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NATURAL BUNDLES AND THEIR GENERAL PROPERTIES

—Geometric objects revisited—

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1. Historical Background

In the early development of the "Ricci calculus", now known as classical differential geometry and tensor analysis, it was realized that objects more general than tensors occurred naturally. The most common examples were affine connections (Christoffel symbols) and what are now called jets of tensor fields and connections. All these occurred in almost every significant formula: the Christoffel symbols are built up from jets of the fundamental tensor field; the curvature tensor is built up from jets of the connection parameters; every covariant derivative is built up from jets of some tensor field.

Tensors were defined by the manner of transformation of their components under a change of coordinates in the underlying manifold, or under a "non-holonomic" change of basis in the tangent space. This invariance-under-transformation type of definition was completely in tune with Felix Klein's *Erlanger Program* of 1872 which defines the geometric properties of a space as the set of invariants of the group of transformations under which the space is invariant. See [1872.1] and [1921.1, p. 460].

The desirability of the formalization of objects more general than tensors was expressed by G. Ricci [1887.1 and 1889.1]. The underlying idea was that these objects would have components which, under a change of coordinates, change in a manner which (unlike tensors) is not necessarily linear and does not necessarily depend only on the first order derivatives of the change of coordinates.

O. Veblen [1927.1, pp. 14, 19] calls these objects *invariants*. Subsequently, J. A. Schouten and E. R. van Kampen [1930.1] propose the less overworked term *geometric object*. Veblen and Whitehead [1932.1, pp. 46–49] adopt the same term and make a few very elucidating comments on the correspondence between geometric objects and geometric structures.

In a 1934 lecture [1937.1] A. Wundheiler gave a more precise definition of geometric objects than our previous references, based as always,

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sisting of a group element (base point) and an element of the Lie algebra (tangent vector at the identity element e) obtained by left translating the tangent vector. More generally, let F be a manifold on which a Lie group G acts simply transitively; a tangent to F is identifiable with a pair consisting of its base point in F and that element of the Lie algebra of G whose "infinitesimal" action on F is that tangent vector. More formally, if $\rho: G \times F \to F$ determines the action of G on F, then the map $\rho_*: T(G) \times T(F) \to T(F)$ can be restricted to the zero tangent vectors of F (i.e. to F itself) and to the tangent space to G at e to yield a map $T_e(G) \times F \to T(F)$: simple transitivity implies this map is a diffeomorphism.

A special case occurs when F is affine and G the group (vector space) of translations of F.

Theorem 5.8. *If $\mathscr{E}(X)$ is a natural bundle of groups, then $\mathscr{V}\mathscr{E}(X)$ (or $\mathscr{C}\mathscr{E}(X)$) is equivalent to the Whitney product of two natural bundles. One of these is $\mathscr{E}(X)$; the other $d\mathscr{E}(X)$ (or* Hom $(\mathscr{T}, d\mathscr{E})(X)$), *where $d\mathscr{E}(X)$ is the natural bundle of Lie algebras of the fibers of $\mathscr{E}(X)$. If $\mathscr{E}'(X)$ is a natural bundle on which $\mathscr{E}(X)$ acts naturally and simply transitively, then $\mathscr{V}\mathscr{E}'(X)$ (or $\mathscr{C}\mathscr{E}(X)$) is equivalent to $(\mathscr{E}' \times d\mathscr{E})(X)$ (or $(\mathscr{E}' \times \mathrm{Hom}(\mathscr{T}, d\mathscr{E}))(X)$). The Whitney products are semi-direct with respect to the group structures (or group actions).*

Remarks. 5.15. The bundle $d\mathscr{E}(X)$ is the pull-back $e^{-1}\mathscr{V}\mathscr{E}(X)$, if $e: \mathscr{I}d(X) \to \mathscr{E}(X)$ is the identity section in $\mathscr{E}(X)$.

5.16. Theorem 5.8 implies that for sections in natural bundles of Lie groups or spaces on which such a bundle of groups acts simply transitively, the Lie derivative or covariant derivative consists of two parts: the section itself and the *essential* part, which is a section in a natural vector bundle. For example, the (essential part of the) Lie derivative of a connection is a tensor field; the (essential part of the) Lie derivative of a frame field (of order 1) can be interpreted as a field of linear transformations in the tangent bundle.

Theorem 5.9. *If $\mathscr{E}(X)$ is a natural bundle of groups then $\mathscr{J}^1\mathscr{E}(X)$ is a natural bundle of groups.*

Proof (sketch). The group properties of $\mathscr{E}(X)$ are expressible by diagrams, cf. Remark 5.8. There are maps μ, ν and e, and relations between them expressing associativity, the fact that e is the identity section and that ν is the inverse map. The last is, for example, expresseed by the commutativity of the diagram (Δ is the diagonal map)

$$\begin{array}{ccccc}
\mathscr{E}(X) & \xrightarrow{\Delta} & (\mathscr{E} \times \mathscr{E})(X) & \xrightarrow{(\mathrm{id},\nu)} & (\mathscr{E} \times \mathscr{E})(X) \\
\downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \mu} \\
\mathscr{I}d(X) & & \xrightarrow{\qquad e \qquad} & & \mathscr{E}(X)
\end{array}$$

Passing to tangent spaces we find new diagrams expressing analogous commutativities for $\mathscr{T}\mathscr{E}(X)$. For example, the above diagram becomes

$$\begin{tikzcd}
\mathscr{T}\mathscr{E}(X) \arrow[r, "\Delta_*"] \arrow[d, "\pi_*"] & (\mathscr{T}\mathscr{E} \times \mathscr{T}\mathscr{E})(X) \arrow[r, "(\mathrm{id}, \nu_*)"] & (\mathscr{T}\mathscr{E} \times \mathscr{T}\mathscr{E})(X) \arrow[d, "\mu_*"] \\
\mathscr{T}(X) \arrow[rr, "e_*"] & & \mathscr{T}\mathscr{E}(X)
\end{tikzcd}$$

Although this one is very similar to the first, there is one property which keeps $\mathscr{T}\mathscr{E}(X)$ from being a natural bundle of groups: e_* is not an identity section. Its domain is no longer X; it is $T(X)$. If, however, in all these new diagrams every natural bundle of tangent spaces (use $\mathscr{T}\mathscr{A}(X)$ for a typical one, where $\mathscr{A}(X) = (A, \alpha, \boldsymbol{D}, X)$) is replaced by the bundle of linear maps $T_x(X) \to T_z(A)$, with $z \in \alpha^{-1}(x)$, which project back to the identity map on $T_x(X)$, then we find (cf. Theorem 5.6) this commutative diagram:

$$\begin{tikzcd}
\mathscr{J}^1\mathscr{E}(X) \arrow[r] \arrow[d] & (\mathscr{J}^1\mathscr{E} \times \mathscr{J}^1\mathscr{E})(X) \arrow[r] & (\mathscr{J}^1\mathscr{E} \times \mathscr{J}^1\mathscr{E})(X) \arrow[d] \\
\mathscr{I}d(X) \arrow[rr] & & \mathscr{J}^1\mathscr{E}(X)
\end{tikzcd}$$

where all the arrows denote naturally induced maps from the previous diagram. The bundle $\mathscr{T}(X)$ has been reduced again to $\mathscr{I}d(X)$ since for each $x \in X$ there is only one map $T_x(X) \to T(X)$ which is admissible, namely the identity map.

Theorem 5.10. *If $\mathscr{E}(X)$ is a natural bundle of order k, then $\mathscr{T}(X)$, $\mathscr{V}\mathscr{E}(X)$ and $\mathscr{C}\mathscr{E}(X)$ are of order $k+1$; $\mathscr{J}^l\mathscr{E}(X)$ is of order $k+l$; $\mathscr{F}^k(X)$ is of order k. A natural bundle $\mathscr{E}(X)$ is of finite order k it and only if it is a differentiable fiber bundle with principal bundle the natural bundle $\mathscr{F}^k(X)$ of frames of order k.*

Remark. 5.17. The last statement can be re-phrased; this re-statement gives a sketch of its proof. Note, first, that $\mathscr{E}(X)$ and $\mathscr{F}^k(X)$ have essentially unique extensions to $\boldsymbol{R}^n$; $n = \dim X$. Further, the fiber F_θ of $\mathscr{F}^k(X \cup \boldsymbol{R}^n)$ over the origin $\mathcal{O}$ of $\boldsymbol{R}^n$ is a group (the same as G_θ), and it acts on the left on E_θ, the corresponding fiber of $\mathscr{E}(X \cup \boldsymbol{R}^n)$. Also, F_θ acts on the right on the total space F of $\mathscr{F}^k(X \cup \boldsymbol{R}^n)$. The principal map on $F \times E_\theta$ which factors out F_θ (cf. Steenrod [1951.1], p. 35ff) produces a natural bundle which is equivalent to $\mathscr{E}(X)$ and whose principal bundle is $\mathscr{F}^k(X)$.

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