\mathbf{B}$ is of the form L_e with a self-adjoint idempotent e. We write $x \geq 0$ if $L_x \geq 0$.

Lemma 2.1. *Let $\{e_\iota\}$ be a maximal orthogonal system of self-adjoint idempotents. Then*

(i) $\mathbf{H} = \sum_\iota \oplus e_\iota \mathbf{H} = \sum_\iota \oplus \mathbf{H} e_\iota$;

(ii) *Put $\phi(A) = \sum_\iota \langle Ae_\iota, e_\iota\rangle$ for $A \in \mathbb{L}^+$. Then $\phi(A)$ is a faithful, essential, normal, pseudo-trace of $\mathbb{L}$. The maximal ideal associated with ϕ is $\mathbb{L}^2_\mathbf{B}$.*

(iii) *$\phi(A)$ is independent of the particular choice of $\{e_\iota\}$.*

(iv) *If we put $\mu(P) = \phi(P)$ for $P \in \mathbb{L}_P$, then $\mu(P) = \|e\|^2$ or $+\infty$ acccording as $P = L_e$ for some e or not* (μ is a canonical gage of **H** in a certain sense of Segal [**26**]).

Proof. (i): Each $e_\iota \mathbf{H}$ is the range of projection $P_\iota = L_{e_\iota}$. $e_\iota \mathbf{H} \perp e_\kappa \mathbf{H}$ for $e_\iota \neq e_\kappa$.

If $I \neq \cup_\iota P_\iota$ we can find a non-zero projection $P=L_e$ such that $P \leq I - \cup_\iota P_\iota$, and therefore $\{e_\iota\}$ will not be maximal. This is a contradiction. Similarly we have $\mathbf{H}=\sum_\iota \oplus \mathbf{H}e_\iota$. We note that for any $x \in \mathbf{H}$, $x=\sum_\iota e_\iota x=\sum_\iota x e_\iota$ and $\|x\|^2=\sum_\iota \|e_\iota x\|^2 = \sum_\iota \|x e_\iota\|^2$.

(ii): That ϕ is linear, normal and positive is clear. Let $\phi(A)=0$ for some $A \geq 0$. Then $Ae_\iota=0$, and therefore $A(e_\iota x)=A(R_x e_\iota)=(Ae_\iota)x=0$. Owing to $x=\sum_\iota e_\iota x$, A must be 0, that is, ϕ is faithful. Let $A=L_{x^2}$ for some $x \in \mathbf{B}^+$. Then $\phi(A)= \sum_\iota \langle L_{x^2} e_\iota, e_\iota \rangle = \sum_\iota \langle x e_\iota, x e_\iota \rangle = \|x\|^2$, while for any $U \in \mathbb{L}_U$, $\phi(UAU^*)=\sum_\iota \langle UL_x U^* e_\iota, UL_x U^* e_\iota \rangle = \sum_\iota \langle (UU^J x) e_\iota, (UU^J x) e_\iota \rangle = \|UU^J x\|^2 = \|x\|^2$ where $U^J=JUJ$. Therefore $\phi(A)=\phi(UAU^*)$ for any $A \in \mathbb{L}_{\mathbf{B}}^{2+}$. Hence by normality of ϕ we have $\phi(A)=\phi(UAU^*)$ for any $A \in \mathbb{L}^+$. If $\phi(A) \neq 0$, then we can take an L_{x^2} such that $0 < x \in \mathbf{B}^+$ and $L_{x^2} \leq A$. Then $\phi(L_{x^2})=\|x\|^2$ is positive and finite. That is, ϕ is essential. The first part of (ii) is proved. To see the last part it suffices to show that if $\phi(A)$ is finite for $A \in \mathbb{L}^+$ then $A \in \mathbb{L}_{\mathbf{B}}^2$, Put $x_\iota = A^{\frac{1}{2}} e_\iota$. Then $x_\iota \in \mathbf{H}e_\iota$ and $\phi(A)=\sum_\iota \|x_\iota\|^2$, and therefore there exists an $x \in \mathbf{H}$ such that $x e_\iota = x_\iota$. $A^{\frac{1}{2}} e_\iota = x e_\iota$. It is easy to see that $A^{\frac{1}{2}}=L_x$ with $x \in \mathbf{B}^+$, that is $A \in \mathbb{L}_{\mathbf{B}}^2$.

(iii). For any choice of $\{e_\iota\}$, $\phi(A)=\|x\|^2$ for $A=L_{x^2} \in \mathbb{L}_{\mathbf{B}}^2$. Hence by normality, ϕ is unique.

(iv) $P \in \mathbb{L}_{\mathbf{B}}^2$ is equivalent to $P \in \mathbb{L}_{\mathbf{B}}$. Hence (iv) follows from the last part of (ii). The proof is completed.

Since $\mathbb{L}$ has a faithful, essential, normal pseudo-trace ϕ, $\mathbb{L}$ is semi-finite, and is known [7] that ϕ is uniquely determined by μ.

Definition 2.1. ϕ in Lemma 2.1 is called the *canonical pseudo-trace* of $\mathbf{H}$.

To make clear the independence of $\phi(A)$ of the particular choice of $\{e_\iota\}$, we give another expression of $\phi(A)$.

Lemma 2.2. *Let* $A=\int_0^\infty \lambda dE_\lambda$ *be the spectral resolution of* $A \in \mathbb{L}^+$. *Then* $\phi(A)= \int_0^\infty \mu(E_\lambda^\perp) d\lambda = -\int_0^\infty \lambda d\mu(E_\lambda^\perp)$.

Proof. Let l be the bound of A. Let $\{\lambda_i\}$ be $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n+1} = l$. $A \geq \sum_{i=1}^n \lambda_i (E_{\lambda_{i+1}} - E_{\lambda_i})$. The set $\{\sum \lambda_i (E_{\lambda_{i+1}} - E_{\lambda_i})\}$ is a directed set converging uniformly to A. Hence by normality of ϕ, $\phi(A)= \lim \sum \lambda_i \phi(E_{\lambda_{i+1}} - E_{\lambda_i}) = \lim \sum \lambda_i \mu(E_{\lambda_{i+1}} - E_{\lambda_i}) = -\int_0^\infty \lambda d\mu(E_\lambda^\perp) = \int_0^\infty \mu(E_\lambda^\perp) d\lambda$.

Corollary. *Let* $A=\int_0^\infty \lambda dE_\lambda$ *be the spectral resolution of* $A \in \mathbb{L}^+$. *In order that*

$A\in \mathbf{L_B}$ *it is necessary and sufficient that* $\int_0^\infty \mu(E_{\sqrt{\lambda}}^{\perp})d\lambda=-\int_0^\infty \lambda^2 d\mu(E_\lambda^{\perp})<\infty$. *In this case the integral equals* $\|x\|^2$, *where* $A=L_x$, $x\in \mathbf{B}^+$.

Proof. $A\in \mathbf{L}_{\mathbf{B}}^{+}$ is equivalent to $A^2\in \mathbf{L}_{\mathbf{B}}^{2}$, and therefore to that $\phi(A^2)<\infty$. Hence the statement of the lemma is true by the preceding two lemmas.

Here we note that $x\in \mathbf{B}^+$ is approximated by $\sum_{i=1}^n \lambda_i e_i$ as nearly as we want, where $\lambda_i\geq 0$ and e_i are orthogonal self-adjoint idempotents. Let $A=L_x=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of L_x. Since $\lim_{\lambda\downarrow 0}\|E_\lambda{}^{\perp}x-x\|=0$, we may assume $E_\delta{}^{\perp}x=x$ for some $\delta>0$. Let $C_\lambda=\int_\lambda^\infty \frac{1}{\lambda}dE_\lambda$ for $\lambda>0$ then $E_\lambda{}^{\perp}=C_\lambda L_x=L_{C_\lambda x}$, therefore $C_\lambda x$ is a self-adjoint idempotent e_λ. Using the notation of the proof of Lemma 2.2 and letting $\lambda_1=\delta$, $\{\sum_{i=1}^n \lambda_i(E_{\lambda_{i+1}}-E_{\lambda_i})e_\delta\}$ converges to Ae_δ. $L_{Ae_\delta}=AL_{e_\delta}=AE_\delta{}^{\perp}=A=L_x$. $L_{(E_{\lambda_{i+1}}-E_{\lambda_i})e_\delta}=E_{\lambda_{i+1}}-E_{\lambda_i}=L_{e_{\lambda_i}-e_{\lambda_{i+1}}}$. Therefore x is approximated by $\sum_{i=1}^n \lambda_i(e_{\lambda_i}-e_{\lambda_{i+1}})$ as near as we want.

2.2. A projection P is called metrically finite [**26**] if $\mu(P)<+\infty$. Such a projection is evidently countably decomposable and the ideal $\mathfrak{m}$ generated by all metrically finite projections is of type $\mathfrak{m}_2$ of **1.2.** We shall use the terms "*μ–restrictedly measurable*" and "*μ–nearly everywhere*" according to the cases. It is wellknown that $\mathfrak{m}$ is dense in $\mathbf{L}$ in the strong topology ($\mathfrak{m}$ is the restricted ideal of the maximal ideal associated with ϕ [7]). Let $\mathfrak{L}_2$ be the set $\{L_x;\ x\in \mathbf{H}\}$. Let us introduce an inner product $\langle L_x, L_y\rangle=\langle x, y\rangle$, then $\mathfrak{L}_2$ is a Hilbert space isometric with $\mathbf{H}$. The element of $\mathfrak{L}_2$, is called *square integrable* with respect to μ.

Lemma 2.3. (i) *L_x is μ–restrictedly measurable.*

(ii) *$\mathfrak{L}_2$ is an invariant linear system of* $\mathbf{L}$.

(iii) $\mathfrak{L}_2\cap\mathbf{L}=\mathbf{L_B}$. *Therefore a projection $P\in\mathbf{L}$ is metrically finite if and only if $P\in\mathfrak{L}_2$.*

Proof. (i): Let $L_x=W|L_x|$ be the polar decomposition of L_x. $|L_x|=W^*L_x=L_{W^*x}$ [**24**]. By Lemma 1.1 we have only to show that L_x, $x\geq 0$ is μ–restrictedly measurable. Let $L_x=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of L_x. Put $A=\int_\delta^\infty \frac{1}{\lambda}dE_\lambda$, $\delta>0$. Then $A\in\mathbf{L}$ and $AL_x\subset L_{Ax}$ and therefore $L_{(Ax)^*}\supset L_xA=E_\delta{}^{\perp}$, that is $L_{(Ax)^*}=E_\delta{}^{\perp}\in\mathbf{L}$. This implies $Ax\in\mathbf{L_B}$. By Lemma 2.1, $E_\delta{}^{\perp}$ is metrically finite. Lemma 1.1 shows that L_x is μ–restrictedly measurable.

(ii): That $\mathfrak{L}_2$ is linear is evident. Let $U\in\mathbf{L}_U$. $U\cdot L_x=UL_x\subset L_{Ux}$. Since UL_x and L_{Ux} are measurable, we obtain $U\cdot L_x=L_{Ux}\in\mathfrak{L}_2$. $L_xU=(U^*\cdot L_{x^*})^*\in\mathfrak{L}_2{}^*=\mathfrak{L}_2$.

(iii) follows from the definition of $\mathbb{L}_{\mathbf{B}}$.

Now we show that $\mathfrak{L}_2$ satisfies the conditions $(\ll)_1$ and $(\ll)_2$ stated in **1.4.** To this end the following lemma is needed.

LEMMA 2.4. *Let $T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of a positive measurable operator $T\eta\mathbb{L}$. $T\in\mathfrak{L}_2$ if and only if $-\int_0^\infty \lambda^2 d\mu(E_\lambda{}^\perp)=\int_0^\infty \mu(E_{\sqrt{\lambda}}^\perp)d\lambda$ is finite. In this case $\langle T, T\rangle=\int_0^\infty \mu(E_{\sqrt{\lambda}}^\perp)d_\lambda$.*

PROOF. First assume that $T\in\mathfrak{L}_2^+$, that is, $T=L_x$ with some $x\in\mathbf{H}^+$. $L_{E_\lambda x}=E_\lambda T=\int_0^\lambda \lambda dE_\lambda$ being bounded, $E_\lambda x\in\mathbf{B}^+$ and $\|E_\lambda x\|^2=-\int_0^\lambda \lambda^2 d\mu(E_\lambda{}^\perp)$ by Cor. of Lemma 2.2. $E_\lambda x\to x$ as $\lambda\uparrow\infty$. Hence $\|x\|^2=-\int_0^\infty \lambda^2 d\mu(E_\lambda^\perp)=\int_0^\infty \mu(E_{\sqrt{\lambda}}^\perp)d\lambda<+\infty$. Now we shall show the converse. $E_\lambda T$ is bounded and $-\int_0^\lambda \lambda^2 d\mu(E_\lambda{}^\perp)<+\infty$. Then by the same Cor., we can write $E_\lambda T=Lx_\lambda$, where $x_\lambda\in\mathbf{B}^+$ and $\|x_\lambda\|^2=-\int_0^\lambda \lambda^2 d\mu(E_\lambda{}^\perp)$. For $\lambda'>\lambda>0$, $\|x_{\lambda'}-x_\lambda\|^2=-\int_\lambda^{\lambda'} \lambda^2 d\mu(E_\lambda{}^\perp)$ since $Lx_{\lambda'}-x_\lambda=Lx_{\lambda'}-Lx_\lambda=(E_{\lambda'}-E_\lambda)T=\int_\lambda^{\lambda'} \lambda dE_\lambda\in\mathbb{L}_{\mathbf{B}}$. $-\int_0^\infty \lambda^2 d\mu(E_\lambda{}^\perp)<+\infty$ implies that $-\int_\lambda^{\lambda'} \lambda^2 d\mu(E_\lambda{}^\perp)\to 0$ as $\lambda'>\lambda\to\infty$, and therefore there exists $x\in\mathbf{H}$ such that $x_\lambda\to x$ as $\lambda\to\infty$. That $x_\lambda=E_\lambda x_{\lambda'}$ for $\lambda'>\lambda$ implies $x_\lambda=E_\lambda x$. Then $E_\lambda T=Lx_\lambda=L_{E_\lambda x}=E_\lambda\cdot Lx$ for every $\lambda>0$. This implies $T=L_x$. The proof is completed.

Let $T=\int_0^\infty \lambda dE_\lambda\in\mathfrak{L}_2{}^+$ and $S=\int_0^\infty \lambda dF_\lambda$ be a positive measurable operator. If $F_\lambda{}^\perp\prec E_\lambda{}^\perp$ for every $\lambda>0$. Then $\mu(F_\lambda{}^\perp)\leq\mu(E_\lambda{}^\perp)$ and therefore $\int_0^\infty \mu(F_{\sqrt{\lambda}}^\perp)d\lambda<+\infty$, which implies $S\in\mathfrak{L}_2{}^+$. Hence $\mathfrak{L}_2$ satisfies $(\ll)_1$. Next assume that $S\in\mathfrak{L}_2{}^+$. Let $G_\lambda=E_\lambda\cap F_\lambda$, then $\mu(G_\lambda^\perp)=\mu(E_\lambda^\perp\cup F_\lambda^\perp)\leq\mu(E_\lambda^\perp)+\mu(F_\lambda^\perp)$ and therefore $\int_0^\infty \mu(G_{\sqrt{\lambda}}^\perp)d\lambda<+\infty$. This means that $T\vee S\in\mathfrak{L}_2{}^+$. $\mathfrak{L}_2$ satisfies $(\ll)_2$.

We define $\mathfrak{L}_\alpha=\mathfrak{L}_2^{2\alpha}$ for $\alpha>0$. $\mathfrak{L}_\alpha$ also satisfies the condition $(\ll)_1$, and $(\ll)_2$. Each $T\in\mathfrak{L}_1$ is expressed by $L_x\cdot L_y$ or more generally by $\sum_{i=1}^m L_{x_i}\cdot L_{y_i}$.

LEMMA 2.5. *$\sum_{i=1}^m L_{x_i}\cdot L_{y_i}=0$ implies $\sum_{i=1}^m\langle y_i, x_i^*\rangle=0$.*

PROOF. Let $\mathfrak{D}$ be the intersection of the domains of $L_{x_i}\cdot L_{y_i}$ and $L_{x_i}(i=1,2,\cdots,m)$. $\mathfrak{D}$ is strongly μ-dense, and therefore there exists a sequence of projections $P_n\in\mathbb{L}$ such that $P_n\mathbf{H}\subset\mathfrak{D}$, $P_n{}^\perp$ is metrically finite and $P_n{}^\perp\downarrow 0$. Since P_n is a least upper bound of metrically finite projections, we can take a maximal orthogonal system

$\{e_\iota\}$ of self-adjoint idempotents such that $e_\iota \mathbf{H} \subset \mathfrak{D}$. $\sum_{i=1}^m \langle y_i, x_i^* \rangle = \sum_{i,\iota} \langle y_i e_\iota, x_i^* e_\iota \rangle = \sum_{i,\iota} \langle L_{x_i} \cdot L_{y_i} e_\iota, e_\iota \rangle = \sum_\iota \langle \sum_i L_{x_i} \cdot L_{y_i} e_\iota, e_\iota \rangle = 0$. The proof is completed.

Definition 2.2. $\mu(T) = \sum_{i=1}^m \langle y_i, x_i^* \rangle$ for $T = \sum_{i=1}^m L_{x_i} \cdot L_{y_i}$ is called the *integral* of T.

Lemma 2.5 shows that $\mu(T)$ is independent of the particular expression of T. If T is a projection $P \in \mathbb{L}$, then $\mu(P)$ coincides with the old one. And if $T \in \mathbb{L}_{\mathbf{B}}^{2+}$, then $\mu(T) = \phi(T)$. It follows from Lemma 2.4 that a positive measurable operator $T = \int_0^\infty \lambda dE_\lambda$ is an element of $\mathfrak{L}_1$ if and only if $T^{\frac{1}{2}} \in \mathfrak{L}_2$, that is, $\int_0^\infty \mu(E_\lambda^\perp) d\lambda < +\infty$. In this case $\mu(T) = \int_0^\infty \mu(E_\lambda^\perp) d\lambda = -\int_0^\infty \lambda d\mu(E_\lambda^\perp)$.

We remark that $\mathfrak{L}_2$ is an H-system isomorphic with $\mathbf{H}$ by the mapping $x \to L_x$. This follows from the facts that (1) if xy is defined and equals z, then $L_x \cdot L_y = L_z$, and (2) if $L_x \cdot L_y$ equals L_z, then xy is defined and equals z. To prove (1) let $\mathfrak{D}$ be the intersection of domains $\mathfrak{D}_{L_z}$ and $\mathfrak{D}_{L_x \cdot L_y}$. $\mathfrak{D}$ is strongly μ-dense. For any $u \in \mathfrak{D}$, we have $L_x \cdot L_y u = x(yu)$ and $L_z u = (xy)u$. It follows from a result of Ambrose [2] that $x(yu) = (xy)u$. Since measurable operators $L_x \cdot L_y$ and L_z agree on a strongly μ-dense domain $\mathfrak{D}$, we must have $L_z = L_x \cdot L_y$. Now we show (2). Let a be any element of A. $\langle z, a \rangle = \langle a^*, z^* \rangle = \langle L_{a^*}, (L_x \cdot L_y)^* \rangle = \mu(L_{a^*} \cdot L_x \cdot L_y) = \mu(L_{a^*x} \cdot L_y) = \langle a^*x, y^* \rangle = \langle x, ay^* \rangle$. Hence xy is defined and equals z. Ambrose [2] defined $\mathbf{H}$ to be *commutative* if so is its bounded algebra. It is easy to see that this definition is equivalent to say that $\mathfrak{L}_2$ is commutative.

Lemma 2.6. *The integral μ has the following properties;*

(i) *μ is linear.*

(ii) $\mu(T^*) = \overline{\mu(T)}$.

(iii) *$\mu(T) \geq 0$ for $T \geq 0$. The equality holds if and only if $T = 0$.*

(iv) *For every $A \in \mathbb{L}$, $\mu(A \cdot L_x \cdot L_y) = \mu(L_y \cdot A \cdot L_x) = \mu(L_x \cdot L_y \cdot A) = \langle Ax, y^* \rangle$. In particular $\mu(A \cdot T) = \mu(TA)$.*

(v) $\text{l. u. b.}_{\|A\| \leq 1} |\mu(A \cdot T)| = \mu(|T|)$.

(vi) *For a fixed T, $\mu(A \cdot T) \geq 0$ for every $A \in \mathbb{L}^+$ if and only if $T \geq 0$.*

(vii) *$L_x, L_y \geq 0$ imply $\mu(L_x \cdot L_y) \geq 0$.*

Proof. (i)–(iii) are evident.

(iv): $\mu(A \cdot L_x \cdot L_y) = \mu(L_{Ax} \cdot L_y) = \langle y, (Ax)^* \rangle = \langle Ax, y^* \rangle = \mu(L_y \cdot A \cdot L_x) = \mu((L_y \cdot A) \cdot L_x) = \mu(L_x \cdot L_y \cdot A)$. Since any T is of the form $L_x \cdot L_y$, we have $\mu(A \cdot T) = \mu(TA)$.

(v): Let $T = W|T|$ be the polar decomposition of T. $\mu(W^* T) = \mu(|T|) = \phi(|T|)$. On the other hand, let $|T| = L_x^2$, $x \in \mathbf{H}^+$, then $|\mu(A \cdot T)| = |\mu(A \cdot W \cdot L_x \cdot L_x)| =$

$|\langle AWx, x\rangle| \leq \|AWx\| \ \|x\| \leq \|x\|^2 = \phi(|T|)$.

(vi) : $\mu(A\cdot T)\geq 0$ for every $A\geq 0$ if $T\geq 0$, since $\mu(A\cdot T)=\mu(A^{\frac{1}{2}}\cdot T\cdot A^{\frac{1}{2}})$ and $A^{\frac{1}{2}}\cdot T\cdot A^{\frac{1}{2}}\geq 0$. Conversely assume that $\mu(A\cdot T)\geqq 0$ for every $A\geq 0$. $\mu(A\cdot T)=\mu(T^*\cdot A)=\mu(A\cdot T^*)$. Since any element of $\mathbb{L}$ is a linear combination of positive ones, it follows from (v) that $T=T^*$. Let $T=\int_{-\infty}^{\infty}\lambda dE_\lambda$ be the spectral resolution of T. $\mu(\int_{-\infty}^{0}\lambda dE_\lambda)=\mu(E_0 T)\geq 0$, while $\int_{-\infty}^{0}\lambda dE_\lambda\leq 0$. Therefore from (iii) we have $\int_{-\infty}^{0}\lambda dE_\lambda=0$ that is, $T\geq 0$.

(vii) $\mu(L_x\cdot L_y)=\langle x, y\rangle$. Let $L_y=\int_{0}^{\infty}\lambda dE_\lambda$ be the spectral resolution of L_y. $(1-E_0)y=y$ and $(E_n-E_{\frac{1}{n}})y\to y$ as $n\to\infty$. $(E_n-E_{\frac{1}{n}})y$ is approximated by an expression $\sum_{i=1}^{m}\lambda_i e_i$, $\lambda_i\geq 0$, as near as we want, where e_i is a self-adjoint idempotent. $\langle x, \sum\lambda_i e_i\rangle=\sum\lambda_i\langle x, e_i\rangle=\sum\lambda_i\langle xe_i, e_i\rangle\geq 0$. Hence $\langle x, y\rangle\geq 0$. This completes the proof.

Lemma 2.7. *If $T\geq S\geq 0$ for $T, S\in\mathfrak{L}_1$, then $T^{\frac{1}{2}}\geq S^{\frac{1}{2}}$.*

Proof. Suppose the contrary. Let $S^{\frac{1}{2}}-T^{\frac{1}{2}}=\int_{-\infty}^{+\infty}\lambda dE_\lambda$ be the spectral resolution of $S^{\frac{1}{2}}-T^{\frac{1}{2}}$. Then for some $\lambda>\delta>0$, $(E_\lambda-E_\delta)(S^{\frac{1}{2}}-T^{\frac{1}{2}})\geq\delta(E_\lambda-E_\delta)>0$. Put $E=E_\lambda-E_\delta$, then $E=L_e$ for some $e\in\mathbf{B}$. Then by (vii) of the above lemma, $\mu(E(S^{\frac{1}{2}}-T^{\frac{1}{2}})\cdot(S^{\frac{1}{2}}+T^{\frac{1}{2}})\geq\delta\mu(L_e\cdot(S^{\frac{1}{2}}+T^{\frac{1}{2}}))\geq 0$, while on the other hand $\mu(E(S^{\frac{1}{2}}-T^{\frac{1}{2}})\cdot(S^{\frac{1}{2}}+T^{\frac{1}{2}}))=\mu(E\cdot(S-T))+\mu(E\cdot S^{\frac{1}{2}}\cdot T^{\frac{1}{2}})-\mu(E\cdot T^{\frac{1}{2}}\cdot S^{\frac{1}{2}})=\mu(E\cdot(S-T))+\mu(T^{\frac{1}{2}}\cdot E\cdot S^{\frac{1}{2}})-\mu(S^{\frac{1}{2}}\cdot E\cdot T^{\frac{1}{2}})$ and $\mu(T^{\frac{1}{2}}\cdot E\cdot S^{\frac{1}{2}})$ is conjugate to $\mu(S^{\frac{1}{2}}\cdot E\cdot T^{\frac{1}{2}})$, and therefore $\mu(E(S^{\frac{1}{2}}-T^{\frac{1}{2}})\cdot(S^{\frac{1}{2}}+T^{\frac{1}{2}}))=\mu(E\cdot(S-T))\leq 0$. Hence $\mu(L_e\cdot(S^{\frac{1}{2}}+T^{\frac{1}{2}}))=0$, that is, $E\cdot(S^{\frac{1}{2}}+T^{\frac{1}{2}})E=0$. This implies that $E\cdot S^{\frac{1}{2}}E=-E\cdot T^{\frac{1}{2}}E$ and therefore $E\cdot S^{\frac{1}{2}}E=-E\cdot T^{\frac{1}{2}}E=0$. Using these equalities, we have $E\cdot(S^{\frac{1}{2}}-T^{\frac{1}{2}})=E\cdot(S^{\frac{1}{2}}-T^{\frac{1}{2}})E=E\cdot S^{\frac{1}{2}}E-\cdot ET^{\frac{1}{2}}E=0$, and therefore $E=0$, since $E\cdot(S^{\frac{1}{2}}-T^{\frac{1}{2}})\geq\delta E$. This is a contradiction. The proof is completed.

Added in proof. This lemma is a special case of a theorem due to E. Heinz, Math. Ann. 123 (1951), p. 425, Satz 2. Cf. also [**22**].

2.3. Now we are ready to show that in the left ring $\mathbb{L}$ of a Hilbert system, (a) ultraweak and weak topologies, and (b) ultrastrong and strong topologies coincide respectively. The following theorem is wellknown [**15**]. But it would seem that much interest lies in the method of proof given here.

THEOREM 3. *Let* $\mathbb{L}$ *be the left ring of a Hilbert system. Then*

(a) *ultraweak and weak topologies of* $\mathbb{L}$ *coincide;*

(b) *ultrastrong and strong topologies of* $\mathbb{L}$ *coincide.*

Let $\Phi(A)$ *be any linear functional continuous in the weak (equivalently strong, ultraweak or ultrastrong) topology, then* $\Phi(A)$ *is of the form* $\langle Ax, y\rangle$ *and if moreover* Φ *is positive we can write* $\Phi(A)=\langle Ax, x\rangle$.

PROOF. Let $\{x_i\}$ be any sequence of elements of $\mathbf{H}$ such that $\sum\|x_i\|^2<\infty$. We can write $\sum_{i=1}^n L_{x_i}\cdot L_{x_i}^* = L_{y_n}^2$ for some $y_n\in\mathbf{H}^+$. $\|y_n\|^2=\sum_{i=1}^n\|x_i\|^2$. For $m>n$, $L_{y_m}^2\geq L_{y_n}^2$ and therefore $L_{y_m}\geq L_{y_n}$ by Lemma 2.7. Therefore $\|y_m-y_n\|^2=\|y_m\|^2+\|y_n\|^2-\langle y_m, y_n\rangle-\langle y_n,y_m\rangle\leq\|y_m\|^2+\|y_n\|^2-\|y_n\|^2-\|y_n\|^2=\|y_m\|^2-\|y_n\|^2$ since $\langle y_m,y_n\rangle-\|y_n\|^2=\langle y_m-y_n, y_n\rangle\geq 0$. Hence $\|y_m-y_n\|^2\leq\sum_{i=n+1}^m\|x_i\|^2\to 0$ as $m>n\to\infty$, that is, $\{y_n\}$ converges to an element $x\in\mathbf{H}$. $\sum_{i=1}^n\langle Ax_i, x_i\rangle=\sum_{i=1}^n\mu(A\cdot L_{x_i}\cdot L_{x_i}^*)=\mu(A\cdot L_{y_n}^2)=\langle Ay_n, y_n\rangle$. Therefore $\sum_{n=1}^\infty\langle Ax_n, x_n\rangle=\langle Ax, x\rangle$. From this equation we see that (a) and (b) hold.

If $\Phi(A)$ is continuous in the weak topology, we can write $\Phi(A)=\sum_{i=1}^n\langle Ax_i, y_i\rangle$ for some x_i, y_i $(i=1, 2, \cdots, n)$. Let $T=L_x\cdot L_{y^*}=\sum_{i=1}^n L_{x_i}\cdot L_{y_i}{}^*$. Then $\Phi(A)=\langle Ax, y\rangle=\mu(A\cdot T)$. If $\Phi(A)$ is positive for every $A\geq 0$, then $T\geq 0$ and therefore we can write $T=L_x^2$ for some $x\in\mathbf{H}^+$, that is $\Phi(A)=\langle Ax, x\rangle$. This completes the proof.

We shall consider some consequences of this theorem.

For every $T\in\mathfrak{L}_1$ we define $\|T\|_1=\mu(|T|)$. Then $\|\ \|_1$ has the norm property by Lemma 2.6 (v) since $|\mu(AT)|$ is a pseudo-norm and $\mu(|T|)=0$ implies $T=0$. We show that $\mathfrak{L}_1$ is complete with respect to this norm, that is, $\mathfrak{L}_1$ is a Banach space.

COROLLARY 3.1. *Let* Φ *be any linear functional on* $\mathbb{L}$ *continuous in the ultraweak topology (=normal),* [**9**], *then there exists a* $T\in\mathfrak{L}_1$ *such that* $\Phi(A)=\mu(A\cdot T)$ *for every* $A\in\mathbb{L}$. *And*

(i) T *is uniquely determined by* Φ.

(ii) Φ *is positive linear if and only if* $T\geq 0$.

(iii) Φ *is central if and only if* $T\in\mathbb{L}^\natural$.

PROOF. By Theorem 3 we can write $\Phi(A)=\langle Ax, y\rangle=\mu(A\cdot L_x\cdot L_{y^*})=\mu(A\cdot T)$ for every $A\in\mathbb{L}$, where $x, y\in\mathbf{H}$ and $T=L_x\cdot L_{y^*}\in\mathfrak{L}_1$. (i) follows from Lemma 2.6 (v). (ii) follows from (iii) of the same lemma. Φ is called central if $\Phi(AB)=\Phi(BA)$ for every $A, B\in\mathbb{L}$. This condition is written as $\mu(A\cdot(B\cdot T-TB))=0$ for every

$A, B \in \mathbb{L}$. By Lemma 2.6 (vi) this condition is equivalent to that $B \cdot T = TB$ for every $B \in \mathbb{L}$, that is, $T \in \mathbb{L}^{\natural}$.

Dixmier [9] proved the following theorem: "Let $\mathbf{M}$ be a ring of operators and denote by $\mathbf{M}_*$ the Banach space of all normal linear functionals Φ. If we identify A with the continuous functional $\langle A, \Phi \rangle = \langle \Phi, A \rangle = \Phi(A)$ then $\mathbf{M}$ is the conjugate space of $\mathbf{M}_*$". Therefore (i) of the above Corollary shows that $\mathfrak{L}_1$ is complete with respect to $\| \ \|_1$, that is, $\mathfrak{L}_1$ is a Banach space. Theorem of Dixmier just stated shows that $\|A\| = \underset{\|T\|_1 \leq 1}{\text{l. u. b.}} |\Phi(A \cdot T)|$ and $\mathbb{L}$ is the conjugate space of $\mathfrak{L}_1$. By this reason we write $\mathbb{L} = \mathfrak{L}_\infty$.

COROLLARY 3.2. *Let $\{T_n\}$ be a monotone increasing sequence of posittive operators $\in \mathfrak{L}_1$. There exists a $T \in \mathfrak{L}_1^+$ such that $T_n \leq T$ and $\lim_{n\to\infty} \mu(T_n) = \mu(T)$ if and only if $\lim_{n\to\infty} \mu(T_n) < +\infty$. In this case T is the* l. u. b. *of $\{T_n\}$ and is the μ-n. e. star convergence limit of $\{T_n\}$.*

PROOF. If $\lim \mu(T_n) = +\infty$, there exists no T stated above. We assume that $\lim \mu(T_n) < +\infty$. Let $T_n = L^2_{y_n}$, $y_n \in \mathbf{H}^+$. We define $L^2_{x_n} = L^2_{y_n} - L^2_{y_{n-1}}$ where $y_0 = 0$. Then $T_n = L^2_{y_n} = \sum_{i=1}^n L^2_{x_i}$ and, $\mu(T_n) = \|y_n\|^2 = \sum_{i=1}^n \|x_i\|^2$. From the proof of Theorem 3, $\{y_n\}$ converges to some $y \in \mathbf{H}$ and $\lim_{n\to\infty} \mu(A \cdot T_n) = \langle Ay, y \rangle = \mu(A \cdot T)$ where $T = L^2_y$. For every $A \in \mathbb{L}_+$, $\mu(A \cdot T_n) \leq \mu(A \cdot T)$. This implies, by Lemma 2.6 (vi), $T_n \leq T$. If we let $A = I$, we have $\lim_{n\to\infty} \mu(T_n) = \mu(T)$. Such T is unique. For otherwise, let T' be such that $T_n \leq T'$ and $\lim_{n\to\infty} \mu(T_n) = \mu(T')$. Then $\|T' - T_n\|_1 = \mu(T' - T_n) \to 0$. Thus T' is the limit of $\{T_n\}$ with respect to $\| \ \|_1$. Thus T is unique. That T is the l. u. b. of $\{T_n\}$ is clear from the discussion just given above. To show that $\{T_n\}$ star converges μ-n. e. to T, it suffices to show that there exists a sequence of integers n_j such that $\{T_{n_j}\}$ converges μ-n. e. to T. To this end it suffices to show under the conditions $\|T - T_n\|_1 \geq \frac{1}{4^n}$ that $\{T_n\}$ converges μ-n.e. to T. Let $T - T_n = \int_0^\infty \lambda dE^{(n)}_\lambda$ be the spectral resolution of $T - T_n$. $\frac{1}{2^{n+1}} \mu(E^{(n)\perp}_{\frac{1}{2^n}}) \leq \int_{2^{-(n+1)}}^{2^{-n}} \mu(E^{(n)\perp}_\lambda) d\lambda \leq \int_0^\infty \mu(E^{(n)\perp}_\lambda) d\lambda = \|T - T_n\|_1 \leq \frac{1}{4^n}$. Therefore $\mu(E^{(n)\perp}_{\frac{1}{2^n}}) \leq \frac{1}{2^{n-1}}$. Put $P_n = \cap_{k=n}^\infty E^{(k)}_{\frac{1}{2^k}}$. Then $P_n^\perp = \cup_{k=n}^\infty E^{(k)\perp}_{\frac{1}{2^k}}$ and $\mu(P_n^\perp) \leq \sum_{k=n}^\infty \frac{1}{2^{k-1}} = \frac{1}{2^{n-2}}$. Therefore $P_n^\perp \downarrow 0$ and $P_n^\perp \in \mathfrak{m}_2$. $\|(T - T_n)P_n\| \leq \frac{1}{2^n}$ $(n = 1, 2, 3, \cdots)$. Thus $\{T_n\}$ converges μ-n. e. to T.

The method of proof used in this lemma is applied to show that if a sequence $\{T_n\}$ of elements of $\mathfrak{L}_1$ converges to T with respect to the norm $\| \ \|_1$, then $\{T_n\}$

converges μ–n. e. in the star sense to T. The details are omitted.

COROLLARY 3.3. *Let $0 \leq T_1 \leq T_2 \leq \cdots$ be a sequence of elements of $\mathfrak{L}_2$ such that $\{\|T_n\|\}$ is bounded. Then there exists the* l. u. b. *T of $\{T_n\}$, $\|T_n - T\|_2 \to 0$ as $n \to \infty$, and T is the μ–n. e. star convergence limit of $\{T_n\}$.*

PROOF. $\{T_n\}$ is a Cauchy sequence in $\mathfrak{L}_2$. In fact, for $m > n$, $\|T_m - T_n\|_2^2 = \|T_m\|_2^2 - \langle T_m, T_n\rangle - \langle T_n, T_m\rangle + \|T_n\|_2 \leq \|T_m\|_2^2 - \|T_n\|_2^2$ since $\langle T_n, T_m\rangle = \langle T_m, T_n\rangle \geq \|T_n\|_2^2$ by Lemma 2.6. $\lim_{n\to\infty} \|T_n\|_2$ exists and is finite. This implies that $\lim_{n,m\to\infty} \|T_n - T_m\|_2 = 0$. Let $T = \lim_{n\to\infty} T_n$. For any $S \in \mathfrak{L}_2^+$, $\langle S, T - T_n\rangle = \lim_{m\to\infty} \langle S, T_m - T_n\rangle \geq 0$. Hence $T \geq T_n$ $(n = 1, 2, 3, \cdots)$. Let T_0 be any measurable operator such that $T \geq T_0 \geq T_n$ $(n = 1, 2, 3, \cdots)$. Then $T_0 \in \mathfrak{L}_2$ and $\|T_0 - T_n\|_2^2 \leq \|T - T_n\|_2^2$ by Lemma 2.6. Therefore we have $T_0 = T$. The last part of the statement of this corollary follows by the same reasoning as in Cor. 3.2, and details are omitted. The proof is completed.

Let T be any positive measurable operator and let $T = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T. Define $\mu'(T) = \text{l. u. b.}_{S \in \mathfrak{L}_1^+,\, S \leq T}\, \mu(S) = \text{l. u. b.}_{A \in \mathfrak{m}_2^+,\, A \leq T}\, \mu(A)$. $\mu'(T)$ is finite if and only if $T \in \mathfrak{L}_1^+$; and then we have $\mu'(T) = \mu(T)$. The "if" part is evident. Now we show the "only if" part. $E_\lambda T = \int_0^\lambda \lambda dE_\lambda \in \mathbb{L}$. $\mu'(E_\lambda T) \leq \mu'(T) < +\infty$. Then $\mu'(E_\lambda T) = \phi(E_\lambda T) < +\infty$ and therefore $E_\lambda T \in \mathbb{L}_{\mathbf{B}}^2$ by Lemma 2.1. Then $\{E_n T\}$ is a monotone increasing sequence of elements of $\mathfrak{L}_1$ and $\lim_{n\to\infty} \mu(E_\lambda T) \leq \mu(T) < +\infty$. $\{E_n T\}$ converges μ–n. e. to T. Then by Cor. 3.2., $T \in \mathfrak{L}_1$. Thus $\mathfrak{L}_1^+$ consists of measurable operators T such that $\mu'(T) < +\infty$.

Let Φ be any linear functional of $\mathbb{L}$ continuous in the ultraweak topology. Φ is uniquely expressed as $\Phi = \Phi_1 + i\Phi_2$, where Φ_1 and Φ_2 are of real type. With regard to Φ of real type we have the following

LEMMA 2.8. *Let Φ be a linear functional on $\mathbb{L}$ of real type continuous in the ultraweak topology. Φ can be expressed uniquely as a difference of two functionals of positive type Φ_+ and Φ_- such that $\Phi = \Phi_+ - \Phi_-$, $\|\Phi\| = \|\Phi_+\| + \|\Phi_-\|$.*

PROOF. $\Phi(A) = \mu(A \cdot T)$, where T is self-adjoint. Let $T_+ = \int_0^\infty \lambda dE_\lambda$, $T_- = -\int_{-\infty}^0 \lambda dE_\lambda$, where $T = \int_{-\infty}^\infty \lambda dE_\lambda$ is the spectral resolution of T. Put $\Phi_+(A) = \mu(A \cdot T_+)$, $\Phi_-(A) = \mu(A \cdot T_-)$. Then $\Phi = \Phi_+ - \Phi_-$ and $\|\Phi\| = \mu(|T|) = \mu(T_+) + \mu(T_-) = \|\Phi_+\| + \|\Phi_-\|$. Next we show the uniqueness of such representation. Let $\Phi = \Psi_1 - \Psi_2$, where Ψ_1 and Ψ_2 are of positive type. $\Psi_1(A) = \mu(A \cdot S_1)$, $\Psi_2(A) = \mu(A \cdot S_2)$ where $S_1, S_2 \in \mathfrak{L}_1^+$. Let

$S_1=\int_0^\infty \lambda dF_\lambda$ and $S_2=\int_0^\infty \lambda dG_\lambda$ be the spectral resolutions of S_1 and S_2 respectively. $\Phi(E_0{}^\perp)=\mu(T_+)=\mu(E_0{}^\perp\cdot S_1)-\mu(E_0{}^\perp\cdot S_2)\leq\mu(E_0{}^\perp\cdot S_1)\leq\mu(S_1)=\|\Psi_1\|$, and therefore $\|\Phi_+\|\leq\|\Psi_1\|$ similarly $\|\Phi_-\|\leq\|\Psi_2\|$. Hence if we require that $\|\Phi\|=\|\Psi_1\|+\|\Psi_2\|$ then we have $\mu(E_0{}^\perp\cdot S_1)=\mu(S_1)$ and therefore $E_0\cdot S_1\cdot E_0=0$. Therefore $E_0\cdot S_1=0$. This is equivalent to $F_0{}^\perp\leq E_0{}^\perp$. Similarly $G_0{}^\perp\leq E_0$. Then $\Phi_+(A)=\mu(A\cdot T_+)=\mu(A\cdot E_0{}^\perp\cdot T)=\Phi(AE_0{}^\perp)=\mu(AE_0{}^\perp\cdot S_1)-\mu(AE_0{}^\perp\cdot S_2)=\mu(A\cdot S_1)=\Psi_1(A)$. Similarly $\Phi_-(A)=\Psi_2(A)$. The proof is completed.

2.4. Some applications to the structure of the left ring $\mathbb{L}$ of an H-system $\mathbf{H}$ are given here. An element $x\in\mathbf{H}$ is called central if $xa=ax$ holds for every $a\in\mathbf{B}$, that is, $L_x\eta\mathbb{L}^\natural$. A central element x is also characterized by the property: $\langle x, ab\rangle=\langle x, ba\rangle$ for every $a, b\in\mathbf{B}$. Let $\mathbf{H}^\natural$ stand for the set of central elements of $\mathbf{H}$. It is clear that $\mathbf{H}^\natural$ is a closed linear manifold of $\mathbf{H}$, since ax and xa are continuous functions of x for each fixed a. Let $x^\natural$ denote the projection of x on $\mathbf{H}^\natural$. Let K_x be the convex closure of $\{UU^Jx\,;\,U\in\mathbb{L}_U\}$. By an ergodic theorem of G. Birkhoff [**3**], $x^\natural$ is just the unique element common to K_x and $\mathbf{H}^\natural$, or the element of K_x whose norm is minimum (cf. [**3**], [**12**]). $x^\natural$ is approximated by forms $\sum_{i=1}^n \alpha_i U_i U_i{}^J x$ as close as we want, where $U_i\in\mathbb{L}_U$, $\alpha_i\geq 0$ and $\sum\alpha_i=1$. It follows then from this remark that if $x\geq y\geq 0$, then $\langle x^\natural a, a\rangle\geq\langle y^\natural a, a\rangle\geq 0$ for every $a\in\mathbf{B}$, that is, $x^\natural\geq y^\natural\geq 0$. For every $B=L_x$, $x\in\mathbf{B}$, we define, after Godement [**13**], $B^\natural=L_{x^\natural}$. It is easy to see [**13**] that $x^\natural\in\mathbf{B}$ and $\|B^\natural\|\leq\|B\|$. $B\to B^\natural$ has the following properties (cf. [**4**], [**13**]):

(a) if $B\in\mathbb{L}_\mathbf{B}\frown\mathbb{L}^\natural$, then $B^\natural=B$;

(b) $B\to B^\natural$ is a positive linear mapping from $\mathbb{L}_\mathbf{B}$ to $\mathbb{L}^\natural$;

(c) $(AB)^\natural=(BA)^\natural$ for every $A\in\mathbb{L}$ and $B\in\mathbb{L}_\mathbf{B}$;

(d) if $A\in\mathbb{L}^\natural$, then $(AB)^\natural=AB^\natural$ for every $B\in\mathbb{L}_\mathbf{B}$;

(e) $B\to B^\natural$ is normal;

(f) $\|B^\natural\|\leq\|B\|$ for every $B\in\mathbb{L}_\mathbf{B}$.

(a), (b) and (f) are clear from the above. To prove (c) it suffices to show that $(UBU^*)^\natural=B^\natural$ for every $U\in\mathbb{L}_U$. This is evident from the defining property of $x^\natural$. (d) follows from $AUU^Jx=UU^JAx$. There remains only to show (e). Let $\{B_\delta\}$ be a directed set $\subset\mathbb{L}_\mathbf{B}^+$ with B as its l. u. b. Put $B=L_x$ and $B_\delta=L_{x_\delta}$. From Lemma 2.6 (vii) we have $\|x_\delta\|\leq\|x\|$. $\lim\limits_\delta\langle x_\delta, ab\rangle=\lim\limits_\delta\langle x_\delta b^*, a\rangle=\lim\limits_\delta\langle B_\delta b^*, a\rangle=\langle Bb^*, a\rangle=\langle x, ab\rangle$. Since $\mathbf{B}^2$ is dense in $\mathbf{H}$, it follows that $\lim\limits_\delta\langle x_\delta, z\rangle=\langle x, z\rangle$ for every $z\in\mathbf{H}$.

of (ii) is clear. Let $T \geq 0$ be $\mathfrak{m}$-restrictedly measurable. Then there exists a projection $P \in \mathbf{M}$ such that TP is bounded and $P^{\perp} \in \mathfrak{m}$. Let $\|TP\| < \lambda_0$. We show that $P \frown E_{\lambda_0}^{\perp} = 0$. If the contrary holds, there exits a non-zero $x \in \mathfrak{H}$ with $P \frown E_{\lambda_0}^{\perp} x = x$. $\|Tx\| = \|TPx\| < \lambda_0 \|x\|$, while $\|Tx\| = \|TE_{\lambda_0}^{\perp} x\| \geq \lambda_0 \|x\|$. This is a contradiction. Since for every projection $Q, R \in \mathbf{M}$, $Q - Q \frown R \sim Q \smile R - R$ [**17**], we have $E_{\lambda_0}^{\perp} = E_{\lambda_0}^{\perp} - P \frown E_{\lambda_0}^{\perp} \sim P \smile E_{\lambda_0}^{\perp} - P \leq P^{\perp} \in \mathfrak{m}$, as desired.

Segal [**26**] proved that if S and T are essentially measurable and agree on a strongly dense domain, then they have identical closures. Next is its slight generalization.

Lemma 1.2. *If two essentially $\mathfrak{m}$-restrictedly measurable operators S and T agree on a dense domain, then they have identical closures.*

Proof. With no loss of generality, we may assume that S and T are $\mathfrak{m}$-restrictedly measurable. The set $\mathfrak{D} = \{x ; Tx = Sx\}$ is obviously invariant under every $U' \in \mathbf{M}'_U$, and is dense in $\mathfrak{H}$. Let T_0 be the restriction of S and T on $\mathfrak{D}$. $T \supset T_0$ implies $T^* \subset T_0^*$. As T^* is $\mathfrak{m}$-restrictedly measurable, as proved below, so is T_0^* by the very definition of measurability. It follows, from the result of Segal above mentioned, that $T^* = T_0^*$ and hence $T = T_0^{**}$. By symmetry $S = T_0^{**}$, and we have $T = S$, as desired.

From Lemma 1.1. if T is $\mathfrak{m}$-restrictedly measurable, then so are T^*T, $|T|^{\alpha}$ $(\alpha > 0)$. We show that T^* is $\mathfrak{m}$-restrictedly measurable if so is T. Let $T = W|T|$ be the polar decomposition of T, where W is a partially isometric operator $\in \mathbf{M}$ with the closure of the range of $|T|$ as the initial set and with the closure of the range of T as the final set. Let $WW^* = E$ and let $|T| = \int_0^{\infty} \lambda dE_{\lambda}$, $|T^*| = \int_0^{\infty} \lambda dF_{\lambda}$ be the spectral resolutions of $|T|$ and $|T^*|$ respectively. $|T^*| = W|T|W^*$ yields $F_{\lambda} = WE_{\lambda}W^* + E^{\perp}$ $(\lambda > 0)$. Hence $F_{\lambda}^{\perp} = WE_{\lambda}^{\perp}W^*$. This implies by Lemma 1.1 that $|T^*|$ is $\mathfrak{m}$-restrictedly measurable. It is clear that the intersection of a finite number of strongly $\mathfrak{m}$-dense domains is so also. After Segal we define the strong sum $S \dotplus T$ and strong product $S \cdot T$ of two $\mathfrak{m}$-restrictedly measurable operators S and T. $S \dotplus T$ and $S \cdot T$ are the closures of $S + T$ and ST respectively. (cf. [**26**], Def. 2.2). But in case of our $\mathfrak{m}$-restrictedly measurable operators, $S + T$ is seen to be essentially $\mathfrak{m}$-restrictedly measurable from the above. That ST is so also, follows from a modification of a proof given in [**26**], and details are omitted. Hence in our case $S \dotplus T$ and $S \cdot T$ are $\mathfrak{m}$-restrictedly measurable. Thus we have the

Lemma 1.3. *The set of all $\mathfrak{m}$-restrictedly measurable operators forms a *-algebra with respect to the strong sum $S \dotplus T$ and product $S \cdot T$, the scalar multiplication (except that*

$\mathbb{L}$ is finite if $\mathbf{H}$ has the unit. This follows from (i). It is easy to see that $\mathbb{L}$ is a finite factor if and only if $\mathbf{H}$ has the unit and $\mathbf{H}^{\natural}$ is of one-dimension. These are all proved by Godement [**13**].

THEOREM 5. *Let $\mathbb{L}$ be the left ring of an H-system $\mathbf{H}$. Then the following conditions* (i)–(iv) *are equivalent:*

(i) $\mathfrak{L}_1$ *is an algebra;*

(ii) $\mathfrak{L}_2$ *is an algebra:*

(iii) $\mathbf{H}$ *is essentially an H^*-algebra;*

(iv) *There exists a positive unmber δ such that $\|e\| \geq \delta$ for every non-zero self-adjoint idemptent $e \in \mathbf{H}$. And if any of these conditions is satisfied, then $\mathbb{L}$ is a direct sum of (generally uncountable number of) factors of type* I.

PROOF. (i) and (ii) are evident from Theorem 2. Owing to the remark given in 2.2, $\mathfrak{L}_2$ and $\mathbf{H}$ are isomorphic and therefore (ii) and (iii) are equivalent. If (iii) holds, there exists a positive number k such that $\|xy\| \leq k\|x\|\|y\|$ for every $x, y \in \mathbf{H}$, and therefore $\|e\| \geq \frac{1}{k}$ for every non-zero slef-adjoint element e of $\mathbf{H}$, that is, (iii) implies (iv). We note that the bound of L_x is $\underset{e \neq 0}{\text{l. u. b.}} \frac{\|xe\|}{\|e\|}$. For, if we let $\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of $L_{x^*} \cdot L_x$, then $\frac{\|xe\|^2}{\|e\|^2} \geq \lambda$ for $L_e = E_\lambda{}^\perp \neq 0$. If (iv) holds, then $\frac{\|xe\|}{\|e\|} \leq \frac{\|x\|}{\delta}$ and therefore L_x is bounded, that is, (iv) implies (ii). There remains to show that last statement of our theorem. In an H^*-algebra every non-zero self-adjont idempotent contains a primitive one e that is, $e\mathbf{H}e =$(complex field)$\times e$ ([**1**], [**17**]). This means that L_e is a primitive abelian projection [**17**]. Then $\mathbb{L}$ is a direct sum of factors of type I. The proof is completed.

COROLLARY 5.1. *$\mathbb{L}$ is a factor of type* I *if and only if $\mathbf{H}$ is simple and consists of bounded elements.*

PROOF. The "if" part is evident from the preceding theorem. As remarked later in 3.5, if $\mathbb{L}$ is a factor of type I, then every measurable operator is bounded and therefore $\mathbf{H}$ consists of bounded elements. For any closed ideal $\mathbf{I}$ of $\mathbf{H}$, the projection P with the range $\mathbf{I}$ is a central projection $\in \mathbb{L}$, and therefore $P = 0$ or I, that is, $\mathbf{I} = \{0\}$ or $\mathbf{H}$. The proof is completed.

Godement (cf. [**13**] Chap. II, II) gave another characterization for $\mathbb{L}$ to be a factor of type I: $\mathbb{L}$ is a factor of type I if and only if $\mathbf{H}$ is $*$-isomorphic with the algebra of operators of Hilbert-Schmidt-type on a Hilbert space. We remark that this follows from Cor. 5.1 and the structure theorem of Ambrose [**1**].

Corollary 5.2. *In order that every measurable operator $\in \mathbb{L}$ is bounded it is necessary and sufficient that $\mathbb{L}$ is a direct sum of finite number of factors of type* I.

Proof. The "if" part is evident since in a factor of type I every measurable operator is bounded. If every measurable operator $\eta\mathbb{L}$ is bounded, the $\mathfrak{L}_1$ is an algebra, and therefore $\mathbb{L}$ is a direct sum of factors of type I from the preceding theorem. The number of these factors is finite, for otherwise, we can construct an unbounded measurable operator $\eta\mathbb{L}$. The proof is completed.

Corollary 5.3. *The following conditions are equivalent :*

(i) $\mathfrak{L}_1=\mathfrak{L}_2$;

(ii) $\mathbb{L}$ *is finite-dimensional* ;

(iii) **H** *is finite-dimensional.*

Proof. It is evident that (ii) and (iii) are equivalent and imply (i). If (i) holds, $\mathfrak{L}_1 \subset \mathfrak{L}_2$ implies that $\mathfrak{L}_1$ is is an algebra, and therefore $\mathbb{L}$ is a direct sum of factors of type I from Theorem 5. Unless each of these factor is finite-dimensional and the number of these factors is finite, we can construct an element of $\mathfrak{L}_2$ but not in $\mathfrak{L}_1$. Therefore $\mathbb{L}$ is finite-dimensional. The proof is completed.

We have shown (Lemma 1.9) that an invariant linear system $\mathfrak{L}$ is commutative if the set $\mathfrak{L}'$ of self-adjoint operators of $\mathfrak{L}$ is a vector lattice by the ordering of operators. The converse is evidently true. Owing to this fact, the following statements are equivalent :

(a) any of $\mathfrak{L}_1^{\frac{1}{p}}$ or $\mathbb{L}$ is commutative;

(b) any of $\mathfrak{L}_1^{\frac{1}{p}\prime}$ or $\mathbb{L}'$ is a vector lattice.

In particular, it follows from the isomorphism between $\mathfrak{L}_2$ and **H** that **H** is commutative if and only if **H**$'$ is a vector lattice.

3. Integrals with respect to a regular gage.

3.1. Let $\mathbb{M}$ be a semi-finite ring of operators on a Hilbert space $\mathfrak{H}$, and let m be a regular gage of $\mathbb{M}$ [**26**]: (a) m is a non-negative valued function defined on $\mathbb{M}_P$; (b) $m(P)=0$ if and only if $P=0$; (c) $m(P+Q)=m(P)+m(Q)$ if $P+Q\in\mathbb{M}_P$; (d) $m(P)=m(UPU^*)$ for every $U\in\mathbb{M}_U$; (e) m is countably additive ; (f) if $m(P)=+\infty$, there exists Q such that $0<Q\leq P$ and $m(Q)<+\infty$. It is shown [7] that m is a restriction on $\mathbb{M}_P$ of a uniquely determined faithful, essential, normal pseudo-trace ψ, and vice versa. Let $\mathfrak{a}$ be the maximal ideal associated with ψ, that is, the set of $A\in\mathbb{M}$ with $\psi(|A|)<+\infty$. ψ is extended to a faithful normal trace of $\mathfrak{a}$ [7]. For any two $B, C \in \mathfrak{a}$, we define $\langle B, C\rangle = \psi(C^*B)$. It is easy to see that $\mathfrak{a}$ becomes a unitary algebra with inner product $\langle B, C\rangle$. Its completion **H** is an H-system as stated in 2.1. For any $A \in \mathbb{M}$, the mapping $\mathfrak{a} \ni B \rightarrow AB$ is continuous since $\langle AB, AB\rangle \leq \|A\|^2 \langle B, B\rangle$ holds. This mapping is uniquely extended to an operator $\theta(A)$ on **H**. Let $\mathbb{L}$ be the left ring of **H**. It is easy to see that $\theta(A) \in \mathbb{L}$, and that θ is a $*$-isomorphic normal mapping. By a theorem of Dixmier [**9**], $\theta(\mathbb{M})$ is a ring of operators on **H**. And it coincides with $\mathbb{L}$ since it contains all $\theta(B)$, $B \in \mathfrak{a}$. Let $\{E_\iota\}$ be a maximal orthogonal system of projections $\in \mathfrak{a}$. Then it is clear that $I = \cup_\iota E_\iota$. Dixmier [**8**] proved that $\psi(A) = \sum_\iota \psi(E_\iota A E_\iota)$ for $A \in \mathbb{M}^+$, and therefore $\psi(A) = \sum_\iota \langle AE_\iota, E_\iota\rangle$. If we put $\phi(\theta(A)) = \psi(A)$ for $A \in \mathbb{M}^+$, ϕ is the canonical pseudo-trace of **H** (Lemma 2.1 and Def. 2.1). Let **B** be the bounded algebra of **H**. By Lemma 2.1 we see that $\theta(\mathfrak{a})$ is the maximal ideal associated with ϕ, and that $\theta(\mathfrak{a}) = \mathbb{L}_{\mathbf{B}}^2$. That is, $\theta(\mathfrak{a}^{\frac{1}{2}}) = \mathbb{L}_{\mathbf{B}}$. If we put $\mu(\theta(P)) = m(P)$, then μ is the canonical gage of **H**. This shows that $\mathbb{M}$ is $*$-isomorphic with the left ring $\mathbb{L}$ of **H** by means of the mapping θ and the regular gage m corresponds to the canonical gage μ. It follows from Theorem 1 that θ is uniquely extended to a $*$-isomorphism θ between measurable operators with respect to $\mathbb{M}$ and $\mathbb{L}$. The theory of integrals with respect to the canonical gage μ developed in the preceding section is now translated into the theory of integrals with respect to the regular gage m. This will be carried out in the sequel.

3.2. Let m be a regular gage of a semi-finite ring $\mathbb{M}$ of operators on a Hilbert space $\mathfrak{H}$, and let θ, μ have the same meaning as described in 3.1. Let $\mathcal{M}$ be the set of all measurable operators $\eta\mathbb{M}$. For every $T \in \mathcal{M}^+$, we put

$$(1) \qquad m(T) = \underset{A \in \mathfrak{a}^+, A \leq T}{\text{l. u. b.}} \psi(A).$$

From the the discussions given in 2.3, $m(T) = \mu(\theta(T))$ and, if we let $\mathbf{L}_1$, denote the set of all T such that $m(|T|) < +\infty$, then $\theta(\mathbf{L}_1) = \mathfrak{L}_1$ the set of all integrable measurable operators $\eta\mathbb{L}$ with respect to μ, and therefore m is uniquely extended to a linear functional on $\mathbf{L}_1$.

Definition 3.1. A measurable operator T is said to be *integrable with respect to* m if $m(|T|) < +\infty$. Let $\mathbf{L}_1$ stand for the set of all integrable operators $\eta\mathbb{M}$, and let m be the extended linear functional on $\mathbf{L}_1$ as described above. $m(T)$ is called the integral of $T \in \mathbf{L}_1$ with respect to the gage m.

From lemmas given in **2**, we have the following theorem.

Theorem 6. *Let* $\mathbf{L}_1$ *be the set of integrable measurable operators* $\eta\mathbb{M}$. *Then* $\mathbf{L}_1$ *is* *an invariant linear system of* $\mathbf{M}$ *satisfying the conditions* $(\ll)_1$ *and* $(\ll)_2$. *And the following statements hold.*

(i) $\mathbf{L}_1$ *is a Banach space with norm* $\|T\|_1=m(|T|)$, $\mathfrak{a}$ *is dense in* $\mathbf{L}_1$, *and has the following properties:*

(a) *if* $T_1, T_2 \in \mathbf{L}_1^+$, *then* $\|T_1+T_2\|_1=\|T_1\|_1+\|T_2\|_1$;

(b) *if* $0 \leq T_1 \leq T_2 \leq \cdots$, *and* $\{\|T_n\|_1\}$ *is bounded, then there exists the* l. u. b. $T \in \mathbf{L}_1^+$ *of* $\{T_n\}$ *and* $\lim\|T-T_n\|_1 \to 0$. $\{T_n\}$ *converges m–n. e. to* T *in the star sense;*

(ii) *the integral* m *is a positive linear functional on* $\mathbf{L}_1$, *with the following properties:*

(a) $m(T^*)=\overline{m(T)}$;

(b) $m(T) \geq 0$ *for* $T \in \mathbf{L}_1^+$. *The equality holds if and only if* $T=0$;

(c) $m(A\cdot T)=m(TA)$ *for* $A \in \mathbf{M}$ *and* $T \in \mathbf{L}_1$. *If* $A \in \mathbf{M}_+$, $T \in \mathbf{L}_1^+$, *then* $m(A\cdot T) \geq 0$;

(b) *For a fixed* T, $m(A\cdot T) \geq 0$ *for every* $A \in \mathbf{M}^+$ *if and only if* $T \in \mathbf{L}_1^+$;

(e) *For a fixed* A, $m(A\cdot T) \geq 0$ *for every* $T \in \mathbf{L}_1^+$ *if and only if* $A \in \mathbf{M}^+$;

(iii) $\|T\|_1 = \underset{\|A\|\leq 1,\, A \in \mathbf{M}}{\text{l. u. b.}} |m(A\cdot T)|$, *and* $\|A\| = \underset{\|T\|_1 \leq 1,\, T \in \mathbf{L}_1}{\text{l. u. b.}} |m(A\cdot T)|$.

(iv) $\Phi_T(A)=m(A\cdot T)$ *is a linear normal functional defined on* $\mathbf{M}$, *and conversely every normal linear functional* Φ *is an* Φ_T, $T \in \mathbf{L}_1$. $\mathbf{M}$ *is a conjugate space of* $\mathbf{L}_1$.

(v) *A positive measurable operator* $T=\int_0^\infty \lambda dE_\lambda$ *is integrable if and only if* $\int_0^\infty m(E_\lambda^\perp)d\lambda < +\infty$. *Then this value equals the integrals of* T.

(vi) *If* $\lim_{n\to\infty}\|T_n-T\|_1=0$, *then* $\{T_n\}$ *converges m–n. e. to* T *in the star sense.*

Segal [**26**] cited (i), (b) the Lebesgue convergence theorem and the second part of the first statement of (iv) the Radon-Nikodym theorem. We remark that the Radon-Nikodym theorem of Dye [**11**] follows from that of Segal.

Corollary. *Let* $\mathbf{M}$ *be a semi-finite ring of operators on a Hilbert space. Let* Φ *and* Ψ *be positive normal linear functionals such that* $\Phi(P)=0$, $P \in \mathbf{M}$, *implies* $\Psi(P)=0$. *Let* θ_Φ *be the canonical representation of* $\mathbf{M}$ *defined by* Φ *and* $<\ ,\ >_\Phi$ *denote the inner product of the representation space* $\mathfrak{H}_\Phi$. *Then* $\Psi(A)$ *is represented as* $\Psi(A)=\langle\theta_\Phi(A)z, z\rangle_\Phi$ *for some* $z \in \mathfrak{H}_\Phi$.

Proof. We may assume without loss of generality that $\mathbf{M}$ is a left ring $\mathbb{L}$ of an H-system. We use the notations in **2**. We may write $\Phi(A)=\langle Ax, x\rangle$, and $\Psi(A)=\langle Ay, y\rangle$ where x, y are positive elements of $\mathbf{H}$. $\Phi(P)=0$ is equivalent to $Px=0$, and in turn to $P^J x=0$. This implies $P^J[\mathbb{L}x]=0$. Put $P^J=P[\mathbb{L}x]^\perp$. Then since $\Psi(P)=0$ and therefore by the same reason as the above $P^J y=0$. It follows that $y \in [\mathbb{L}x]$. By a theorem of Murray and v. Neumann (BT–Theorem called by

Dye [11], [16]), $y=BTx$ where $B\in\mathbf{L}$ and T is a closed operator $\eta\mathbf{L}$. We may take $T\geq 0$. Let $T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T and put $T_n=\int_0^n \lambda dE_\lambda$. $\|T_nx-Tx\|\to 0$ as $n\to\infty$. Since $\Phi(B^*A)=\langle A, B\rangle_\Phi=\langle Ax, Bx\rangle$, there exists $u\in\mathfrak{H}_\Phi$ such that $\langle u-T_nu-T_n\rangle_\Phi\to 0$. It follows therefore $\Psi(A)=\langle\theta_\Phi(A)z, z\rangle_\Phi$, where $z=Bu$. The proof is completed.

In the above corollary, if $\mathbf{M}$ is finite, then y is written as $y=Tx$ since $B\cdot T$ is a closed measurable operator $\eta\mathbf{L}$. Let $T=W|T|$ be its polar decomposition. And consider the spectral resolution $\int_0^\infty \lambda dE_\lambda$ of $|T|$ and put $|T|_n=\int_6^n \lambda dE_\lambda$ and $T_n=W|T|_n$. $\{T_n\}$ converges m-n. e. to T. It is easy to see that $\Psi(A)=\lim_{n\to\infty}\Phi(T_n^*AT_n)$ which is defined as $\Phi(T^*\cdot A\cdot T)$ [11].

Definition 3.2. A measurable operator $T\eta\mathbf{M}$ is called *square-integrable* with respect to the gage m if $T^*T\in\mathbf{L}_1$. Let $\mathbf{L}_2$ be the set of all square-integrable operators. For any two $S, T\in\mathbf{L}_2$ we define $\langle S, T\rangle=m(T^*\cdot S)$ and $\|T\|_2=m(T^*T)^{\frac{1}{2}}$.

It is clear from the discussions given in **2** that $\theta(\mathbf{L}_2)=\mathfrak{L}_2$. Therefore we have the following theorem.

Theorem 7. *$\mathbf{L}_2$ is an invariant linear system of $\mathbf{M}$ satisfying the conditions $(\ll)_1$ and $(\ll)_2$, and $\mathbf{L}_2^2=\mathbf{L}_1$. $\mathbf{L}_2$ has the following properties:*

(i) *$\mathbf{L}_2$ is an H-system with inner product $\langle S, T\rangle=m(T^*\cdot S)$. The bounded algebra of $\mathbf{L}_2$ is $\mathfrak{a}^{\frac{1}{2}}$;*

(ii) (a) *$\langle S, T\rangle\geq 0$ for $S, T\in\mathbf{L}_2^+$,*

(b) *if $\langle S, T\rangle\geq 0$ for every $T\in\mathbf{L}_2^+$, then $S\geq 0$,*

(c) *if $S\cdot T^*=0$, then $\|S+T\|_2^2=\|S\|_2^2+\|T\|_2^2$,*

(d) *if $|S|\leq|T|$, then $\|S\|_2\leq\|T\|_2$,*

(e) *$\|T\|_2=\|U\cdot TU^*\|_2$ for every $U\in\mathbf{M}_U$,*

(f) *$|m(S\cdot T)|\leq m(|S\cdot T|)\leq\|S\|_2\|T\|_2$,*

(g) *$\|A\cdot T\|_2\leq\|A\|\,\|T\|_2$ for $A\in\mathbf{M}$ and $T\in\mathbf{L}_2$;*

(iii) *Let $0\leq T_1\leq T_2\leq\cdots$ be a sequence of elements of $\mathbf{L}_2$ such that $\{\|T_n\|_2\}$ is bounded. Then there exists the l. u. b. T of $\{T_n\}$ and $\|T_n-T\|_2\to 0$ as $n\to\infty$. $\{T_n\}$ converges m-n. e. to T in the star sense;*

(iv) *if $\|T_n-T\|_2\to 0$ as $n\to\infty$, then $\{T_n\}$ converges m-n. e. to T in the star sense;*

(v) *Let T be a positive measurable operator $\eta\mathbf{M}$. Let $T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T. Then $T\in\mathbf{L}_2$ if and only if $\int_0^\infty m(E^\perp_{\sqrt{\lambda}})d\lambda<+\infty$; and $\|T\|_2^2$ equals this value.*

COROLLERY. *If T is a measurable operator such that $T\cdot S\in \mathbf{L}_1$ for every $S\in \mathbf{L}_2^+$, then $T\in \mathbf{L}_2$.*

PROOF. Let $T=W|T|$ be the polar decomposition of T. Since $|T|\cdot S=W^*T\cdot S\in \mathbf{L}_1$ for every $S\in \mathbf{L}_2$ we may assume without loss of generality that $T\geq 0$. Let $T=\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T. $E_\lambda{}^\perp$ is metrically finite for $\lambda>0$. For otherwise, $E_\lambda{}^\perp$ is metrically infinite for some $\lambda_0>0$ and we can write $E_{\lambda_0}^\perp=\sum_\iota P_\iota$ where P_ι are metrically finite projections. We can choose a sequence $\{P_{\iota_n}\}$ such that $\sum_n m(P_{\iota_n})=+\infty$. Let $\{\alpha_n\}$ be an arbitrary sequence of positive numbers such that $\sum_n \alpha_n{}^2 m(P_{\iota_n})<+\infty$. Put $S=\sum_n \alpha_n P_{\iota_n}$. Clearly $S\in \mathbf{L}_2^+$. Put $A=\int_{\lambda_0}^\infty \frac{1}{\lambda}E_\lambda \in \mathbf{M}$. Then $S=E_{\lambda_0}^\perp\cdot S=A\cdot T\cdot S\in \mathbf{L}_1$, and therefore $\sum_n \alpha_n m(P_{\iota_n})<+\infty$. Since $\alpha_n m(P_{\iota_n})=\{\alpha_n m(P_{\iota_n})^{\frac{1}{2}}\}m(P_{\iota_n})^{\frac{1}{2}}$, we can conclude that $\sum_n m(P_{\iota_n})<+\infty$, a contradiction. Let $S=\int_0^\infty \varphi(\lambda)dE_\lambda$ such that $-\int_0^\infty |\varphi(\lambda)|^2 dm(E_\lambda{}^\perp)<+\infty$, where $\varphi(\lambda)$ is a Baire function of λ. $T\cdot S=\int_0^\infty \lambda\varphi(\lambda)dE_\lambda \in \mathbf{L}_1$. This implies $|\int_0^\infty \lambda\varphi(\lambda)dm(E_\lambda{}^\perp)|<+\infty$. It follows from a classical result concerning square-integrable functions that $-\int_0^\infty \lambda^2 dm(E_\lambda{}^\perp)<\infty$. Hence $T\in \mathbf{L}_2$. The proof is completed.

Similarly we can show that if T is a measurable operator such that $T\cdot S\in \mathbf{L}_1$, for every $S\in \mathbf{L}_1^+$, then $T\in \mathbf{M}$.

3.3. We give some remarks on "normed" operators. An operator A is called *normed* ([**19**], [**14**]) if $A\in \mathbf{L}_2\cap \mathbf{M}$. Let $\mathbf{M}$ be a semi-finite ring of operator with a regular gage m. If a sequence of normed operators T_n with bounded uniform norms converges to T in $\mathbf{L}_2$, then, by the remark after Lemma 1.6, T is a normed operator and $\{T_n\}$ converges strongly to T. If the converse of this statement holds, that is, strong convergence entails $\mathbf{L}_2$-convergence for every sequence of normed operators with bounded uniform norms, then $m(I)<+\infty$. To prove this, write $I=\sum_\iota E_\iota$ where E_ι are metrically finite projections. For any sequence $\{\iota_j\}$ from $\{\iota\}$, $\{\sum_{j=1}^n E_{\iota_j}\}$ converges strongly to $\sum_{j=1}^\infty E_{\iota_j}$. Hence $\sum_{j=1}^\infty E_{\iota_j}\in \mathbf{L}_2\cap \mathbf{M}$ and $\sum_{j=1}^\infty m(E_{\iota_j})=m(\sum_{j=1}^\infty E_{\iota_j})<+\infty$. It follows from this that $m(E_\iota)>0$ for atmost countable ι, and therefore $m(I)<+\infty$. Conversely, let $m(I)<+\infty$, then strong convergence entails $\mathbf{L}_2$-convergence for every sequence of normed operators T_n with bounded uniform norms. We may assume that $\mathbf{M}$ is a left ring $\mathbb{L}$ of an H-system, since two topologies, ultrastrong and strong, have the same effect on the sequential convergence. Then $\langle A, B\rangle=\langle AI, BI\rangle$ for any operators $A, B\in \mathbf{M}$. If $\{T_n\}$ converges strongly to T, then

$\langle T_n - T,\ T_n - T\rangle \to 0$ as $n\to\infty$. The proof is completed.

These facts are remarked by Dixmier [9] on a topological stand point, but the method of our proof is different. See also [19], p. 106–.

3.4. DEFINITION 3.3. A measurable operator $T\eta\mathbf{M}$ is said to be p^{th} power integrable with respect to m if $|T|^p\in\mathbf{L}_1$. Let $\mathbf{L}_p(1\leq p<+\infty)$ stand for the set of p^{th} power integrable operators $\eta\mathbf{M}$. The $\mathbf{L}_p$-norm of $T\in\mathbf{L}_p$ is defined as $m(|T|^p)^{\frac{1}{p}}$ and designated by $\|T\|_p$. If $p=+\infty$, we shall identify $\mathbf{M}$ with $\mathbf{L}_\infty$.

From this definition a measurable operator T belongs to $\mathbf{L}_p$ $(1\leq p<+\infty)$ if and only if T is m-restrictedly measurable and $-\int_0^\infty \lambda^p dm(E_\lambda{}^\perp)<+\infty$, where $\int_0^\infty \lambda dE_\lambda$ is the spectral resolution of $|T|$.

LEMMA 3.1. (cf. [9]). *Let* $\frac{1}{p}+\frac{1}{q}=1$ *where* $1\leq p,\ q\leq+\infty$. *Then*

(i) $m(S\cdot T)=m(T\cdot S)$ *for* $S\in\mathbf{L}_p$ *and* $T\in\mathbf{L}_q$. *If furthermore* $S, T\geq 0$, *then* $m(S\cdot T)\geq 0$; *and conversely, if* $m(S\cdot T)\geq 0$ *for every* $T\geq 0$, *then* $S\geq 0$.

(ii) $|m(T_1\cdot T_2\cdot\cdots\cdot T_n)|\leq m(|T_1\cdot T_2\cdot\cdots\cdot T_n|)\leq\|T_1\|_{p_1}\|T_2\|_{p_2}\cdots\|T_n\|_{p_n}$ *for* $T_i\in\mathbf{L}_{p_i}$ *with* $\sum_{i=1}^n\frac{1}{p_i}=1,\ p_i\geq 1$ $(i=1, 2, \cdots, n)$.

(iii) $\|S\|_p=\underset{T\in\mathbf{L}_q,\ \|T\|_q\leq 1}{\text{l. u. b.}}|m(S\cdot T)|$ *for* $S\in\mathbf{L}_p$ *where the* l. u. b. *is attained by some* T *if* $1\leq p<+\infty$;

(iv) $\mathbf{L}_p$ *is a normed linear space, and* $\|T\|_p=\|T^*\|_p=\|U\cdot T\cdot U^*\|_p$ *for* $T\in\mathbf{L}_p$ *and* $U\in\mathbf{M}_U$. $\|T\|_p\leq\|S\|_p$ *for* $S, T\in\mathbf{L}_p$ *such that* $|T|\leq|S|$.

(v) $|m(S\cdot T)|^2\leq m(|S^*|\cdot|T|)m(|S|\cdot|T^*|)\leq m(|S\cdot T|)m(|T\cdot S|)$ *for* $S\in\mathbf{L}_p$ *and* $T\in\mathbf{L}_q$.

PROOF. The lemma will be proved with necessary modifications along the similar lines as Dixmier [9], and the details are omitted.

LEMMA 3.2. *Let T be an m-restrictedly measurable operator $\eta\mathbf{M}$ and* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ *where* $1\leq p, q, r\leq+\infty$. *If* $T\cdot S\in\mathbf{L}_r$ *for every* $S\in\mathbf{L}_q^+$, *then* $T\in\mathbf{L}_p$.

PROOF. The proof will be carried out along the similar line as Cor. of Theorem 7 and the details are omitted.

THEOREM 8. *$\mathbf{L}_p$ is complete.*

PROOF. Let $\{T_n\}$ be any Cauchy sequence of elements of $\mathbf{L}_p$. It is easy to see that $\{T_n\}$ converges m-n. e. in the star sense to an m-restrictedly measurable operator

T. Let S be any element of $\mathbf{L}_q$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Lemma 3.1 $\|T_m\cdot S-T_n\cdot S\|_1\leq\|T_m-T_n\|_p\,\|S\|_q$, which implies that $\{T_n\cdot S\}$ converges m–n. e. in the star sense to $T\cdot S$ and $\|T_n\cdot S-T\cdot S\|_1\to 0$ as $n\to\infty$ since $\mathbf{L}_1$ is complete. Lemma 3.2 shows that $T\in\mathbf{L}_p$. Let ε be any given positive number. Choose n_ε such that $\|T_m-T_n\|_p\leq\varepsilon$ for every $m>n\geq n_\varepsilon$. If $\|S\|_q\leq 1$, then $\|(T-T_n)\cdot S\|_1\leq\varliminf_{m\to\infty}\|(T_m-T_n)\cdot S_n\|_1\leq\varepsilon$. Hence by Lemma 3.1 we have $\|T-T_n\|_p\leq\varepsilon$ for $n\geq n_\varepsilon$, that is, $\lim_{n\to\infty}\|T-T_n\|_p=0$. The proof is completed.

As Dixmier [9] did, we can show that $\mathbf{L}_p$ is reflexive if $1<p<+\infty$. Using this fact we show

Theorem 9. *If $0\leq T_1\leq T_2\leq\cdots$ is a sequence of elements of $\mathbf{L}_p$ $(1\leq p<+\infty)$ such that $\{\|T_n\|_p\}$ is bounded, then there exists the* l. u. b. *T of $\{T_n\}$ and $\|T-T_n\|_p\to 0$ as $n\to\infty$. And $\{T_n\}$ converges m–n. e. to T in the star sense.*

Proof. It is sufficient to show the theorem for $1<p<+\infty$. Let S be any operator $\in\mathbf{L}_q$ with $\frac{1}{p}+\frac{1}{q}=1$. It follows from Lemma 3.1 that $0=m(T_n\cdot S)\leq m(T_{n+1}\cdot S)\leq\|T_{n+1}\|_p\|S\|_q\leq k\|S\|_q$ for some constant $k>0$. Since every operator $\in\mathbf{L}_q$ is a linear combination of positive ones $\in\mathbf{L}_q$, $\{T_n\}$ converges weakly (=in the topology $\sigma(\mathbf{L}_p,\mathbf{L}_q)$) to $T\in\mathbf{L}_q$, that is, $m(T\cdot S)=\lim_{n\to\infty}m(T_n\cdot S)$. $m(T\cdot S)\geq mT_n\cdot S)$ for every $S\in\mathbf{L}_q^+$. Therefore $T\geq T_n(n=1,2,3,\cdots)$ by Lemma 3.1. For every $\varepsilon>0$ there exist non-negative numbers $\alpha_j(j=1,2,\cdots,m)$ with $\sum_{j=1}^m\alpha_j=1$ such that $\|T-\sum_{j=1}^m\alpha_jT_j\|_p<\varepsilon$. $0\leq T-T_n\leq T-\sum_{j=1}^m\alpha_jT_j$ for every $n\geq m$. This implies that $\|T-T_m\|_p\leq\|T-\sum\alpha_jT_j\|_p\leq\varepsilon$ for $n\geq m$, that is, $\|T-T_n\|_p\to 0$ as $n\to\infty$. The other parts of the theorem will be proved by the same way as in Cor. 3.2.

3.5. Let $\mathbf{M}$ be an arbitrary ring of operators. There exists a central projection Q such that $Q\mathbf{M}$ is semi-finite and $Q^\perp\mathbf{M}$ is of type III (cf. [17]). Any measurable operator $\eta Q^\perp\mathbf{M}$ is bounded since there exists no non-zero finite projection in a ring of type III. It follows from 2.4 that every measurable operator $\eta\mathbf{M}$ is bounded if and only if $\mathbf{M}$ is a direct sum of a ring of type III and a finite number of factors of type I. In the rest of this section we assume that $\mathbf{M}$ is a semi-finite ring with a regular gage m.

Lemma 3.3. *Let $1\leq p<r\leq+\infty$. The following conditions* (i)–(iii) *are equivalent:*

(i) $\mathbf{L}_p\supset\mathbf{L}_r$;

(ii) $\mathbf{M}\cap\mathbf{L}_p\supset\mathbf{M}\cap\mathbf{L}_r$;

(iii) $\mathbf{M}$ *is finite and* $m(I)<+\infty$.

Proof. (i)→(ii) is evident. (ii)→(iii): We may assume that $r<+\infty$. Put $q=\frac{r}{p}$. $\mathbf{M}\frown\mathbf{L}_p\supset\mathbf{M}\frown\mathbf{L}_r$ is equivalent to $\mathbf{M}\frown\mathbf{L}_1\supset\mathbf{M}\frown\mathbf{L}_r^p=\mathbf{M}\frown\mathbf{L}_q$ (Theorem 2). We can write $I=\sum_\iota E_\iota$ where E_ι are metrically finite projections. If, for some $\varepsilon>0$, the set $\{E_\iota; m(E_\iota)>\varepsilon\}$ is infinite, we can take a sequence $\{E_n\}$ from the set. Let $\{\alpha_n\}$ be a sequence of positive numbers such that $\sum_n\alpha_n{}^q m(E_n)<+\infty$, and put $T=\sum_n\alpha_n E_n$. Evidently $T\in\mathbf{M}\frown\mathbf{L}_q$, and therefore $T\in\mathbf{M}\frown\mathbf{L}_1$, that is, $\sum_n\alpha_n m(E_n)<+\infty$. Let $\{\beta_n\}$ be an arbitrary sequence of positive numbers such that $\sum_n\beta_n{}^q<+\infty$. We can find $\{\alpha_n\}$ satisfying the above condition and such that $\alpha_n m(E_n)^{\frac{1}{q}}=\beta_n$. Then $\alpha_n m(E_n)=\beta_n m(E)^{\frac{1}{q'}}$ for $\frac{1}{q}+\frac{1}{q'}=1$. Hence we must obtain $\sum_n m(E_n)<+\infty$, a contradiction. Hence $\{E_\iota\}$ is at most countable. If $\sum_\iota m(E_\iota)=+\infty$, we may assume that $m(E_\iota)\geq 1$. If we repeat the above discussion, we reach a contradiction. Hence $m(I)<+\infty$. iii)→(i): We may assume $r<+\infty$ since the case $r=+\infty$ is evident. $I\in\mathbf{L}_{q'}$, where $\frac{1}{q}+\frac{1}{q'}=1$. Therefore from Theorem 2 we have $\mathbf{L}_q=\mathbf{L}_q I\subset\mathbf{L}_1$. The proof is completed.

Lemma 3.4. *Let* $1\leq p<r\leq+\infty$. *The following conditions are equivalent:*

(i) $\mathbf{L}_p\subset\mathbf{L}_r$;

(ii) $\mathbf{M}$ *is a direct sum of factors of type* I *and there exists a positive number* δ, *such that* $m(E)\geq\delta$ *for every non-zero projection* $E\in\mathbf{M}$.

Proof. (i)→(ii): Suppose that $r<+\infty$. Put $q=\frac{r}{p}$. $\mathbf{L}_p\subset\mathbf{L}_r$ is equivalent to $\mathbf{L}_1\subset\mathbf{L}_1^{\frac{1}{q}}=\mathbf{L}_q$ (Theorem 2) and in turn we obtain $\mathbf{L}_1\subset\mathbf{L}_q\subset\mathbf{L}_{q^2}\subset\cdots$. We may assume that $q\geq 2$. Let T be any measurable operator $\in\mathbf{L}_1^+$ and $\int_0^\infty\lambda dE_\lambda$ be its spectral resolution. Put $T_1=TE_1$ and $T_2=TE_1^\perp$. Then $T_1,T_2\in\mathbf{L}_1$, and therefore $T_2\in\mathbf{L}_q$. $T_1\in\mathbf{M}\frown\mathbf{L}_1$ implies that $T_1\in\mathbf{L}_2$. $-\int_1^\infty\lambda^q dm(E_\lambda{}^\perp)\geq-\int_1^\infty\lambda^2 dm(E_\lambda{}^\perp)$ implies that $T_2\in\mathbf{L}_2$. Hence we obtain $T\in\mathbf{L}_2$. Thus $\mathbf{L}_1\subset\mathbf{L}_2$. If $r=+\infty$, then for any $T\in\mathbf{L}_1^+$ we have $T^{\frac{1}{p}}\in\mathbf{L}_p\subset\mathbf{M}$ and therefore $T\in\mathbf{M}$, which implies that that $T^2\in\mathbf{L}_1$. In any case we have $\mathbf{L}_1\subset\mathbf{L}_2$. It follows from Theorem 5 that the left ring $\mathbf{L}$ considered in 3.1 satisfies (ii), and therefore $\mathbf{M}$ satisfies (ii).

(ii)→(i): Owing to Theorem 5 and. Cor. 5.2, the isomorphic mapping θ considered in 3.1 shows that $\mathbf{L}_2\subset\mathbf{M}$. Then it is clear that $\mathbf{L}_p\subset\mathbf{L}_r$ for any $1\leq p<r\leq+\infty$.

Using these lemmas we obtain the condition under which every measurable operator is integrable.

THEOREM 10. *Let* M *be a semi-finite ring with a regular gage m. The following conditions* (i)–(iii) *are equivalent:*

(i) M *is finite-dimensional;*

(ii) $\mathbf{L}_p=\mathbf{L}_r$ *for some* $p\neq r$;

(iii) *Every measurable operator is integrable.*

PROOF. The preceding two lemmas shows that (i) and (ii) are equivalent. (i)→(iii) is evident. (iii) implies that $\mathbf{L}_2\subset\mathbf{L}_1$ and $\mathbf{L}_1^2\subset\mathbf{L}_1$. The latter is equivalent to $\mathbf{L}_1\subset\mathbf{L}_2$. Thus we have $\mathbf{L}_1=\mathbf{L}_2$. The proof is completed.

Let $\mathscr{B}$ be the ring of all bounded operators on a Hilbert space $\mathfrak{H}$, and let $\{f_\iota\}$ be a complete orthonormal system of $\mathfrak{H}$. If we put $\phi(A)=\sum_\iota \langle Af_\iota, f_\iota\rangle$ for $A\in\mathscr{B}^+$, it is easy to see that ϕ is a faithful, essential, normal pseudo-trace, and that any other such pseudo-trace is a multiple of ϕ since $\mathscr{B}$ is a factor. The corresponding gage $m(P)$ is the dimension of $P\mathfrak{H}$. It follows that in a factor of type I every measurable operator is bounded. $\mathbf{L}_p^+$ $(1\leq p<+\infty)$ consists of positive operators, the sums of the p^{th} powers of whose proper values counted as their multiplicities are finite. Therefore every operator $\in\mathbf{L}_p(1\leq p<+\infty)$ is completely continuous, and $\mathbf{L}_p\subset\mathbf{L}_q$ for $p<q$. Since (c_0) is not the union of (l_p), $1\leq p<+\infty$, we see that $\mathfrak{H}$ is finite-dimensional if and only if every completely continuous operator is integrable. These considerations suggest the following generalization.

THEOREM 11. *Let* M *be a semi-simple ring with a regular gage m. The following statements are epuivalent:*

(i) *every operator* $\in\mathbf{L}_p\cap$M $(1\leq p<+\infty$, *p fixed*) *is a w. c. c. element of* M;

(ii) M *is a direct sum of factors of type* I *and there exists a positive number* δ *such that* $m(E)>\delta$ *for every non-zero projection* $E\in$M.

If any of (i) *and* (ii) *holds, and furthermore if every w. c. c. element of* M *is* $\in\mathbf{L}_p$ *for some* $1\leq p<+\infty$, *then* M *is finite-dimensional.*

PROOF. An operator $A\in$M is called w. c. c. [**23**] if the right (or left) multiplication by A is a completely continuous operator on M in the topology σ(M, M*). The set $\mathscr{I}$ of all w. c. c. elements of M forms a closed ideal of M. We note that the second part of (ii) implies the first part of (i).

(i)→(ii): Let $\mathfrak{a}$ be the maximal ideal of M associated with m, that is, $\mathfrak{a}=\mathbf{L}_1\cap$M. $\mathfrak{a}^{\frac{1}{p}}=\mathbf{L}_p\cap$M. $\mathfrak{a}$ and $\mathfrak{a}^{\frac{1}{p}}$ have identical uniform closure J which is an ideal of

M contained in $\mathcal{J}$. Therefore (i) holds for every p if so does for some p. J is a w. c. c C^*-algebra. This shows [23] that every non-zero projection $\in J$ contains a primitive one, that is, $PJP=$(complex field)$\times P$. Since $\mathfrak{a}$ is dense in M in the strong topology, it follows that $PMP=$ (complex field) $\times P$, and therefore P is a primitive abelian projection $\in$ M. Every non-zero projection $\in$ M contains a non-zero metrically finite projection, and a fortiori a primitive abelian projection. It follows that M is a direct sum of factors of type I. If the second part of (ii) does not hold, we can choose a sequence of orthogonal primitive projections E_n such that $m(E_n)\leq\frac{1}{n^2}$. Put $A=\sum_n E_n$. Then $A\in\mathfrak{a}^{\frac{1}{p}}$, but not a w. c. c. element of M.

(ii)$\rightarrow$(i): M is assumed to be a direct sum of $\mathcal{B}_\alpha$ where $\mathcal{B}_\alpha$ is the ring of all bounded operators on a certain Hilbert space. A is determined by its components A_α and $\|A\|=\text{l. u. b.}_{\alpha}\|A_\alpha\|$. A w. c. c. element A is characterized by the properties that each A_α is a completely continuous operator on $\mathfrak{H}$ and the set $\{\alpha;\|A_\alpha\|>\varepsilon\}$ is finite for every $\varepsilon>0$. The pseudo-trace of M in question uniquely determined by m is of the form $\sum_\alpha c_\alpha\phi_\alpha$ where each ϕ_α is the ordinary pseudo-trace of $\mathcal{B}_\alpha$, c_α is a positive number $>\delta$ and $\phi(A)=\sum_\alpha c_\alpha\phi_\alpha(A)$ for $A\in\text{M}^+$. $\phi(A_\alpha)<+\infty$ implies that each A_α is completely continuous and $\{\alpha;\phi_\alpha(A_\alpha)>\varepsilon\}$ is finite. Since $\|A_\alpha\|\leq\phi_\alpha(A_\alpha)$ holds, we see that A is a w. c. c. element of M.

Now we show the rest part of the theorem. We follow the notations of the proof of (ii)$\rightarrow$(i). Each $\mathcal{B}_\alpha$ is finite-dimensional as remarked above. We have only to show that the index set $\{\alpha\}$ is finite. Otherwise we can choose a sequence $\{\beta_n\}$ of positive numbers such that $\beta_n\downarrow 0$ and $\sum_n\beta_n{}^p\phi_{\alpha_n}(E_{\alpha_n})=+\infty$, where E_{α_n} is a primitive projection $\in\mathcal{B}_{\alpha_n}$. Put $A=\sum_n\beta_n E_{\alpha_n}$. Then A is a w. c. c. element of M, but $\phi(A^p)=\infty$. The proof is completed.

Corollary. *$\mathbf{L}_p(1\leq p<+\infty)$ coincides with the set of w. c. c. elements of* M *if and only if* M *is finite-dimensional.*

Let $\mathbf{L}'_p$ be the set of self-adjoint operators $\in\mathbf{L}_p$. If $\mathbf{L}'_p$ is a vector lattice by the ordering of operators, $\mathbf{L}_p$ is commutative by Theorem 2, and vice versa.

4. Analogies to (AL).

4.1. Let V be a normed vector lattice with norm $\|x\|$. We say [20] that V is an (AL) if

(a) if $|x|\leq|y|$ holds, $\|x\|\leq\|y\|$;

(b) if $x\frown y=0$ holds, $\|x+y\|=\|x\|+\|y\|$;

(c) if $0 \leq x_1 \leq x_2 \leq \cdots$ and $\{\|x_n\|\}$ is bounded, then there exists the l. u. b. $x \in V$ of $\{x_n\}$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Then it is shown [20] that V is a Banach lattice, and is representable as an L_1-space on a measure space. The same is true for a complex vector lattice [21].

4.2. Let $\mathfrak{L}$ be an invariant linear system consisting of measurable operators η **M**, where **M** is an arbitrary ring of operators. Let $\mathfrak{L}$ be a normed linear space with norm $\|T\|$ and have the following properties:

(α) if $|S| \leq |T|$ holds, $\|S\| \leq \|T\|$. And $\|T\| = \|UTU^*\|$ for every $U \in \mathbf{M}_U$;

(β) if $S \cdot T = 0$ for $S, T \in \mathfrak{L}^+$, then $\|S+T\| = \|S\| + \|T\|$;

(γ) if $0 \leq T_1 \leq T_2 \leq \cdots$ be a sequence of mutually commutative operators $\in \mathfrak{L}$ such that $\{\|T_n\|\}$ is bounded, then there exists the l. u. b. $T \in \mathfrak{L}$ of $\{T_n\}$ such that $\|T_n - T\| \to 0$ as $n \to \infty$.

Let $\mathfrak{m}$ be the ideal of **M** generated by the projections $\in \mathfrak{L}$. $\mathfrak{m}$ is the union of $P\mathbf{M}P$, $P \in \mathfrak{L}$. Let $\mathbf{M}_1$ be the closure of $\mathfrak{m}$ in the strong topology. It is known [7] that there exists a central projection $Q \in \mathbf{M}$ such that $\mathbf{M}_1 = Q\mathbf{M}$. Let $T \in \mathfrak{L}^+$ and $\int_0^\infty \lambda dE_\lambda$ be the spectral resolution of T. Put $T_n = \sum_{k=1}^{2^{2n}} \frac{k}{2^n}(E_{(k+1)/2^n} - E_{k/2^n})$. Since $0 \leq T_n \leq T$, it follows that $T_n \in \mathfrak{L}$ and $E_{(k+1)/2^n} - E_{k/2^n} \in \mathfrak{L}$. It is clear that T is the l. u. b. of $\{T_n\}$. Hence it is easy to see that $Q^\perp T = 0$ since $Q^\perp T_n = 0$. Therefore we may assume $Q^\perp \mathbf{M} = 0$, that is, $Q = I$. And $\|T\| = \lim_{n \to \infty} \sum \frac{k}{2^n} \|E_{(k+1)/2^n} - E_{k/2^n}\|$.

From (α)—(γ) we see that $E_\lambda^\perp \in \mathfrak{L}$ for erery $\lambda > 0$, and $\|T\| = \int_0^\infty \|E_\lambda^\perp\| d\lambda$. Conversely if for a given positive measurable operator $T = \int_0^\infty \lambda dE_\lambda$, $E_\lambda^\perp \in \mathfrak{L}$ for every $\lambda > 0$ and $\int_0^\infty \|E_\lambda^\perp\| d\lambda < +\infty$, then $T \in \mathfrak{L}$. The proof is easy. Put for any $P \in \mathbf{M}_P$, $m(P) = \underset{E \subset P, E \in \mathfrak{m}_P}{\text{l. u. b.}} \|E\|$. It is easy to see that $m(P)$ is finite if and only if $P \in \mathfrak{m}_P$, and that $m(P) = \|P\|$ for $P \in \mathfrak{m}_P$. It follows easily from (α)—(γ) that m is a regular gage of **M** and that **M** is semi-finite. Therefore from Theorem 6 (v) we conclude that $\mathfrak{L}$ is the set of all integrable operators η **M**. Moreover it is clear from Theorem 6 that any $\mathbf{L}_1$ satisfies (α)—(γ).

Thus (α)—(γ) are the characteristic properties for an $\mathfrak{L}$ to be $\mathbf{L}_1$. Compare (α)—(γ) with (a)—(c) of 4.1. Let $\mathfrak{L}'$ be a vector lattice, then $Q\mathbf{M}$ and therefore $\mathfrak{L}$ is commutative. This is proved in **2.** And (β) is reduced to (b). $\|T\| = \|UTU^*\|$ is always satisfied, and therefore (α) is reduced to (a). Thus $\mathbf{L}_1$ is considered as a non-commutative extension of (AL).

In like manner, we can state characteristic properties for $\mathbf{L}_p$ and find an analogy to (AL_p). It suffices to replace (β) by $(\beta)_p$: if $S\cdot T=0$ fcr $S, T\in\mathfrak{L}^+$, then $\|S+T\|^p=\|S\|^p+\|T\|^p$. The details are omitted.

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