Algebraic Homotopy Theory

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1. SEMI-SIMPLICIAL COMPLEXES

In classical algebraic topology one studies simplicial complexes. However, modern developments have shown that these are inadequate, particularly for problems in homotopy theory. In recent years there has been a tendency to study the total singular complex of a space (cf. example 2 below) instead of simplicial complexes; but this method is also inconvenient from the point of view of homotopy. A more useful procedure seems to be the study of abstract semi-simplicial complexes, introduced by Eilenberg and Zilber [1], and of the sub-class consisting of semi-simplicial complexes satisfying the extension condition of Kan (cf. definition 1.2 below).

Let Z^+ denote the set of non-negative integers.

Definition 1.1: A semi-simplicial complex consists of the following:

(i) A set $X = \bigcup_{q \in Z^+} X_q$,
where the X_q are disjoint sets (an element of X_q is called a q-simplex of X);

(ii) functions $\partial_i : X_{q+1} \longrightarrow X_q$, $i = 0,\ldots,q+1$, called face operators;
functions $s_i : X_q \longrightarrow X_{q+1}$, $i = 0,\ldots,q$, called degeneracy operators, satisfying the relations

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad i < j ,$$
$$s_i s_j = s_{j+1} s_i \qquad i \leq j ,$$

$$\partial_i s_j = s_{j-1} \partial_i \qquad i < j ,$$

$$\partial_j s_j = \partial_{j+1} s_j = \text{identity}$$

$$\partial_i s_j = s_j \partial_{i-1} \qquad i > j + 1$$

We shall usually denote a semi-simplicial complex by its set X of simplexes.

A simplex $x \in X_{n+1}$ is called <u>degenerate</u> if there exists $y \in X_n$ and a degeneracy operator s_j such that $x = s_j y$; otherwise x is called <u>non-degenerate</u>.

<u>Example 1</u>: Recall that a simplicial complex K is a set whose elements are finite subsets of a given set $\bar{K}$, subject to the condition that if $x \in K$ and y is a non-empty subset of x, then $y \in K$. Sets with n+1 elements are called n-simplexes, and the set of n-simplexes of K is denoted by K_n.

We now define a semi-simplicial complex $X(K)$ which arises from K in a natural manner. An n-simplex of $X(K)$ is a sequence $(a_0,\ldots,a_n)$ of elements of $\bar{K}$ such that the set $\{a_0,\ldots,a_n\}$ is an r-simplex of K for some $r \leq n$. Define

$$\partial_i(a_0,\ldots,a_n) = (a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n),$$

$$s_i(a_0,\ldots,a_n) = (a_0,\ldots,a_i,a_i,a_{i+1},\ldots,a_n).$$

<u>Example 2</u>: Let Δ_n denote the standard n-simplex, so that a point of Δ_n is an (n+1)-tuple $(t_0,\ldots,t_n)$ of real numbers such that $0 \leq t_i \leq 1$, $i = 0,\ldots,n$, and $\sum t_i = 1$. Let A be a topological space. A singular n-simplex of A is a map* $u: \Delta_n \longrightarrow A$. Let $S_n(A)$ be the set of singular n-simplex in

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*by "map" we shall always mean a continuous function, provided both the domain and image are topological spaces.

A, and set $S(A) = \bigcup_{n \in Z^+} S_n(A)$. Define

$$\partial_i : S_n(A) \longrightarrow S_{n-1}(A)$$

by $\partial_i \, u(t_0,\ldots,t_{n-1}) = u(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1})$,

and define

$$s_i : S_n(A) \longrightarrow S_{n+1}(A)$$

by $s_i \, u(t_0,\ldots,t_{n+1}) = u(t_0,\ldots,t_{i-1},t_i+t_{i+1},t_{i+2},\ldots,t_{n+1})$.

It is easy to verify that $S(A)$ is a semi-simplicial complex, the <u>total singular complex</u> of the space A, [2].

In the examples we have seen two ways in which semi-simplicial complexes arise; henceforth we shall consider abstract semi-simplicial complexes. For problems in homotopy theory it is convenient to restrict attention to semi-simplicial complexes satisfying the following condition:

<u>Definition 1.2</u> A semi-simplicial complex X is said to satisfy the <u>extension condition</u> if given $x_0,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+1} \in X_n$ such that $\partial_i x_j = \partial_{j-1} x_i$, $i < j$, $i, j \neq k$, then there exists $x \in X_{n+1}$ such that $\partial_i x = x_i$, $i \neq k$. Such a complex will be called a <u>Kan complex</u>.

<u>Proposition 1.3</u>: If A is a topological space, then the total singular complex $S(A)$ satisfies the extension condition.

The proposition follows from the fact that the union of n+1 faces of Δ_{n+1} is a retract of Δ_{n+1}; thus a given map defined on the union of the n+1 faces can always be extended to Δ_{n+1}.

Although it has long been realized that the total singular complex satisfies the extension condition, it was only recently that D.M.Kan pointed out that the extension condition is sufficient for the definition of homotopy groups.

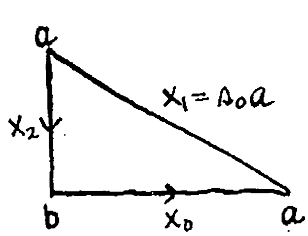
Definition 1.4: Let X be a semi-simplicial complex. A point of X is a 0-simplex, i.e. an element of X_0; and a path in X is a 1-simplex, i.e. an element of X_1. If x is a path in X, then $\partial_1 x$ is the initial point or origin of x, and $\partial_0 x$ is the final or terminal point of x.

Note that if A is a topological space, then a path in A is a map $u : \Delta_1 \longrightarrow A$, and therefore a path in $S(A)$. Further, the initial and final points of the path considered as an element of $S(A)$ are the same as when considered as the map $u : \Delta_1 \longrightarrow A$.

Let X be a Kan complex. The point $a \epsilon X$ is said to be in the same path component as the point $b \epsilon X$ if there exists a path with initial point a and final point b.

Proposition 1.5: The relation "to be in the same path component" is an equivalence relation.

Proof: (i) To show that the relation is symmetric, let x_2 be a path from a to b, and let $x_1 = s_0 a$. Now $\partial_1 x_2 = a = \partial_1 s_0 a = \partial_1 x_1$. Consequently there exists $x \epsilon X_2$ such that $\partial_i x = x_i$, $i = 1,2$. Let $x_0 = \partial_0 x$. Then $\partial_0 x_0 = \partial_0 \partial_0 x = \partial_0 \partial_1 x = \partial_0 x_1 = \partial_0 s_0 a = a$, and $\partial_1 x_0 = \partial_1 \partial_0 x = \partial_0 \partial_2 x = \partial_0 x_2 = b$. Therefore x_0 is a path from b to a.



(ii) To show that the relation is transitive, let x_1, x_0 be paths from a to b and b to c respectively. Then $\partial_0 x_2 = b = \partial_1 x_0$. Let x be a 2-simplex such that $\partial_0 x = x_0, \partial_2 x = x_2$, and let $x_1 = \partial_1 x$. Then x_1 is a path in X from a to c.

(iii) That the relation is reflexive is clear.

Let $\pi_0(X)$ denote the set of path components of X. X is called <u>connected</u> if $\pi_0(X)$ has only one element.

<u>Definition 1.6</u>: If X is a semi-simplicial complex, and $x^* \in X_0$, define $\Omega(X; x^*)$ as follows:

i) $\Omega_n(X,x^*) = \{ x | x \in X_{n+1}, \partial_0 x = s_0^n x^*, \partial_{i_0} \cdots \partial_{i_n} x = x^* \}$

where $0 \leq i_k \leq n + 1$, $k = 0,\ldots,n$.

ii) $\partial_i : \Omega_{n+1}(X,x^*) \longrightarrow \Omega_n(X,x^*)$ is the function determined by $\partial_{i+1} : X_{n+2} \longrightarrow X_{n+1}$, $i = 0,\ldots,n + 1$

iii) $s_i : \Omega_n(X,x^*) \longrightarrow \Omega_{n+1}(X,x^*)$ is the function determined by $s_{i+1} : X_{n+1} \longrightarrow X_{n+2}$, $i = 0,\ldots,n$

iv) $\Omega(X,x^*) = \bigcup_{n \in Z^+} \Omega_n(X, x)$

<u>Theorem 1.7</u>: If X is a semi-simplicial complex, and $x^* \in X_0$, then

i) $\Omega(X,x^*)$ is a semi0simplcial complex

ii) If X satisfies the extension condition, then so does $\Omega(X,x^*)$.

The proof of this theorem is straightforward, and will be left to the reader.

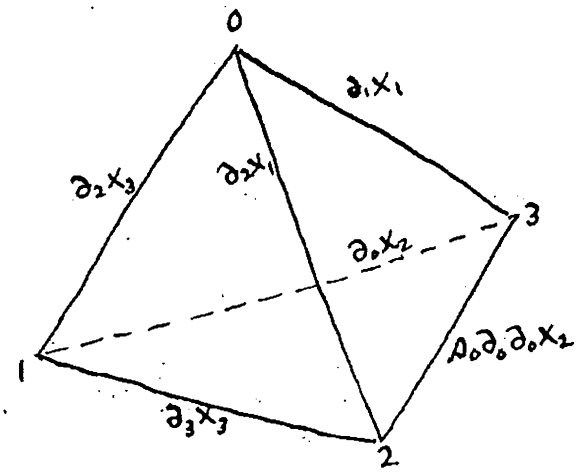
Proposition 1.8: If X is a Kan complex, and $x_2, x_3 \in X_2$ are such that $\partial_0 x_3 = \partial_0 x_2$, $\partial_2 x_3 = \partial_2 x_2$, then there exists $x_1 \in X_2$ such that $\partial_0 x_1 = s_0\partial_0\partial_0 x_2$, $\partial_1 x_1 = \partial_1 x_2$, $\partial_2 x_1 = \partial_1 x_3$

Proof: Let $x_0 = s_1\partial_0 x_2$. Then $\partial_0 x_3 = \partial_0 x_2 = \partial_2 x_0$, $\partial_0 x_2 = \partial_1 x_0$, and there exists $x \in X_3$ such that $\partial_i x = x_i \quad i \neq 1$. Let $x_1 = \partial_1 x$. Then

$\partial_0 x_1 = \partial_0\partial_1 x = \partial_0\partial_0 x = \partial_0 s_1 \partial_0 x_2 = s_0\partial_0\partial_0 x_2$,

$\partial_1 x_1 = \partial_1\partial_1 x = \partial_1\partial_2 x = \partial_1 x_2$, and

$\partial_2 x_1 = \partial_2\partial_1 x = \partial_1\partial_3 x = \partial_1 x_3$.



Notation and Convention: If X is a semi-semplicial complex, and x^* is a point of X, let $\Omega^0(X,x^*) = X$, and let $\Omega^{n+1}(X,x^*) = \Omega(\Omega^n(X,x^*), s_0^n x^*)$. The point $s_0^n x^* \in \Omega^n{}_0(X,x^*)$ (here s_0 denotes the degeneracy operator in X) is the natural base point for $\Omega^n(X,x^*)$.

Definition 1.9: If X is a Kan complex, and x^* is a point of X, define $\pi_n(X,x^*)$ to be $\pi_0(\Omega^n(X,x^*))$.

Now $\pi_n(X,x^*)$ is the set we wish to make into the n-dimensional homotopy group of X. Therefore it remains to define a multiplication in $\pi_n(X,x^*)$ for $n > 0$. However, to do this it is sufficient to define a multiplication in $\pi_1(X,x^*)$ since $\pi_{n+1}(X,x^*) = \pi_1(\Omega^n(X,x_0),\ s_0^n x^*)$.

Let X be a Kan complex and $x^* \in X_0$. According to the preceding proposition there is a map of $\Omega_0(X,x^*) \times \Omega_0(X,x^*) \longrightarrow \pi_0(\Omega(X,x^*))$ defined as follows:

if $x,y \in \Omega_0(X,x^*) \subset X$, then there exists $w \in X_2$ and $z \in \Omega_0(X,x^*)$ such that $\partial_2 w = x, \partial_0 w = y, \partial_1 w = z$. Let $[z]$ denote the image of z in $\pi_0(\Omega(X,x^*))$. Although z is not unique, $[z]$ is so, according to the preceding proposition. We therefore denote $[z]$ by $x \cdot y$, and the desired map is given by $(x,y) \longrightarrow x \cdot y$.

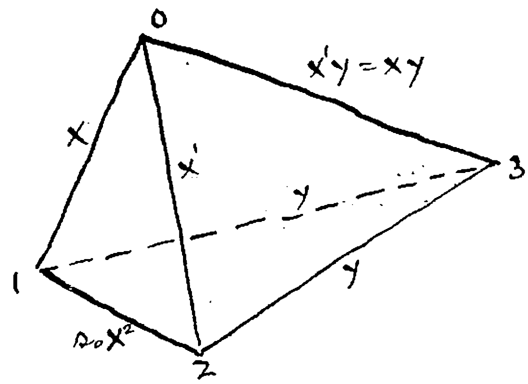
<u>Proposition 1.10</u>: If $x,x^1,y \in \Omega_0(X,x^*)$, and x,x^1 represent the same element of $\pi_0(\Omega(X,x^*))$, then $x \cdot y = x^1 \cdot y$.

<u>Proof</u>: Since $[x] = [x^1]$, there exists $z \in X_2$ such that $\partial_1 z = x^1, \partial_2 z = x, \partial_0 z = s_0 x^*$. By the extension condition, there exists $a \in X_2$ such that $\partial_0 a = y, \partial_2 a = x^1, \partial_1 a = x^1 y$, and there exists $b \in X_3$ such that $\partial_0 b = s_0 y, \partial_1 b = a, \partial_3 b = z$. Setting $c = \partial_2 b$, we have

$$\partial_0 c = \partial_0 \partial_2 b = \partial_1 \partial_0 b = \partial_1 s_0 y = y,$$

$$\partial_2 c = \partial_2 \partial_2 b = \partial_2 \partial_3 b = \partial_2 z = x.$$

therefore $\partial_1 c = x \cdot y$; but $\partial_1 c = \partial_1 \partial_2 b = \partial_1 \partial_1 b = \partial_1 a = x^1 y$, and the proposition follows.



<u>Proposition 1.11</u>: If $x, y, y' \in \Omega_0(X, x^*)$, and $[y] = [y']$, then $x \cdot y = x \cdot y'$.

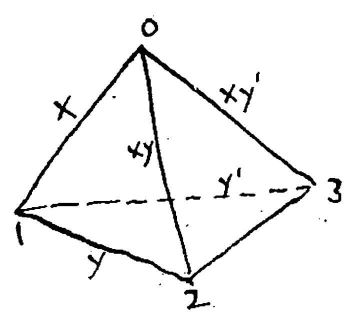
<u>Proof</u>: By hypothesis there exist $a, b, z \in X_2$ such that

$$\partial_0 a = y, \partial_2 a = x, \partial_1 a = xy$$
$$\partial_0 b = y', \partial_2 b = x, \partial_1 b = xy'$$
$$\partial_0 z = s_0 x_0, \partial_1 z = y', \partial_2 z = y$$

Then by the extension condition there exists $c \in X_3$ such that

$$\partial_0 c = z, \partial_2 c = b, \partial_3 c = a.$$

Let $d = \partial_1 c$; then



$$\partial_2 d = \partial_2 \partial_1 c = \partial_1 \partial_3 c = \partial_1 a = xy$$
$$\partial_1 d = \partial_1 \partial_1 c = \partial_1 \partial_2 c = \partial_1 b = xy'$$
$$\partial_0 d = \partial_0 \partial_1 c = \partial_0 \partial_0 c = \partial_0 z = \partial_0 s_0 x_0 = x_0.$$

Therefore $x \cdot y = x \cdot y'$.

According to propositions 1.10, 1.11 there is a map

$$\pi_0(\Omega_0(X,x^*) \times \pi_0(\Omega_0(X,x^*)) \longrightarrow \pi_0(\Omega(X,x^*))$$

given by $[x] \cdot [y] = x \cdot y$.

If X is a Kan complex, we shall denote by $\pi_n(X,x^*)$ the set previously defined together with this multiplication. We shall use Greek letters to denote the elements of $\pi_n(X,x^*)$.

<u>Theorem 1.12</u>: If X is a Kan complex, $x^* \in X_0$, then $\pi_n(X,x^*)$ is a group for $n \geq 1$.

<u>Proof</u>: Let $\alpha, \beta, \gamma \in \pi_n(X,x^*) = \pi_0(\Omega^n(X,x^*)) = \pi_0(\Omega(\Omega^{n-1}(X,x^*)))$.

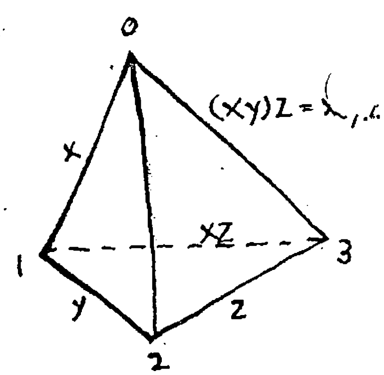
have representatives $x,y,z \in \Omega_0(\Omega^{n-1}(X,x^*)) \subset \Omega_1^{n-1}(X,x^*)$

i) Associativity: There exist $a_0, a_1, a_3 \in \Omega_2^{n-1}(X,x^*)$ such that $\partial_0 a_0 = z$, $\partial_2 a_0 = y$, $\partial_1 a_0 = yz$

$\partial_0 a_1 = z$, $\partial_2 a_1 = xy$, $\partial_1 a_1 = (yx)z$

$\partial_0 a_3 = y$, $\partial_2 a_3 = x$, $\partial_1 a_3 = xy$

By the extension condition there exists $b \in \Omega_3^{n-1}(X,x^*)$ such that $\partial_i b = a_i$, $i = 0,1,3$. Set $a_2 = \partial_2 b$. Then



$\partial_0 a_2 = \partial_0 \partial_2 b = \partial_1 \partial_0 b = \partial_1 a_0 = yz$

$\partial_2 a_2 = \partial_2 \partial_2 b = \partial_2 \partial_3 b = \partial_2 a_3 = x$

and therefore $\partial_1 a_2 = x(yz)$. But

$x(y_1 z) = \partial_1 a_2 = \partial_1 \partial_2 b = \partial_1 \partial_1 b = \partial_1 a_1 = (xy)z$.

ii) A left identity is furnished by $s_0 x_0$; for $s_0 x \in \Omega_2^{n-1}(X,x^*)$ has as faces $\partial_0 s_0 x = x$, $\partial_2 s_0 x = s_0 \partial_1 x = s_0 x_0$, $\partial_1 s_0 x = x$; and hence $(s_0 x)x = x$.

iii) Left inverse:

By the extension condition there exists $a \in \Omega_2^{n-1}(X,x^*)$ such that $\partial_0 a = x$, $\partial_1 a = s_0 x_0$;

then by definiton

$$(\partial_2 a)x = s_0 x_0,$$

so that $\partial_2 a$ is a left inverse for x.

In order to see the connection between the homotopy groups of a Kan complex X and homotopy groups as classically defined it is convenient to define $\Omega^n(X,x^*)$ directly, instead of inductively. We therefore write down the explicit definition of $\Omega^n(X,x^*)$ using elements of X, and face and degeneracy operators of X.

$\Omega^n_q(X,x^*) = \{x \mid x \in X_{n+q}, \partial_i x = s_0^{n+q-1}x^*$ for $i < n$, and $\partial_{i_0} \cdots \partial_{i_q} x = s_0^{n-1}x^*\}$. This definition is easily seen to coincide with that originally given. Now $\Omega^n_0(X,x^*) = \{x \mid x \in X_n, \partial_i x = s_0^{n-1}x^*$ for $i < n$, and $\partial_{i_0} x = s_0^{n-1}x^*\}$. Therefore an element of $\Omega^n_0(X,x^*)$ is an n-simplex of X all of whose faces are at the base point, and an element of $\pi_n(X,x^*)$ is an equivalence class of such simplexes. Two such simplexes x, x' are equivalent if there exists $z \in X_{n+1}$ such that $\partial_{n+1} z = x, \partial_n z = x'$, and $\partial_i z = s_0^n x^*$ for $i < n$. Futher, if x, x' are two n-simplexes all of whose faces are at the base point, then $[x \cdot x']$ is represented as follows: By the extension condition there exists $z \in X_{n+1}$ such that $\partial_{n+1} z = x, \partial_{n-1} z = x'$, and $\partial_i z = s_0^n x^*$ for $i < n$. $[x \cdot x']$ is represented by $\partial_n z$.

<u>Definition 1.13:</u> If X,Y are semi-simplicial complexes, then $f : X \longrightarrow Y$ is a <u>semi-simplicial map</u> if

1) $f(X_q) \subset Y_q$,

2) $f\partial_i = \partial_i f$, all i, and

3) $s_i f = f s_i$, all i .

We shall often denote $f|X_q$ by f_q .

Definition 1.14: If X, Y are Kan complexes and $f:X \longrightarrow Y$ is a semi-simplicial map, then for every $q \geq 0$ f induces a function

$$f_q^{\#} : \pi_q(X,x^*) \longrightarrow \pi_q(Y,f(x^*))$$

by $f_q^{\#}[x] = [f_q x]$, for $x \in \Omega_0^q(X,x^*)$.

Proposition 1.15: The function $f_q^{\#}$ is a homomorphism for $q > 0$.

The proof is evident from the definition.

Proposition 1.16: Let A, B, C be Kan complexes

i) If $f: A \longrightarrow B$, $g : B \longrightarrow C$, are semi-simplicial maps, and $a^* \in A_0$, then

$$(gf)_q^{\#} = g_q^{\#} f_q^{\#} : \pi_q(X,a^*) \longrightarrow \pi_q(C,gf(a^*)).$$

ii) If 1 is the identity map of A, then $1_q^{\#}$ is the identity automorphism of $\pi q(A,a^*)$.

It is convenient to derive some of the relations between the faces of a 3-simplex. For the following five propositions let X be a Kan complex, $x \in X_0$.

Proposition 1.17: Let x_3 be 3-simplex such that $\partial_i \partial_j x_3 = s_0 x$, all i, j. Let the faces of x_3 be a, b, c, $s_0^2 x$, in order. Then [a] [c] = [b].

Proof: It is straightforward to check that the following four 3-simplexes satisfy the extension condition:

$x_0 = s_2 a$

x_1 has faces $\partial_0 x_1 = s_0^2 x, \partial_2 x_1 = b, \partial_3 x_1 = a$, and is then obtained by the extension condition. Set $w = \partial_1 x_1$.

x_3 as given

$x_4 = s_0 a$.

Therefore there exists a 4-simplex z such that $\partial_i z = x_i$, $i \neq 2$. Let $x_2 = \partial_2 z$.

Then $\partial_0 x_2 = s_0^2 x, \partial_1 x_2 = w, \partial_2 x_2 = c, \partial_3 x_2 = s_0^2 x$.

Therefore, by the rule for addition, $[w] = [c]$. But from x_1 we have $[a][w] = [b]$; therefore $[a][c] = [b]$.

Proposition 1.18: Let x_2 be a 3-simplex such that $\partial_i \partial_j x_2 = s_0 x$, all i, j. Let the faces of x_2 be $a, s_0^2 x, c, d$, in order. Then $[c][a] = [d]$.

Proof: The following four 3-simplexes satisfy the extension condition:

$x_0 = s_0 a$

x_1 has faces $\partial_0 x_1 = a, \partial_1 x_1 = \partial_3 x_1 = s_0^2 x$, and is obtained by extension. Let $y = \partial_2 x_1$.

x_2 as given

$x_4 = s_2 d$

Therefore there exists a 4-simplex z such that $\partial_i z = x_i$, $i \neq 3$. Set $x_3 = \partial_3 z$

Then $\partial_0 x_3 = s_0^2 x, \partial_1 x_3 = y$, $\partial_2 x_3 = c, \partial_3 x_3 = d$. Therefore $[d][y] = [c]$. From x_1 and 1.17 we have $[y] = [a]^{-1}$. Therefore $[d] = [c][a]$.

Proposition 1.19: Let x_4 be a 3-simplex such that $\partial_i \partial_j x_4 = s_0 x$, all i, j. Let the faces of x_4 be a,b,c,d in order. Then $[d][b][a]^{-1} = [c]$.

Proof: The following four 3-simplexes satisfy the extension condition:

x_0 has faces $\partial_1 x_0 = \partial_2 x_0 = s_0 x, \partial_3 x_0 = a$, and is obtained by extension.

set $v = \partial_0 x_0$.

x_1 has faces $\partial_0 x_1 = v, \partial_1 x_1 = s_0 x, \partial_3 x_0 = b$, and is obtained by extension.

set $w = \partial_2 x_1$.

$x_2 = s_2 c$

x_4 as given.

Therefore there exists a 4-simplex z such that $\partial_i z = x_i, i \neq 3$. Set $x_3 = \partial_3 z$.

By 1.18, $[v] = [a]$, and $[w] = [b][v]^{-1} = [b][a]^{-1}$.

x_3 has faces $\partial_0 x_3 = s_0^2 x, \partial_1 x_3 = w, \partial_2 x_3 = c, \partial_3 x_3 = d$.

Therefore $[c] = [d][w] = [d][b][a]^{-1}$.

Setting $d = s_0^2 x$ in 1.19, x_4 then has faces $a, b, c, s_0^2 x$, in order, and the relation $[c] = [b][a]^{-1}$ holds. But 1.17 applies to the simplex x_4 to give the relation $[c] = [a]^{-1}[b]$. Therefore, for arbitrary $[a]$ and $[b]$, $[b][a]^{-1} = [a]^{-1}[b]$, or

$[a][b] = [b][a]$, and π_2 is therefore abelian. Since the higher homotopy groups were defined by iteration, we have

Corollary 1.20: $\pi_n(X,x)$ is abelian for $n \geq 2$.

We shall henceforth write π_n additively for $n \geq 2$.

Proposition 1.21: Let $z \in X_{q+1}, q \geq 2$, be such that

(1) $\partial_r z = a, \partial_{r+1} z = b, \quad 0 \leq r \leq q$

(2) $\partial_i z = s_0^q x, \ i \neq r, r+1$

(3) $\partial_j \partial_k z = s_0^{q-1} x$, all j,k.

Then $[a] = [b]$.

Proof: If $r = q$, the proposition follows from the definition of homotopy classes. Suppose $r < q$; then the following set of $q+1$ $(q+1)$-simplexes satisfies the extension condition:

$y_i = s_0^{q+1} x$ for $i < r$ and $i > r+3$

$y_{r+1} = s_{r+1} b$

$y_{r+z} = z$

$y_{r+3} = s_r b$

Then there exists $y \in X_{q+2}$ such that $\partial_i y = y_i$, $i \neq r$. $y_r = \partial_r y$ has faces $\partial_i y_r = s_0^q x$, $i \neq r+1, r+2$; $\partial_{r+1} y = a$, $\partial_{r+2} y = b$.

If we iterate this process $q-r$ times we obtain a $(q+1)$-simplex y' such that

$$\partial_i y' = s_0^q x, \ , i < q, \ \partial_q y' = a, \ \partial_{q+1} y' = b.$$

Hence $[a] = [b]$.

<u>Proposition 1.22:</u> Let X be a Kan complex $x \in X_0$. Let $z \in X_{q+1}$, $q \geq 2$, be such that (1) $\partial_{r-1} z = a$, $\partial_r z = b$, $\partial_{r+1} z = c$, where $1 \leq r \leq q$, (2) $\partial_i z = s_0^q x$, $i \neq r-1, r, r+1$. (3) $\partial_j \partial_k z = s_0^{q-1} x$, all j,k.

Then $[b] = [c][a] = [a][c]$.

<u>Proof</u>: Hypothesis (3) implies that a,b,c represent elements of $\pi_q(X,x)$; and since this group is abelian, $[c][a] = [a][c]$. If $r = q$, $[b] = [c][a]$ is just the definition of the group operation. If $r < q$, then the following set of q+1 (q+1)-simplexes satisfies the extension condition:

$y_i = s_0^{q+1} x$ for $i < r$ and $i > r + 4$

$y_i = s_{r+2} a$

y_{r+2} has faces $\partial_i y_{r+2} = s_0^q x$, $i \neq r+1, r+2$, $\partial_{r+2} y_{r+2} = c$, and is obtained by extension. Let $w = \partial_{r+1} y_{r+2}$.

$y_{r+3} = z$

$y_{r+4} = s_r a$.

Then there exists $y \in X_{q+2}$ such that $\partial_i y = y$, $, i \neq r+1$.

$y_{r+1} = \partial_{r+1} y$ has faces $\partial_i y_{r+1} = s_0^q x$, $i < r + 1$ or $i > r + 3$, $\partial_{r+1} y_{r+1} = w$, $\partial_{r+2} y_{r+1} = b$, $\partial_{r+3} y_{r+1} = a$.

By the previous proposition, $[w] = [c]$.

It is easy to see that by iterating this process q-r-1 times we obtain a (q+1)-simplex y' such that

$\partial_i y' = s_0^q x$, $i < q - 1$, ${}_{q-1} y' = w'$,

$\partial_q y' = b$, $\partial_{q+1} y' = w''$, and such that either $[w'] = [a], [w''] = [c]$, or $[w'] = [c], [w''] = [a]$.

In either case $[b] = [c][a] = [a][c]$.

Definition 1.23: A semi-simplicial fiber space is a triple (E, p, B) where E, B are semi-simplicial complexes, and $p: E \longrightarrow B$ is a semi-simplicial map, satisfying the following condition: if $x \in B_{q+1}$, $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_{q+1} \in E_q$ are such that $p(y_i) = \partial_i x$ for $i \neq k$, and $\partial_i y_j = \partial_{j-1} y_i$, for $i < j$, $i, j \neq k$, then there exists $y \in E_{q+1}$ such that $p(y) = x$, and $\partial_i y = y_i$ for $i \neq k$.

Let b be a point of B, and let $F_q = \{x | x\ E_q, p(x) = s_0^q b\}$. Let $F = \bigcup F_g$, and defines $\partial_i : F_{g+1} \longrightarrow F_q$ to be the function induced by $\partial_i : E_{q+1} \longrightarrow E_q$, and $s_i : F_q \longrightarrow F_{q+1}$ to be the function induced by $s_i : E_q \longrightarrow E_{q+1}$. Now F is a semi simplicial complex called the fibre over b.

Proposition 1.24: F is a Kan complex.

Proof: Suppose $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1} \in F_q$ are such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$, $i, j \neq k$. Then $p(x_i) = s_0^q b$; and since (E, p, B) is a fibre space, there exists $x \in E_{q+1}$ such that $p(x) = s_0^{q+1} b$, and $\partial_i x = x_i$ for $i \neq 1$. Since $p(x) = s_0^{q+1} b$, $x \in F_{q+1}$, which proves the proposition.

Now let (E, p, B) be a semi-simplicial fibre space in which E and B are Kan complexes, and let F be the fibre over a point b of B. Let a be a point of F, which we assume to be non-empty.

For $q \geq 2$ we define a homomorphism

$$\partial^{\#} : \pi_q(B,b) \longrightarrow \pi_{q-1}(F,a)$$

as follows. Recall that an element $\alpha \in {}_q(B,b)$ is represented by $x \in B_q$ such that $\partial_i = s_0^{q-1}b$ for all i. Since p is a fibre map, there exists $y \in E_q$ such that $p(y) = x$ and $\partial_i y = s_0^{q-1}a$ for $i > 0$. Then $\partial_0 y$ is contained in F_{q-1}, and represents an element of $\pi_{q-1}(F,a)$. Suppose $x' \in B_q$ also represents α. Then there exists $z \in B_{q+1}$ such that $\partial_i z = s_0^q b$, $i < q$, $\partial_q z = x$, $\partial_{q+1} z = x'$. Let $y' \in E_q$ be such that $p(y) = x'$ and $\partial_i y = s_0^{q-1}a$ for $i > 0$. Since p is a fibre map, there exists $w \in E_{q+1}$ such that $p(w) = z$, $\partial_i w = s_0^q a$, $0 < i < q$, $\partial_q w = y$, $\partial_{q+1} w = y'$. Now $p(\partial_0 w) = s_0^q b$: and $\partial_i \partial_0 w = \partial_0 \partial_{i+1} w = s_0^q a$, $i < q-1$, $\partial_{q-1}\partial_0 w = \partial_0 y$, $\partial_q \partial_0 w = \partial_0 y'$. Therefore $[\partial_0 y] = [\partial_0 y']$ in $\pi_{q-1}(F,a)$. Since in particular we may take $x' = x$, the element $[\partial_0 y]$ is independent of both the choice of x representing $[x]$ and the choice of y. We set $\partial^{\#}\alpha = [\partial_0 y]$.

We now show that $\partial^{\#}$ is a homomorphism. Let $\alpha, \beta \in \pi q(B,b)$ have representatives x, x' respectively. Let $z \in B_{q+1}$ have faces

$$\partial_i z = s_0^q b, \quad i < q-1, \quad \partial_{q-1} z = x', \quad \partial_{q+1} z = x.$$

Then $\partial_q z$ represents $\alpha + \beta$. Let $v \in E_{q+1}$ be such that $p(v) = z$, $\partial_i v = s_0^q a$, $0 < i < q-1$, and

$\partial_i \partial_j v = s_0^{q-1} a$ for $j = q-1, q, q+1$, and $i > 0$.

Then $\partial_0 v \in F_q$, and

$\partial_i \partial_0 v = s_0^{q-1} a$ for $i < q-2$, $\partial_{q-2} \partial_0 v = \partial_0 \partial_{q-1} v$,

$\partial_{q-1} \partial_0 v = \partial_0 \partial_q v, \partial_q \partial_0 v = \partial_0 \partial_{q+1} v$.

Since $\partial_0 \partial_{q-1} v, \partial_0 \partial_q v, \partial_0 \partial_{q+1} v$ represent

$\partial^{\#}\beta$, $\partial^{\#}(\alpha + \beta)$, $\partial^{\#}\alpha$ respectively, from their relationship as faces of $\partial_0 v$ it follows that

$\partial^{\#}(\alpha + \beta) = \partial^{\#}\alpha + \partial^{\#}\beta$.

Theorem 1.25: Let (E,p,B) be a semi-simplicial fibre space in which E and B are Kan complexes. Let $b \in B_0$, F the fibre over b, $a \in F_0$ (we assume F non-empty). Let $i : F \longrightarrow E$ be the inclusion map. Then the following sequence is exact:

$$\cdots \longrightarrow \pi_q(F,a) \xrightarrow{i^{\#}} \pi_q(E,a) \xrightarrow{p^{\#}} \pi_q(B,b) \xrightarrow{\partial^{\#}} \pi_{q-1}(F,a) \longrightarrow \cdots$$

Proof: Let x represent $\alpha \in \pi_q(F,a)$. Then $pix = s_0^q b$, and consequently $p^{\#} i^{\#} = 0$. If x represents $\alpha \in \pi_q(E,a)$, then $\partial_0 x = s_0^{q-1} a$ represents $\partial^{\#} p^{\#} \alpha$ and $\partial^{\#} p^{\#} = 0$. Again, let x represent $\alpha \in \pi_q(B,b)$. Let $y \in E_q$ be such that $\partial_i y = s_0^{q-1} a$, $0 < i$, and $p(y) = x$. Then $\partial_0 y$ represents $i^{\#}\partial^{\#}\alpha$; but as an element of $\pi_{q-1}(E,a)$, by proposition 1.21, $[\partial_0 y] = [\partial_1 y] = [s_0^{q-1} a] = 0$, and $i^{\#}\partial^{\#} = 0$.

If x represents $\alpha \in \pi_q(F,a)$ and $i^{\#}(\alpha) = 0$, then there exists $y \in E_{q+1}$ such that $\partial_i y = s_0^q a$, $i < q+1$, and $\partial_{q+1} y = x$. Therefore $\partial_i p(y) = s_0^q b$, $i \leq q+1$, and $\partial^{\#}[p(y)] = \alpha$.

Suppose that x represents $\alpha \in \pi_q(E,a)$ such that $p^{\#}\alpha = 0$. Then we may assume that $x \in F_q$, and thus $i^{\#}[x] = \alpha$.

Finally suppose that x represents $\alpha \in \pi_q(B,b)$ such that $\partial^{\#}\alpha = 0$. Then there exists $y \in E_q$ such that $p(y) = x, \partial_i y = s_0^{q-1}a$, $0 < i$, and $[\partial_0 y] = [s_0^{q-1}a]$ in $\pi_{q-1}(F,a)$. Therefore, since p is a fibre map, there exists $y' \in E_q$ such that $p(y') = x$ and $\partial_i y' = s_0^{q-1}a$, all i. Then $p^{\#}[y'] = \alpha$.

This completes the proof of the theorem.

<u>Proposition 1.26:</u> Let (E,p,B) be a fibre space, $p:E \longrightarrow B$ be onto, $x \in B_q$, and let $y_{i_0},\dots,y_{i_r} \in E_{q-1}, 0 \leq i_0 < \dots < i_r \leq q$, be such that $\partial_{i_s} y_{i_t} = \partial_{i_t - 1} y_{i_s}$ for $s < t$, $\{i_0,\dots,i_r\} \neq \{0,\dots,q\}$, and $p(y_{i_s}) = \partial_{i_s} x$; then there exists $y \in E_q$ such that $p(y) = x$, and $\partial_{i_s} y = y_{i_s}$, $s = 0,\dots,r$.

<u>Proof</u>: If $q = 1$, then the proposition follows immediately from the definition of fibre space. Consequently suppose that the proposition is true for $q \leq n$, and that $q = n+1$. If the set $\{i_0,\dots,i_r\}$ has q elements, the result follows immediately from the definition of fibre space. In this case $r = q-1$. Suppose then that the proposition is true for $r \geq m$, $m \leq q-1$, and that $r = m-1 \geq 0$. Let $t \in \{0,\dots,q\}$ be the least integer such that $t \notin \{i_0,\dots,i_r\}$. Define

$j_s = i_s$ for $i_s < t$. Let s' be the largest integer s such that $i_s < t$. Define $j_{s'+1} = t$, and $j_s = i_{s-1}$ for $s' + 1 < s \leq r+1 = m$. We now wish to define y_t such that

$$\partial_{j_s} y_t = \partial_{t-1} y_{j_s} \quad s \leq s', \quad \partial_{j_{s-1}} y_t = \partial_t y_{j_s} \quad s \geq s'+2.$$

The set $\{j_0, \ldots, j_{s'}, j_{s'+2}, \ldots, j_{r+1}\}$ has at most $(q-1)$ elements. Therefore, using the inductive hypothesis, we may choose $y_t = y_{j_{s'+1}}$ such that $\partial_{j_s} y_t = \partial_{t-1} y_{j_s}$ $s \leq s'$, $\partial_{j_{s-1}} y_t = \partial_t y_{j_s}$ $s \geq s'+2$, and $p(y) = \partial_t x$. Now the set $\{j_0, \ldots, j_{r+1}\}$ has m elements; therefore by inductive hypothesis there exists $y \in E_q$ such that $p(y) = x$, and $\partial_{j_s} y = y_{j_s}$ $s = 0, \ldots, r+1$. Then $p(y) = x$, and $\partial_{i_s} y = y_{i_s}$ $i = 0, \ldots, r$.

Proposition 1.27: If (E,p,B) is a fibre space, and p is onto, then B is a Kan complex if and only if E is a Kan complex.

Proof: Let E be a Kan complex, and let $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ be elements of B_{q-1} such that $\partial_i x_j = \partial_{j-1} x_i$ $i < j$, $i,j \neq k$. Choose $y_0 \in E_{q-1}$ such that $p(y_0) = x_0$, choose $y_1 \in E_{q-1}$ such that $p(y_1) = x_1$ and $\partial_0 y_1 = \partial_0 y_0$, and continue in this manner until $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q$ have been chosen such that $\partial_i y_j = \partial_{j-1} y_i$, $i < j$, $i, j \neq k$, and $p(y_i) = x_i$ $i \neq k$. This procedure is possible by the preceding proposition. Now choose $y \in E_q$

such that $\partial_i y = y_i$ for $i \neq k$, and let $x = p(y)$. Then $\partial_i x = x_i$ for $i \neq k$, and B is a Kan complex

Now let B be a Kan complex, and let $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q$ be elements of E_{q-1} such that $\partial_i y_j = \partial_{j-1} y_i$, $i < j$, $i, j \neq k$. Let $x_i = p(y_i)$, and let $x \in B_q$ be an element such that $\partial_i x = x_i$ for $i \neq k$. Since p is a fibre map, there exists $y \in E_q$ such that $p(y) = x$ and $\partial_i y = y_i$ for $i \neq k$. Therefore E is a Kan complex.

Definition 1.28: Let X be a semi-simplicial complex. If $x, x' \in X_q$; and n is a non-negative integer then $x \overset{n}{\sim} x'$ if and only if $\partial_{i_1} \ldots \partial_{i_r} x = \partial_{i_1} \ldots \partial_{i_r} x'$ for every iterated face operator $\partial_{i_1} \ldots \partial_{i_r}$ such that $n + r \geq q$.

Lemma 1.29: If $x, x', x'' \in X_q$, then

1) $\overset{n}{\sim}$ is an equivalence relation
2) If $x \overset{n}{\sim} x'$, then $\partial_i x \overset{n}{\sim} \partial_i x'$, and
3) If $x \overset{n}{\sim} x'$, then $s_i x \overset{n}{\sim} s_i x'$.

Definition 1.30: Let X be a semi-simplicial complex. Define a semi-simplicial complex $X^{(n)}$ as follows:

1) An element of $X_q^{(n)}$ is an equivalence class of q-simplexes of X, $x, x' \in X_q$ being equivalent if $x \overset{n}{\sim} x'$,
2) $\partial_i : X_{q+1}^{(n)} \longrightarrow X_q^{(n)}$ is induced by $\partial_i : X_{q+1} \longrightarrow X_q$ and
3) $s_i : X_q^{(n)} \longrightarrow X_{q+1}^{(n)}$ is induced by $s_i : X_q \longrightarrow X_{q-1}$

Let $X^{(\infty)} = X$, and let $p_k^n : X^{(n)} \longrightarrow X^{(k)}$ be the natural map for $n \geq k$, where $\infty \geq k$, for every k. When there is no danger of confusion, p_k^n will be abbreviated by p.

Theorem 1.31: If X is a Kan complex, then $(X^{(n)}, p_k^n, X^{(k)})$ is a fibre space for $n \geq k$, and $X^{(n)}$ is a Kan complex.

Proof: We will first prove that $(X^{(\infty)}, p_k^{\infty}, X^{(k)})$ is a fibre space. Suppose that $x \in X_q^{(k)}$, and that $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q \in X_{q-1}$ are such that $\partial_i y_j = \partial_{j-1} y_i$, $i, j \neq k$, $i < j$, and $p(y_i) = \partial_i x$. Now if $q \leq k$, then $X_q^{(k)} = X_q$, and $y_i = \partial_i x$. Therefore if we choose $y = x$, then $y \in X_q, \partial_i y = y_i$, and $p(y) = x$. Assume therefore that $q > k$. Since X is a Kan complex there exists $y \in X_q$ such that $\partial_i y = y_i$ for $i \neq k$. Further any face of dimension $\leq n$ of y is also a face of some y_i. Therefore $p(y) = x$, and p is a fibre map. Now $X^{(\infty)} = X$ is a Kan complex, and p_k^{∞} is a fibre map. Therefore, $X^{(k)}$ is a Kan complex. The fact that $(X^{(n)}, p_k^n, X^{(k)})$ is a fibre space follows similarly, and the details will be left to the reader.

The fibre spaces $(X, p, X^{(n)})$ are closely related to the construction (II) of Cartan and Serre [2].

Notation: If X is a Kan complex, $x \in X_0$, let $E_n(X,x)$ denotes the fibre of $p : X \longrightarrow X^{(n-1)}$.

The complex $E_n(X,x)$ is the n-th Eilenberg subcomplex of X based at x.[3].

Theorem 1.32. Let X be a Kan complex, $x \in X_0$, and $i : E_{n+1}(X,x) \longrightarrow X$ the natural inclusion map. Then

1) $p^{\#} : \pi_q(X,x) \xrightarrow{\approx} \pi_q(X^{(n)},x)$ for $q \leq n$,

2) $\pi_q(X^{(n)},x) = 0$ for $q > n$,

3) $i^{\#} : \pi_q(E_{n+1}(X,x),x) \xrightarrow{\approx} \pi_q(X,x)$ for $q > n$.

4) $\pi_q(E_{n+1}(X,x),x) = 0$ for $q \leq n$.

Proof: Notice that $E_{n+1}(X,x)_q$ has a single element for $q \leq n$. This implies (4), and (4) implies (1) since $(X,p,X^{(n)})$ is a fibre space with fibre $E_{n+1}(X,x)$.

Let y represent $\alpha \in \pi_q(X^{(n)},x)$; then $\partial_i y = s_0^{q-1}x$ for all i. Now y is an equivalence class of simplexes $z \in X_q$, and the above condition on the faces of y implies that all faces of dimension $r \leq n$ of z are $s_0^r x$. Therefore $s_0^q x$ is in the class y, and $\alpha = 0$. This proves (2), which implies (3), using the exact sequence of the fibre space.

Definition 1.33: If X is a Kan complex, let $\mathfrak{X}^n = (X^{(n+1)}, p, X^{(n)})$. The sequence $\mathfrak{X} = (\mathfrak{X}^0, \mathfrak{X}^1, \ldots, \mathfrak{X}^n, \ldots)$ is defined to be the natural Postnikov system of X. [4].

Theorem 1.34. If X is a Kan complex, $\mathfrak{X}$ is the natural Postnikov system of X, x is a part of X, and if $F^{(n+1)}$ is the fibre over x in the fibre space $\mathfrak{X}^n$, then

$$\pi_q (F^{n+1}, x) = 0 \quad \text{for} \quad q \neq n+1$$
$$\pi_{n+1}(F^{(n+1)}, x) = \pi_{n+1} (X, x).$$

The proof, which follows easily from the previous theorems, will be omitted.

Definition 1.35: If X is a connected Kan complex, n is a positive integer, $\pi_q(X,x) = 0$ for $q \neq n$, and $\pi_n(X,x) = \pi$; then X will be called an Eilenberg-MacLane complex of type (π, n).

Thus what we have shown is that, in some sense, any Kan complex X can be constructed from Eilenberg-MacLane complexes, and that this is done by means of the natural Postnikov system of X.

References

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Chapter 1. Appendix A.

In Chapter 1, no general definition was given of homotopy between maps of one semi-simplicial complex into another. The purpose of this appendix is to rectify that situation, and further to prove after the manner of Eilenberg-Zilber ([1]), that every Kan complex is equivalent to a minimal subcomplex.

Definition: If X and Y are semi-simplicial complexes, the Cartesian product of X and Y is the semi-simplicial complex $X \times Y$ given by

1) $(X \times Y)_q = \{ (a,b) \mid a \in X_q, b \in Y_q \}$,
2) if $(a,b) \in (X \times Y)_{q+1}$, then $\partial_i(a,b) = (\partial_i a, \partial_i b)$ for $i = 0,\ldots,q+1$, and
3) if $(a,b) \in (X \times Y)_q$, then $s_i(a,b) = (s_i a, s_i b)$ for $i = 0,\ldots,q$.

Notation and Convention: Let Δ_q denote the semi-simplicial complex defined by the following:

1) an n-simplex is an (n+1)-tuple $(a_0,\ldots,a_n)$ of integers a_i such that $0 \leq a_0 \leq \cdots \leq a_i \leq a_{i+1} \leq \cdots \leq a_n \leq q$,
2) $\partial_i (a_0,\ldots,a_n) = (a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$, and
3) $s_i(a_0,\ldots,a_n) = (a_0,\ldots,a_{i-1},a_i,a_i,a_{i+1},\ldots,a_n)$.

The semi-simplicial complex Δ_q is the standard q-simplex, and itself has a canonical element of dimension q, namely

$(0,\ldots,q)$. If X is any semi-simplicial complex, and $x \in X_q$, there is a unique semi-simplicial map $f : \Delta_q \longrightarrow X$ such that $f((0,\ldots,q)) = x$. The semi-simplicial complex Δ_1 will also be denoted by I.

<u>Definition:</u> If X, Y are semi-simplicial complexes, $f_0, f_1 : X \longrightarrow Y$ are homotopic if there exists $F : X \times I \longrightarrow Y$ such that for any simplex σ of X,

1) $F(\sigma \times (0,\ldots,0)) = f_0(\sigma)$, and
2) $F(\sigma \times (1,\ldots,1)) = f_1(\sigma)$

The map F is a <u>homotopy from</u> f_0 to f_1. If A is a subcomplex of X, and $f_0|A = f_1|A$, then f_0 is said to be <u>homotopic to</u> f_1 <u>relative to</u> A if there exists a homotopy F from f_0 to f_1 such that $F(\sigma \times \tau) = f_0(\sigma)$ for $\sigma \in A$. The subcomplex A is a <u>deformation retract</u> of X, if the identity map of $X \longrightarrow X$ is homotopic relative to A to a map of X into A.

<u>Proposition 1:</u> If X and Y are semi-simplicial complexes, then $f_0, f_1 : X \longrightarrow Y$ are homotpic if and only if there exist functions $k_i : X_q \longrightarrow Y_{q+1}$ defined for $i = 0,\ldots,q$, and all q such that

1) $\partial_0 k_0 = f_1$,
2) $\partial_{q+1} k_q = f_0$,
3) $\partial_i k_j = k_{j-1} \partial_i$ $\quad i < j$.
4) $\partial_{j+1} k_{j+1} = \partial_{j+1} k_j$,
5) $\partial_i k_j = k_j \partial_{i-1}$ $\quad$ for $i > j+1$,
6) $s_i k_j = k_{j+1} s_i$ $\quad$ for $i \leq j$, and
7) $s_i k_j = k_j s_{i-1}$ $\quad$ for $i > j$.

If A is a subcomplex of X, and $f_0|A = f_1|A$, then f_0 is homotopic to f_1 relative to A if and only if $k_i(\sigma) = f_0(s_i(\sigma))$ for $\sigma \in A$.

Proof: Suppose that F is a homotopy connecting f_0 and f_1. Define $k_i(\sigma) = F(s_i\sigma \times s_q \ldots s_{i+1}s_{i-1} \ldots s_0(0,1))$ for $\sigma \in X_q$, $i = 0,\ldots,q$. The verification that the k_i's satisfy relation 1) - 7) is now a routine matter.

Suppose that there exist functions k_i satisfying 1) -7). Define $F(\sigma \times s_{q-1} \ldots s_{i+1}s_{i-1} \ldots s_0(0,1)) = \partial_{i+1}k_i(\sigma)$ for $\sigma \in X_q$, $i = 0,\ldots,q-1$, $F(\sigma \times s_0^q(0)) = f_0(\sigma)$ and $F(\sigma \times s_0^q(1)) = f_1(\sigma)$. Using relations 1) - 7), one sees readily that F is a semi-simplicial map, and hence a homotpy from f_0 to f_1.

<u>Notation and Convention:</u> For $i = 0,\ldots,q+1$, let $\lambda^i : \{0,\ldots,q\} \longrightarrow \{0,\ldots,q+1\}$ be the function defined by

$$\lambda^i(j) = j \qquad j < i, \quad \text{and}$$

$$\lambda^i(j) = j+1 \qquad j \geq i .$$

Similarly let $\eta^i : \{0,\ldots,q+1\} \longrightarrow \{0,\ldots,q\}$ be defined by

$$\eta^i(j) = j \qquad j \leq i, \quad \text{and}$$

$$\eta^i(j) = j-1 \qquad j > i \text{ for } i = 0,\ldots,q.$$

Further denote by $\lambda^i : \Delta_q \longrightarrow \Delta_{q+1}$ the semi-simplicial map defined by the function λ^i, and by $\eta^i : \Delta_{q+1} \longrightarrow \Delta_q$ the map defined by η^i.

We now wish to translate these definitions into a slightly different framework. In ordinary topology, if A and B are spaces, a map of A into B is a point in the function-space of maps of A into B, and this function-space is usually denoted by B^A. Following an idea of A. Heller, we shall now define the semi-simplicial analogue of a function-space.

Definition: If X and Y are semi-simplicial complexes, then Y^X is the semi-simplicial complex defined as follows:

1) $(Y^X)_q$ is the set of semi-simplicial maps $f : X \times \Delta_q \longrightarrow Y$, and

2) if $f : X \times \Delta_q \longrightarrow Y$, then $\partial_i f : X \times \Delta_{q-1} \longrightarrow Y$ is defined by

$\partial_i f = f(1 \times \lambda^i)$, where $1 : X \longrightarrow X$ is the identity map, and $s_i f : X \times \Delta_{q+1} \longrightarrow Y$ is defined by $s_i f = f(1 \times \eta^i)$.

Now, as in the geometric case, a homotopy between between $f_0, f_1 : X \longrightarrow Y$ is just a path in Y^X which starts at the point f_0 and ends at the point f_1. Consequently, for homotopy to be an equivalence relation it would suffice for Y^X to be a Kan complex. (cf. definition of π_0 in Chapter 1). This is indeed the case if Y is a Kan complex. The next few pages will therefore be devoted to the proof of this theorem.

<u>Definition:</u> A <u>(p,q) "shuffle"</u> is a partition (μ,ν) of the set $\{0,\ldots,p+q-1\}$ of integers into two disjoint sets such that $\mu_1 < \ldots < \mu_p$ and $\nu_1 < \ldots < \nu_q$. The (p,q) shuffle is determined by μ or ν.

The reason for introducing (p,q)-shuffles is the following: If τ is a non-degenerate p-simplex of K, let $\hat{\tau}$ denote the smallest subcomplex of K containing τ. Then the non-degenerate (p+q)-simplexes of $\bar{\tau} \times \Delta_q$ are of the form

$$s_{\nu_q} \ldots s_{\nu_1} \times s_{\mu_p} \ldots s_{\mu_1} (0,\ldots,q)$$

where (μ,ν) is a (p,q)-shuffle; and the set of such simplex is thus in a natural 1-1 correspondence with the set of (p,q)-shuffles.

Let $i \in \{0,\ldots,p+q\}$. The (p,q) shuffle (μ,ν) is of type I relative to i if either

1) $i < \mu_1$, or
2) $i, i-1 \in \{\nu_1,\ldots,\nu_q\}$, or
3) $i = p+q$, $i-1 = \nu_q$

It is of type II relative to i if either

1) $i < \nu_1$, or
2) $i, i-1 \in \{\mu_1,\ldots,\mu_p\}$, or
3) $i = p+q$, $i-1 = \mu_p$.

If the (p,q) shuffle (μ,ν) is not of type I or II relative to i, then it is said to be of type III relative to i.

In this case $\max \{\mu_1, \nu_1\} \leq i < p+q$ and either

1) $i \in \{\mu_1, \ldots, \mu_p\}$ and $i-1 \in \{\nu_1, \ldots, \nu_q\}$, on
2) $i \in \{\nu_1, \ldots, \nu_q\}$ and $i-1 \in \{\mu_1, \ldots, \mu_p\}$.

Now we wish to define a new shuffle $(\bar{\mu}, \bar{\nu})$ associated with (μ, ν) and i.

If (μ, ν) is of type I relative to i, then $(\bar{\mu}, \bar{\nu})$ is a $(p, q-1)$ shuffle. Let k be the integer such that $\nu_k = i$ in case 1 or case 2, and let $k = q$ in case 3. Let $\bar{\nu}_j = \nu_j$ for $j < k$, $\bar{\nu}_j = \nu_{j+1} - 1$ for $k \leq j \leq q-1$; $(\bar{\mu}, \bar{\nu})$ is the corresponding $(p, q-1)$ shuffle. There is an integer r, called the index of i in (μ, ν), such that $\bar{\mu}_j = \mu_j$ for $j \leq r$, and $\bar{\mu}_j = \mu_j - 1$ for $r < j \leq p$.

If (μ, ν) is of type II relative to i, then $(\bar{\mu}, \bar{\nu})$ is a $(p-1, q)$ shuffle. Let k be the integer such that $\mu_k = i$ in case 1 or case 2, and let $k = p$ in case 3. Let $\bar{\mu}_j = \mu_j$ for $j < k$, $\bar{\mu}_j = \mu_{j+1} - 1$ for $k \leq j \leq p-1$; $(\bar{\mu}, \bar{\nu})$ is the corresponding $(p-1, q)$ shuffle. There is an integer r, called the index of i in (μ, ν) such that $\bar{\nu}_j = \nu_j$ for $j \leq r$, and $\bar{\nu}_j = \nu_j - 1$ for $r < j \leq q$.

If (μ, ν) is of type III relative to i, then in case 1, $i = \mu_r$, $i-1 = \nu_s$. Let $\bar{\mu}_j = \mu_j$ for $j \neq r$, $\bar{\mu}_r = i-1$, and let $(\bar{\mu}, \bar{\nu})$ be the corresponding (p,q) shuffle. In case i, $i-1 = \mu_i$, $i = \nu_s$. Let $\bar{\mu}_j = \mu_j$ for $j \neq r$, $\bar{\mu}_i = i$, and let $(\bar{\mu}, \bar{\nu})$ be the corresponding (p,q) shuffle.

Now the associated shuffle $(\bar{\mu}, \bar{\nu})$ of (μ, ν) relative to i is defined for all (μ,ν) and i. However, we want a <u>second associated shuffle</u> $(\bar{\mu},\bar{\nu})$ relative to i; it is to be a (p+1,q) shuffle, defined as follows. If r is the largest integer such that $\mu_j < i$ for $j < r$, then $\bar{\mu}_j = \mu_j$ for $j < r$; $\bar{\mu}_r = i$, and $\bar{\mu}_j = \mu_{j-1}+1$ for $j > r$. The <u>second index</u> of i in (μ,ν) is the number of ν_j such that $i > \mu_j$.

<u>Definition:</u> Let X and Y be semi-simplicial complexes, and $F : X \times \Delta_q \longrightarrow Y$ a semi simplicial map. If (μ,ν) is a (p,q) shuffle, define

$$F_{(\mu,\nu)} : X_p \longrightarrow Y_{p+q} \qquad \text{by}$$

$$F_{(\mu,\nu)}\, a = s_{\nu_q} \cdots s_{\nu_1} a \times s_{\mu_p} \cdots s_{\mu_1}(0,\ldots,q).$$

Further define

$$F^i_{(\mu',\nu')} : X_p \longrightarrow Y_{p+q-1}$$

by $F^i_{(\mu',\nu')}\, a = s_{\nu'_{q-1}} \cdots s_{\nu'_1} a \times s_{\mu'_p} \cdots s_{\mu'_1}(0,\ldots,i-1,i+1,\ldots,q)$ where (μ',ν') is a (p,q-1) shuffle, and $i = 0,\ldots,q$.

<u>Proposition 2:</u> If $F : X \times \Delta_q \longrightarrow Y$ is a semi simplicial map, then

1) $\partial_i F_{(\mu,\nu)} = F_{(\bar{\mu},\bar{\nu})} \partial_{i-r}$ is (μ,ν)

is a (p,q) shuffle of type II relative to i, r is the index of i in (μ,ν), and $(\bar{\mu},\bar{\nu})$ is the associated shuffle of (μ,ν) relative to i,

2) $\partial_i F_{(\mu,\nu)} = \partial_i F_{(\bar{\mu},\bar{\nu})}$ if (μ,ν)

is a (p,q) shuffle of type III relative to i, and

$(\overline{\mu},\overline{\nu})$ is the associated shuffle of (μ,ν) relative to i,

3) $s_i F_{(\mu,\nu)} = F_{(\overline{\mu},\overline{\nu})}$ is the second associated shuffle of (μ,ν) relative to i, and r is the second index of i in (μ,ν), and

4) $\partial_i F_{(\mu,\nu)} = F^{i-r}_{(\overline{\mu},\overline{\nu})}$ if (μ,ν) is a (p,q) shuffle of type I relative to i, $(\overline{\mu},\overline{\nu})$ is the associated (p,q-1) shuffle, and r is the index of i in (μ,ν).

Further, a set $\{F_{(\mu,\nu)}\}$ of functions $F_{(\mu,\nu)} : X_p \longrightarrow Y_{p+q}$ indexed on the (p,q) shuffles for fixed q, and satisfying conditions 1)-3) above, determine a map $F : X \times \Delta_q \longrightarrow Y$.

The proof is entirely similar to the proof of the first proposition of this appendix, but more tedious. It will be omitted.

<u>Theorem 3:</u> If X is a semi-simplicial complex, and Y is a Kan complex, then Y^X is a Kan complex.

<u>Proof</u>: Let $F_0,\ldots,F_{k-1},F_{k+1},\ldots,F_q \in (Y^X)_{q-1}$ be such that $\partial_i F_j = \partial_{j-1} F_i$, $i < j$, $i, j \neq k$. Let $\{F^i_{(\mu,\nu)}\}$ be the functions indexed on the (p,q-1) shuffles determined by F_i for $i \neq k$. We wish to produce a set of functions $F_{(\mu,\nu)}$, indexed on the (p,q) shuffles, and satisfying relation 1)-4). Order the shuffls as follows: an (r,q) shuffle precedes a (p,q) shuffle if $r < p$. A (p,q) shuffle

(μ, ν) precedes a (p,q) shuffle (μ^*, ν^*) if $\mu_i = \mu_i^*$ for $i < j$, and $\mu_j < \mu_j^*$. The first shuffle is a (0,q) shuffle, and this unique. Therefore, if $a \in X_0$ we must find an element $b \in Y_p$ such that $\partial_i b = F^i_{(0,\ldots,q-1)} a$ for $i \neq k$; and we can do so since Y is a Kan complex. Define $F_{(0,\ldots,q)} a = b$.

Suppose now that $F_{(\mu,\nu)}$ is defined for $(\mu,\nu) < (\mu^*, \nu^*)$.

<u>Case 1:</u> (μ^*, ν^*) is the first (p,q) shuffle; i.e. $\mu^*_i = i-1$ for $i = 1,\ldots,p$, $\nu^*_2 = i+p-1$. This shuffle is of type III with respect to p, and $(\overline{\mu}, \overline{\nu})$ the associated (p,q) shuffle relative to p, is given by $\overline{\mu}_i = i-1$ for $i < p$, $\overline{\mu}_p = p$, $\overline{\nu}_1 = p-1$, $\overline{\nu}_i = i+p-1$ for $i > 1$. Therefore (μ,ν) precedes $(\overline{\mu}, \overline{\nu})$. Consequently if $a \in X_p$, $\partial_p F_{(\mu^*, \nu^*)} a$ is not specified. Therefore if a is non-degenerate we may use the extension condition to define $F_{(\mu^*, \nu^*)} a$; while if a is degenerate we may condition 3) of the proposition to make the definition.

<u>Case 2:</u> For some integer i, and for some $r \in \{1,\ldots,p\}$ $s \in \{1,\ldots,q\}$, we have $\mu^*_r = i-1$, and $\nu^*_s = i$. Now (μ^*, ν^*) precedes $(\overline{\mu}, \overline{\nu})$, the associated (p,q) shuffle relative to i, and $\partial_i F_{(\mu^*, \nu^*)}$ is not specified. The proof for this case is then completed as in case 1.

<u>Case 3</u>: $\mu_i^* = i + q - 1$, $\nu_i^* = i-1$, $k < q$. In this case we must have $\partial_k F_{(\mu^*, \nu^*)} = F^k_{(\bar{\mu}, \bar{\nu})}$ where $(\bar{\mu}, \bar{\nu})$ is associated with (μ^*, ν^*) relative to k. But $F^k_{(\bar{\mu}, \bar{\nu})}$ is undefined, so that ${}_kF_{(\mu^*, \nu^*)}$ is free, and we may proceed as before.

If $k < q$, cases 1,2,3 are exhaustive. Therefore it remains to prove the extension condition in case $k = q$. To do this we reorder the $F_{(\mu, \nu)}$'s by simply reversing the ordering of the (p,q)-shuffles for each fixed p.

Now in the inductive step, $\mu_i^* = i + q - 1$, $\nu_i^* = i-1$ is the first case to be considered, and this may be carried through. The reverse of the previous case 2) is now case 2), i.e. for some i,r,s, $r \in \{1, \ldots, p\}$, $s \in \{1, \ldots, q\}$ $\mu_r^* = i$, $\nu_s^* = i-1$, and we proceed as in case 2. The last case is now $\mu_i^* = i-1$, $\nu_i^* = i+p-1$, and by the relations we see that $\partial_{p+q} F_{(\mu^*, \nu^*)}$ is unspecified, and the proof may be completed.

Theorem 4: If X is a semi simplicial complex, A is a subcomplex of X, and Y is a Kan complex, then the map $p : Y^X \longrightarrow Y^A$ given by $p(f) = f|A \times \Delta_q$, where $f : X \times \Delta_q \longrightarrow Y$, is a fibre map.

Proof: The proof of this theorem is essentially the same as the proof of the preceding theorem.

Corollary 5: (Homotopy Extension Theorem) Let (X,A) be a semi-simplicial pair, Y a Kan complex. Let $f : X \longrightarrow Y$, and let $F : A \times I \longrightarrow Y$ be a homotopy such that $F(\tau \times (0_0,\ldots,0_r)) = f(\tau)$ for $\tau \in A_r$, all r. Then there exists a homotopy $\overline{F} : X \times I \longrightarrow Y$ which agrees with F on $A \times I$ and such that $F(\sigma \times (0_0,\ldots,0_r) = f(\sigma)$ for $\sigma \in X_r$, all r.

Now following Eilenberg and Zilber ([1]) we shall show the existence of a minimal subcomplex of any Kan complex which is equivalent to that Kan complex up to homotopy. We first give some preliminary definitions and lemmas.

Definition: If X is a semi-simplicial complex, then $x,y \in X_q$ are compatible if $\partial_i x = \partial_i y$ for $i = 0,\ldots,q$. Now x defines a unique map $\overline{x} : \Delta_q \longrightarrow X$, determined by $\overline{x}(0,\ldots,q) = x$, and similarly for y. The simplexes x and y are said to be homotopic if $\overline{x}$ and $\overline{y}$ are homotopic rel $\dot{\Delta}_q$.

Lemma 6: If X is a Kan complex, then X is minimal if and only if for each compatible pair $x,y \in X_q$ such that x is homotopic to y, we have $x = y$.

<u>Proof:</u> Suppose first that X is minimal, and $x,y \in X_q$ with x homotopic to y. Let $k_i : (\Delta_q)_r \longrightarrow X_{r+1}$ be functions satisfying the conditions of Proposition 1 generating a homotopy from $\bar{x}$ to $\bar{y}$ rel $\dot{\Delta}_q$.

We then have $\partial_0 k_0(0,\ldots,q) = x$,

$$\partial_i k_0(0,\ldots,q) = k_0(0,\ldots,i-2,i,\ldots,q) = \bar{x}(s_0(0,\ldots,i-2,i,\ldots,q))$$
$$= s_0 \partial_{i-1} x = \partial_i s_0 x \quad \text{for} \quad i > 1.$$

Therefore $k_0(0,\ldots,q)$ has the same faces, other than the first, as does $s_0 x$. Since X is minimal, we have therefore

$$\partial_1 k_1(0,\ldots,q) = \partial_1 k_0(0,\ldots,q) = \partial_1 s_0 x = x.$$

By an inductive argument of this nature it is easy to show that $\partial_{i+1} k_i(0,\ldots,q) = x$ for all i. Hence $x = \partial_{q+1} k_q(0,\ldots,q) = y$.

The converse is proved in a similar manner.

<u>Lemma 7:</u> If X is a semi-simplicial complex, $x,y \in X_q$, and x and y are compatible and degenerate, then $x = y$.

<u>Proof</u>: Let $x = s_m z$, $y = s_n z'$. Then either $m = n$, in which case $\partial_m x = z$ and $\partial_m y = z'$ implies $z = z'$, or $m \neq n$. In this latter case suppose $m < n$. Now $z = \partial_m s_m z = \partial_m x = \partial_m y = \partial_m s_n z' = s_{n-1} \partial_m z'$. Therefore $x = s_m s_{n-1} \partial_m z' = s_n s_m \partial_m z'$, and $\partial_n x = s_m \partial_m z'$. Since $z' = \partial_n y = \partial_n x = s_m \partial_m z'$, $z' = s_m \partial_m z'$. Then $x = s_n s_m \partial_m z' = s_n z' = y$.

Now let X be a Kan complex, and define a new seim-simplicial complex M as follows. For each component of X choose a representative point. These are to be the elements of M_0. Suppose now that M_r is is defined with face operators for $r \leq n$, so that $M_r \subset X_r$ and the face operators agree. Consider the homotopy classes of (n+1)-simplexes of X, each simplex having all its faces in M_n. We choose one representative from each such class, always choosing a degenerate representative if such exists; these are to be the elements of M_{n+1}. ∂_i and s_i are induced by the corresponding operators in X. Thus we obtain by induction a semi-simplicial complex $M \subset X$ which is clearly minimal. We now define by induction a set of functions

$$k_i : X_n \longrightarrow X_{n+1}, \ i = 0, \ldots, n$$

for each dimension $n = 0, 1, \ldots,$ satisfying the relations of proposition 1, and such that $\partial_0 k_0(x) = x$, $\partial_{n+1} k_n(x) \in M_n$ for $x \in X_n$, and $k_i(x) = s_i(x)$ if $x \in M_n$.

1) If $x \in X_0$, $k_0(x)$ is to be a path such that $\partial_0 k_0 = x, \partial_1 k_0(x) \in M_0$.
   Further, if $x \in M_0$, we take $k_0(x) = s_0(x)$.
2) Suppose that the functions k_i have been defined for X_n for $n \leq r$, satisfying the above conditions. Let $x \in X_{r+1}$. If x is degenerate, then $k_0(x)$ is defined by the relations, while if $x \in M_{r+1}$ we set $k_0(x) = s_0(x)$. Otherwise we must find an element $y = k_0(x)$ such that $\partial_0 y = x$ and $\partial_i y = k_0(\partial_{i-1} x)$ for $i > 1$. We may choose such a y using the extension condition.

3) Suppose further that $k_i : X_{r+1} \longrightarrow X_{r+2}$ has been defined for $i < j$. Then for $x \in X_{r+1}$ we must find $y = k_j(x)$ such that
$\partial_i y = k_{j-1} \partial_i x$ for $i < j$, $\partial_j k_j(x) = \partial_j k_{j-1}(x)$, and $\partial_i k_j(x) = k_j \partial_{i-1} x$ for $i > j+1$. If x is degenerate, define $k_j(x)$ using the relations. If $x \in M_{r+1}$, set $k_j(x) = s_j(x)$. Otherwise apply the extension condition and choose $k_j(x)$ arbitrarily, provided $j \neq r+1$. If $j = r+1$, we must have the further condition $\partial_{r+2} k_{r+1}(x) \in M_{r+1}$. First choose $y = k_{n+1}(x)$ by the extension condition to satisfy all the above conditions except that on $\partial_{r+1} y$.
Then

$$\partial_i \partial_{r+2} k_{r+1}(x) = \partial_{r+1} \partial_i k_{r+1}(x) = \partial_{r+1} k_r(\ _i x) \in M_r \text{ for } i < r+1;$$

$$\partial_{r+1} \partial_{r+2} k_{r+1}(x) = \partial_{r+1} \partial_{r+1} k_{r+1}(x) = \partial_{r+1} \partial_{r+1} k_r(x)$$
$$= \partial_{r+1} \partial_{r+2} k_r(x) = \partial_{r+1} k_r(\partial_{r+1} x) \in M_r.$$

Thus $\partial_{r+2} y$ has all its faces in M_r, and there is therefore a unique $z \in M_{r+1}$ which is compatible with and homotopic to $\partial_{r+2} y$. Then by an obvious modification of the homotopy extension theorem, there exists $y' \in X_{r+2}$ such that $\partial_i y = \partial_i y'$, $i < r+2$, and $\partial_{r+2} y' = z$. We finally define $k_{r+1}(x) = y'$. This completes the induction.

<u>Theorem 8:</u> If X is a Kan complex, then there exists a minimal subcomplex M of X which is a deformation retract of X. Further, if M' is another such subcomplex, then M is isomorphic to M'.

Proof: The existence of M has already been proved, so suppose that M' is another such complex

Let $r : X \longrightarrow M$, $r' : X \longrightarrow M'$ be deformation retractions. Then we have maps

$$M \xrightarrow{i} X \xrightarrow{r'} M', \quad \text{and}$$
$$M' \xrightarrow{i'} X \xrightarrow{r} M$$

where i and i' are inclusions.

The map $i \circ r$ is homotopic to the identity map of X, and hence $r' \circ i \circ r \circ i' \simeq r' \circ i' = 1$. One verifies readily that the identity is the only map of a minimal complex into itself which is homotopic to the identity, and hence $r' \circ i \circ r \circ i' = 1$.
Similarly $r \circ i' \circ r' \circ i = 1$, and hence $r' \circ i$ is an isomorphism. This completes the proof.

Reference

[1] S. Eilenberg and J. A. Zilber, Semi-simplicial complexes and singular homology, Annals of Math. 51 (1950), pp. 499-513.

Appendix 1 B. Definition of Homotopy Groups by Mappings of Spheres

W. Barcus

Let Δ_q denote the semi-simplicial complex on the standard q-simplex; an r-simplex of Δ_q is a sequence $(i_0,\ldots,i_r)$ with $0 \leq i_0 \leq \ldots \leq i_r \leq q$, the i_j being the "vertices" of the simplex. We shall also denote the complex Δ_1 by I. Similarly, let $\dot{\Delta}_q$ denote the usual semi-simplicial complex on the boundary of the standard q-simplex, so that $\dot{\Delta}_q$ is a subcomplex of Δ_q. $\dot{\Delta}_{q+1}$ is the analogue of a q-sphere, for semi-simplicial theory. Let σ_i denote the simplex $(0,\ldots,i-1,i+1,\ldots,q+1)$ of $\dot{\Delta}_{q+1}$, and let $\partial_i \Delta_{q+1}$ denote the subcomplex of Δ_{q+1} consisting of simplexes which do not comtain the vertex i. We may embed Δ_q in Δ_{q+1} as $\partial_{q+1} \Delta_{q+1}$.

Let X be a Kan complex, $x^* \in X_0$. It is clear that $\pi_q(X,x^*)$, the q^{th} homotopy group of X based at x^*, as previously defined, may be considered as the set of equivalence classes of maps[1] $h:(\Delta_q, \dot{\Delta}_q) \longrightarrow (X,\bar{x}^*)$, two maps h, k being equivalent ("simplicially homotopic") if there exists a map $F:\Delta_{q+1} \longrightarrow X$ such that $F(\partial_j \sigma_i) = s_0^{q-1} x^*$, all i, j; $F(\sigma_i) = s_0^q x^*$, $i \neq q, q+1$; $F(\sigma_q) = h(0,\ldots,q)$, $F(\sigma_{q+1}) = k(0,\ldots,q)$.

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[1] For any simplex $\gamma \in X$, let $\bar{\gamma}$ denote the smallest subcomplex of X containing γ.

Let $\bar{\pi}_q(X,x^*)$ denote the set whose elements are the homotopy classes[1] rel(q+1) of maps $f:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$, $(q+1)$ being the 0-simplex consisting of just the vertex q+1.

<u>Lemma 1B.1:</u> Any map $g:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$ is homotopic rel (q+1) to a map $\bar{g}:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$ such that $\bar{g}(\sigma_i) = s_0^q x^*$ for $i < q+1$.

<u>Lemma 1B.2:</u> Let $h,k:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$ be maps such that $h(\sigma_i) = k(\sigma_i) = s_0^q x^*$, $i < q+1$, and suppose that $h \sim k$ rel (q+1). Then $h \sim k$ rel $\sigma_0 \cup \ldots \cup \sigma_q$.

The proofs of the above two lemmas are straightforward; one need only extend maps defined on subcomplexes of $\dot{\Delta}_{q+1} \times I$ and $\dot{\Delta}_{q+1} \times I \times I$. Details will be omitted.

<u>Lemma 1B.3:</u> Let $h,k:(\Delta_q,\dot{\Delta}_q) \longrightarrow (X,\bar{x}^*)$. Then $h \sim k$ rel $\dot{\Delta}_q$ if and only if $h \underset{s}{\simeq} k$.

<u>Proof</u>: Suppose that $h \sim k$ rel $\dot{\Delta}_q$ under a homotopy $F:\Delta_q \times I \longrightarrow X$. The non-degenerate (q+1)-simplexes of $\Delta_q \times I$ are

$$\tau_i = (0,\ldots,i-1,i,i,i+1,\ldots,q) \times (0_0,\ldots,0_i,1_{i+1},\ldots,1_{q+1}).$$

For each i, $\partial_k \partial_j \tau_i \in \dot{\Delta}_q \times I$ for all k, j, and $\partial_j \tau_i \in \dot{\Delta}_q \times I$ for $j \neq i, i+1$. Applying lemma (1.21), from τ_0 we have

[1] In the sense of Appendix 1A. Homotopy in this sense will be denoted $\sim$; in the simplicial sense, $\underset{s}{\simeq}$.

$k \underset{s}{\simeq} F|\partial_1\tau_0$; from τ_1 we have $F|\partial_1\tau_0 = F|\partial_1\tau_1 \underset{s}{\simeq} F|\partial_2\tau_1$; hence by the transitivity of $\underset{s}{\simeq}$, $k \underset{s}{\simeq} F|\partial_2\tau_1$. Proceeding inductively, $k \underset{s}{\simeq} F|\partial_{q+1}\tau_q = h$.

Conversely, let $h \underset{s}{\simeq} k$. Then we define $F: \Delta_q \times I \longrightarrow X$ as follows. $\beta_q = F(\tau_q)$ is to have faces $\partial_{q+1}\beta_q = h(0,\ldots,q), \partial_q\beta_q = k(0,\ldots,q), \partial_i\beta_q = s_0^q x^*, i \leq q$. Let $\beta_i = F(\tau_i) = s_i k(0,\ldots,q), i < q$. F is then determined, and is a homotopy from h to k rel $\dot{\Delta}_q$.

Define a function $\psi: \pi_q \longrightarrow \overline{\pi}_q$ as follows: $\psi[h]$ is represented by the map $h': (\dot{\Delta}_{q+1}, (q+1)) \longrightarrow (X, x^*)$ determined by $h'(\sigma_i) = s_0^q x^*, i \neq q+1$; $h'(\sigma_{q+1}) = h(0,\ldots,q)$.

<u>Theorem 1B.4:</u> ψ is 1-1.

A group structure is therefore induced in $\overline{\pi}_q$ such that ψ is an isomorphism.

<u>Proof of 1B.4:</u> To show that ψ is single-valued, suppose that $h \underset{s}{\simeq} k$. Then by (1B.3), $h \sim k$ rel $\dot{\Delta}_q$. If the homotopy is $F: \Delta_q \times I \longrightarrow X$, then F can be extended to $F': \dot{\Delta}_{q+1} \times I \longrightarrow X$ by setting $F'(\omega_r) = s_0^r x^*$ for any simplex ω_r of $\dot{\Delta}_{q+1} \times I - \Delta_q \times I$. F' is then a homotopy from h' to k'.

Define $\phi: \overline{\pi}_q \longrightarrow \pi_q$ by $\phi[g] = [\bar{\bar{g}}]$, where $\bar{\bar{g}}: (\Delta_q, \dot{\Delta}_q) \longrightarrow (X, \bar{x}^*)$ is the restriction of the map $\bar{g}$ of (1B.1). ϕ is single-valued by (1B.2),(1B.3). It is clear that $\phi\psi$ = identity, and by (1B.1) ψ is onto. Therefore ψ is 1-1, which proves (1B.4).

Using the representation of the elements of π_q as homotopy classes of mappings of $\dot{\Delta}_{q+1}$, it is easy to define the isomorphism[1] induced by a path α in X from x_0 to x_1:

$$\alpha_{\#} : \pi_q(X,x_0) \xrightarrow{\approx} \pi_q(X,x_1).$$

Let $\xi \in \pi_q(X,x_0)$ have representative map $f_0:(\dot{\Delta}_{q+1},(q+1)) \rightarrow (X,x_0)$. Define $F: \dot{\Delta}_{q+1} \times I \longrightarrow X$ by $F(\tau \times (0_0,\ldots,0_r)) = f_0(\tau)$, $\tau \in (\dot{\Delta}_{q+1})_r$, all r; $F((q+1,q+1)\times(0,1)) = \alpha$; and extend by the homotopy extension theorem. Define $f_1:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x_1)$ by $f_1(\tau) = F(\tau \times (1_0,\ldots,1_r))$; then $\alpha_{\#}\xi = [f_1]$. That $\alpha_{\#}$ is an isomorphism follows by applying the homotopy extension theorem. The usual properties of the induced isomorphism may also be demonstrated.

1) It is more convenient to define this isomorphism rather than its inverse, as is usually done.

In the preceding parts of chapter 1, a good deal of elementary homotopy theory has been developed, but some standard and necessary properties have not yet been stated. This section will first take up a few of these, and then pass on to a proof of the Hurewicz Theorem.

Theorem: If X,Y are Kan complexes, and $f,g:X \longrightarrow Y$ are semi-simplicial maps homotopic relative to $[x]$ (the subcomplex of X generated by $x \in X_0$), then $f^{\#} = g^{\#} : \pi_q(X,x) \longrightarrow \pi_q(Y,f(x))$.

Proof: The theorem follows immediately from the fact that elements of $\pi_q(X,x)$ correspond to homotopy classes of maps $\phi:(\Delta_q, \dot{\Delta}_q) \longrightarrow (X,x)$ (see appendix B); since f,g are homotopic relative to $[x]$, $f \circ \phi$, $g \circ \phi : (\Delta_q, \dot{\Delta}_q) \longrightarrow (Y,f(x)$ are homotopic.

Definition: Two Kan complexes X and Y are said to have the same homotopy type if and only if there exist maps $f:X \longrightarrow Y$ and $g:Y \longrightarrow X$ such that fg is homotopic to the identity map of Y and gf is homotopic to the identity map of X.

Proposition: If X and Y are connected minimal Kan complexes such that $\pi_q(X,x) = \pi_q(Y,y) = 0$ for $g \neq n$, and $\phi: \pi_n(X,x) \longrightarrow \pi_n(Y,y)$ is a homomorphism, then there is a unique semi-simplicial map $f:X \longrightarrow Y$ such that

$f^{\#} = \phi: \pi_n(X,x) \longrightarrow \pi_n(Y,y)$.

Proof: Since X and Y are minimal they both have exactly one simplex in each dimension $< n$. Further there is a natural 1:1 correspondence between $\pi_n(X,x)$ and X_n, and between $\pi_n(Y,y)$ and Y_n. Therefore f is defined and is unique in dimension $\leq n$. Suppose now that f is defined in dimension $\leq q$, where $q \geq n$, and let $\sigma \in X_{q+1}$. Then $f(\partial_i \sigma)$ is defined for $i = 0,\ldots,q+1$, and there is a unique element τ of Y_{q+1} such that $\partial_i \tau = f(\partial_i \sigma)$ for $i = 0,\ldots,q$. Set $f(\sigma) = \tau$. Thus f is defined inductively and satisfies the condition $\partial_i f = f \partial_i$. Suppose that $s_i f = f s_i$ in dimension $\leq q$ (we may suppose that $q \geq n$), and $\sigma \in X_{q+1}$. Then $\partial_j s_i f(\sigma) = s_{i-1} \partial_j f(\sigma) = s_{i-1} f(\partial_j \sigma) = f(s_{i-1} \partial_j \sigma) = f(\partial_j s_i \sigma) = \partial_j f(s_i \sigma)$ for $j < i$, $\partial_i s_i f(\sigma) = f(\sigma) = f(\partial_i s_i \sigma) = \partial_i f(s_i \sigma)$, $\partial_{i+1} s_i f(\sigma) = f(\sigma) = f(\partial_{i+1} s_i \sigma) = \partial_{i+1} f(s_i \sigma)$, $\partial_j s_i f(\sigma) = s_i \partial_{j-1} f(\sigma) = s_i f(\partial_{j-1} \sigma) = f(s_i \partial_{j-1} \sigma) = \partial_j f(s_i \sigma)$ for $j > i+1$. Consequently $s_i f(\sigma)$ and $f(s_i \sigma)$ have the same faces, and since $q+1 > n$, and $\pi_{q+1}(Y,y) = 0$, we have $f(s_i \sigma) = s_i f(\sigma)$. This last assertion completes the inductive step in the proof.

Corollary: If X and Y are connected minimal Kan complexes such that $\pi_q(X,x) = \pi_q(Y,y) = 0$ for $q \neq n$, and $\pi_n(X,x) \simeq \pi_n(Y,y)$, then X and Y are isomorphic.

<u>Theorem</u>: If X and Y are connected minimal Kan complexes, and $f: X \longrightarrow Y$ is a semi-simplicial map such that $f^{\#}: \pi_q(X,x) \xrightarrow{\approx} \pi_q(Y,y)$ for all q, then f is an isomorphism.

<u>Proof:</u> Let $\mathfrak{X}^n = (X^{(n+1)}, p, X^{(n)})$ be the n'th term in the natural Postnikov system of X, and $\mathfrak{Y}^n = (Y^{(n+1)}, p', Y^{(n)})$ that for Y (Chapter 1, p. 23). Now it is evident that all the terms in the Postnikov system of a minimal complex are minimal. Using the preceding corollary, we may make the inductive hypothesis that $f^{(n)}: X^{(n)} \longrightarrow Y^{(n)}$ is an isomorphism. There is a commutative diagram

$$\begin{array}{ccc} X^{(n+1)} & \xrightarrow{f^{(n+1)}} & Y^{(n+1)} \\ \big\downarrow p & & \big\downarrow p' \\ X^{n} & \xrightarrow{f^{(n)}} & Y^{(n)} \end{array}$$

Suppose that $\sigma, \tau \in X_q^{(n+1)}$, and that $f^{(n+1)}(\sigma) = f^{(n+1)}(\tau)$. Then $p'f^{(n+1)}(\sigma) = p'f^{(n+1)}(\tau)$, and $p(\sigma) = p(\tau)$. Therefore $\sigma = \tau$ if $q \leq n$. Suppose we have proved that $f^{(n+1)}(\sigma') = f^{(n+1)}(\tau')$ implies $\sigma' = \tau'$ when $\dim \sigma' = \dim \tau' < q$. We then have $\partial_i \sigma = \partial_i \tau$, $i = 0, \ldots, q$, and $\sigma = \tau$ unless $q = n+1$. If $q = n+1$ we recall that the simplexes of dimension $(n+1)$ with a given boundary in a minimal complex are in a natural 1:1 correspondence with π_{n+1}. Let $[\sigma]$, $[\tau]$ be the element of π_{n+1} corresponding to

σ and τ respectively. Since $f(\sigma) = f(\tau)$, by naturality $f^{\#}[\sigma] = f^{\#}[\tau]$; since $f^{\#}$ is an isomorphism, $[\sigma] = [\tau]$, and hence σ is homotopic to τ. Since σ and τ are compatible (1A-11) and homotopic, $\sigma = \tau$.

The fact that $f^{(n+1)}$ is onto may be proved similarly. It then follows that f is an isomorphism, since $X_q = X_q^{(n)}$ for $q \leq n$.

__Theorem:__ Let X and Y be connected Kan complexes. Then the following conditions are equivalent

1) X and Y have the same homotopy type,
2) there is a map $f:X \longrightarrow Y$ such that $f^{\#}:\pi_q(X,x) \xrightarrow{\approx} \pi_q(Y,f(x))$ for all q, where $x \in X_0$, and
3) X and Y have isomorphic minimal subcomplexes.

The proof is straightforward, using the earlier theorems of the appendix and the fact that every Kan complex has a minimal subcomplex which is a deformation retract of the original complex (1A-14 Theorem 8).

The fact that 1) and 2) in the preceding theorem are equivalent is in the topological case a theorem of J. H. C. Whitehead [1].

__Corollary:__ If X is a connected Kan complex, $x \in X_0$, $\pi_q(X,x) = 0$ for $q < n$, and $E_n(X,x)$ is the n-th Eilenberg subcomplex of X based at x, then the inclusion map

$i:E_n(X,x) \longrightarrow X$ is a homotopy equivalence.

<u>Definitions and Notations:</u> If X is a semi-simplicial complex, then $C_n(X)$, the <u>group of n-chains of X,</u> is the free abelian group generated by the elements of X_n. $C(X) = \sum C_n(X)$ is the <u>chain group of X</u>. Let $\partial: C_{n+1}(X) \longrightarrow C_n(X)$ be the homomorphism defined by $\partial x = \sum_{i=0}^{n+1} (-1)^i \partial_i x$ for $x \in X_{n+1}$. $C(X)$, together with the endomorphism ∂, is the <u>chain complex of X</u>. Let $Z_n(X)$ be the kernel of $\partial: C_n(X) \longrightarrow C_{n-1}(X)$, $B_n(X)$ the image of $\partial: C_{n+1}(X) \longrightarrow C_n(X)$. The group $Z_n(X)$ is the <u>group of n-cycles of X,</u> and $B_n(X)$ is the group of <u>n-dimensional boundaries</u> of X. The endomorphism ∂ of $C(X)$ has the property that $\partial\partial = 0$. Therefore $B_n(X) \subset Z_n(X)$, and the <u>n-dimensional homology group of X</u> is $H_n(X) = Z_n(X)/B_n(X)$. The <u>homology group of X</u> is $H(X) = \sum_{n \geq 0} H_n(X)$.

<u>Theorem:</u> If X and Y are semi-simplicial complexes, and $f,g: X \longrightarrow Y$ are homotopic maps, then $f_* = g_*: H(X) \longrightarrow H(Y)$.

<u>Proof:</u> Let $k_i: X_q \longrightarrow Y_{q+1}$ be functions determining a homotopy between f and g (1A-2, proposition 2), and define $k: C_q(X) \longrightarrow C_{q+1}(Y)$ by $k(x) = \sum_{i=0}^{q} (-1)^i k_i(x)$ for $x \in X_q$. Now $\partial k(x) + k\partial(x) = f(x) - g(x)$, and the result follows.

The preceding theorem is the usual statement that homology is an invariant of homotopy type.

<u>Theorem</u>: If X is a Kan complex, then $H_0(X) = Z(\pi_0(X))$, the free abelian group generated by $\pi_0(X)$.

<u>Proof</u>: There is a natural map $X_0 \longrightarrow \pi_0(X)$, which induces a homomorphism $C_0(X) \longrightarrow Z(\pi_0(X))$. Clearly this map is an epimorphism (homomorphism onto). Suppose that $x \in X_1$; then $\partial_0 x$ and $\partial_1 x$ are in the same component of X, so that the above epimorphism induces an epimorphism $\phi: H_0(X) \longrightarrow Z(\pi_0(X))$. For $x \in \pi_0(X)$, let $\hat{x}$ be an element of X_0 which represents x. Suppose that $\hat{y}$ also represents x, then there exists $z \in X$, such that $\partial_0 z = \hat{x}$, $\partial_1 z = \hat{y}$, and $\hat{x} - \hat{y} \in B_0(X)$. Consequently $x \longrightarrow \hat{x}$ induces a homomorphism $\psi: Z(\pi_0(X)) \longrightarrow H_0(X)$. Since $\psi\phi$ and $\phi\psi$ are the respective identities, ϕ is an isomorphism.

<u>Definition</u>: Let X be a Kan complex, $x \in X_0$, and define a homomorphism

$$\phi: \pi_n(X,x) \longrightarrow H_n(X,x) \quad \text{for } n > 0$$

as follows:

Let $\alpha \in \pi_n(X,x)$ have representative $a \in X_n$ such $\partial_i a = s_0^{n-1} x$ for $i = 0, \ldots, n$. Now if n is odd, $\partial a = 0$, while if n is even, $\partial a = s_0^{n-1} x$. Therefore we may take $\phi(\alpha)$ to have representative a if n is odd, and $a - s_0^n x$ if n is even.

To show that ϕ is single-valued, suppose that $a' \in X_n$ also represents α. Then there exists $w \in X_{n+1}$

such that

$$\partial_n w = a,\ \partial_{n+1} w = a',\ \partial_i w = s_0^n x \text{ for } i < n.$$

Then if n is even, $\partial w = a - a'$, while if n is odd, $\partial w = s_0^n x - a + a'$; and since $s_0^n x$ is a boundary, in either case a' is homologous to a.

To show that ϕ is homomorphism, suppose that $a, b \in X_n$ represent $\alpha, \beta \in \pi_n(X,x)$. There exists $v \in X_{n+1}$ such that

$$\partial_{n+1} v = a,\ \partial_{n-1} v = b, \text{ and } \partial_i v = s_0^n x \text{ for } i < n-1,$$

and $\partial_n v$ then represents $\alpha + \beta$. If n is odd, $\phi(\alpha + \beta)$ is represented by $\partial_n v$; but since $\partial v = b - \partial_n v + a$, $\partial_n v$ is homologous to $a + b$, which represents $\phi(\alpha) + \phi(\beta)$. Similarly if n is even, $\phi(\alpha + \beta)$ is represented by $\partial_n v - s_0^n x$; and since $\partial v = s_0^n x - b + \partial_n v - a$, this is homologous to $a + b$, which represents $\phi(\alpha) + \phi(\beta)$.

<u>Theorem (Poincaré):</u> If X is a connected Kan complex and $x \in X_0$, then $\phi: \pi_1(X,x) \longrightarrow H_1(X)$ induces an isomorphism $\phi': \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)]$.

<u>Proof</u>: We may assume that $X = E_1(X,x)$. Then there is a natural map $\eta: Z_1(X) = C_1(X) \longrightarrow \pi_1/[\pi_1, \pi_1]$, and as natural map $\lambda: Z_1(X) \longrightarrow H_1(X)$. We thus have a diagram

$$\begin{array}{ccc} Z_1(X) & \xrightarrow{\quad\lambda\quad} & \\ \downarrow \eta & & \\ \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)] & \longrightarrow & H_1(X) \end{array}$$

and we know that $\pi_1/[\pi_1, \pi_1,] \longrightarrow H_1(X)$ is an epimorphism.

If $a \in B_1(X)$, $a = \partial b$, $b \in X_2$, then a is represented by $\partial_0 b - \partial_1 b + \partial_2 b$, which is already 0 in $\pi_1(X,x)$; hence $\eta(B_1(X)) = 0$, and η induces a homomorphism $\eta' : H_1(X) \longrightarrow \pi_1/[\pi_1, \pi_1]$. Clearly $\phi' \eta'$ and $\eta' \phi$ are the respective identities, and the result follows.

<u>Definition</u>: A Kan complex X is <u>n-connected</u> if for $x \in X_0$, $\pi_q(X,x) = 0$ for $q \leq n$.

<u>Theorem (Hurewicz)</u>: Let X be a Kan complex, $x \in X_0$. If X is $(n-1)$ connected, $n \geq 2$, then $H_q(X) = 0$ for $0 < q < n$, and $\phi : \pi_n(X,x) \xrightarrow{\approx} H_n(X)$.

The proof of this theorem is similar to that of the preceding theorem. Here it may be assumed that $X = E_{n-1}(X,x)$ so that X has only one simplex in each dimension $< n$.

<u>References</u>

[1] Whitehead, J. H. C., <u>Combinatorial homotopy</u>, Bull. American Math. Soc., 55(1949), 213-145.

[2] Poincaré, Henri, Oeuvres, (Paris, 1928), vol. 6, 142-293.

[3] Hurewicz, W., Beitrage zur Topologie der Deformationen I - IV, K. Akademie van Wetenschappen, Amsterdam, Proceedings, 38(1935), 112-19, 521-8; 39(1936), 117-26, 215-24.

<u>errata:</u>

1A-7 line 7, should be $i > \nu_j$ instead of $i > \mu_j$.

1A-8 line 2, should start $s_i F(\mu, \nu) = F(\bar{\mu}, \bar{\nu})\, s_{i-r}$

instead of $s_i F(\mu, \nu) = F(\bar{\mu}, \bar{\nu})$

The geometric realization of a semi-simplicial complex

John Milnor

Corresponding to each (complete) semi-simplicial complex K, a topological space $|K|$ will be defined. This construction will be different from that used by Giever [4] and Hu [5] in that the degeneracy operations of K are used. This difference is important when dealing with product complexes.

If K and K' are countable it is shown that $|K \times K'|$ is canonically homeomorphic to $|K| \times |K'|$. It follows that if K is a countable group complex then $|K|$ is a topological group. In particular $|K(\pi,n)|$ is an abelian group.

The terminology for semi-simplicial complexes will follow John Moore [7].

1. The definition

As standard n-simplex Δ_n take the set of all (n+2)-tuples $(t_0, \ldots, t_{n+1})$ satisfying $0 = t_0 \leq t_1 \leq \cdots \leq t_{n+1} = 1$. The face and degeneracy maps $\partial_i : \Delta_{n-1} \longrightarrow \Delta_n$ and $s_i : \Delta_{n+1} \longrightarrow \Delta_n$ are defined by

$$\partial_i(t_0, \ldots, t_n) = (t_0, \ldots, t_i, t_i, \ldots, t_n)$$

$$s_i(t_0, \ldots, t_{n+2}) = (t_0, \ldots, t_i, t_{i+2}, \ldots, t_n).$$

Let $K = \bigcup_{i \geq 0} K_i$ be a semi-simplicial complex. Giving K the discrete topology, form the topological sum

$$\bar{K} = (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \cdots + (K_n \times \Delta_n) + \cdots.$$

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Thus $\bar{K}$ is a disjoint union of open sets $k_i \times \Delta_i$. An equivalence relation in $\bar{K}$ is generated by the relations

$$(\partial_i k_n, \delta_{n-1}) \sim (k_n, \partial_i \delta_{n-1})$$

$$(s_i k_n, \delta_{n+1}) \sim (k_n, s_i \delta_{n+1}),$$

for $i = 0,1,\ldots,n$. The identification space $|K| = \bar{K}/(\sim)$ will be called the geometric realization of K. The equivalence class of (k_n, δ_n) will be denoted by $|k_n, \delta_n|$.

Theorem 1. $|K|$ is a CW-complex having one n-cell corresponding to each non-degenerate n-simplex of K.

For the definition of CW-complex see Whitehead [8].

lemma 1. Every simplex $k_n \in K_n$ can be expressed in one and only one way as $k_n = s_{j_p} \ldots s_{j_1} k_{n-p}$ where k_{n-p} is non-degenerate and $0 \leq j_1 < \ldots < j_p < n$. The indices j_α which occur are precisely those j for which $k_n \in s_j K_{n-1}$

The proof is not difficult. See [3] 8.3. Similarly it can be shown that every $\delta_n \in \Delta_n$ can be written in exactly one way as $\delta_n = \partial_{i_q} \ldots \partial_{i_1} \delta_{n-q}$ where δ_{n-q} is an interior point (that is $t_0 < t_1 < \ldots < t_{n-q+1}$) and $0 \leq i_1 < \ldots < i_q \leq n$.

By a non-degenerate point of $\bar{K}$ will be meant a point (k_n, δ_n) with k_n non-degenerate and δ_n interior.

<u>lemma 2.</u> Each $(k_n, \delta_n) \in \bar{K}$ is equivalent to a unique non-degenerate point.

Define the map $\lambda : \bar{K} \longrightarrow \bar{K}$ as follows. Given k_n choose $j_1, \ldots, j_p, k_{n-p}$ as in lemma 1 and set

$$\lambda(k_n, \delta_n) = (k_{n-p}, s_{j_1} \ldots s_{j_p} \delta n).$$

Define the discontinuous function $\rho : \bar{K} \longrightarrow \bar{K}$ by choosing $i_1 \ldots i_q, \delta_{n-q}$ as above and setting

$$\rho(k_n, \delta_n) = (\partial_{i_1} \ldots \partial_{i_q} k_n, \delta_{n-q})$$

Now the composition $\lambda\rho : \bar{K} \longrightarrow \bar{K}$ carries each point into an equivalent, non-degenerate point. It can be verified that if $x \sim x'$ then $\lambda\rho(x) = \lambda\rho(x')$; which proves lemma 2.

Take as n-cells of $|K|$ the images of the non-degenerate simplexes of $\bar{K}$. By lemma 2 the interiors of these cells partition $|K|$. Since the remaining conditions for a CW-complex are easily verified, this proves theorem 1.

<u>lemma 3.</u> A semi-simplical map $f : K \longrightarrow K'$ induces a continuous map $|K| \longrightarrow |K'|$

In fact the map $|f|$ defined by $|k_n, \delta_n| \longrightarrow |f(k_n), \delta_n|$ is clearly well defined and continous.

As an example of the geometric realization, let C be an ordered simplicial complex with space $|C|$. (See [2] pg. 56 and 67). From C we can define a semi-simplicial complex K, where K_n is the set of all (n+1)-tuples $(a_0,\ldots,a_n)$ of vertices of C which (1) all lie in a common simplex, and (2) satisfy $a_0 \leq a_1 \leq \ldots \leq a_n$. The operations ∂_i, s_i are defined in the usual way.

Assertion The space $|C|$ is homeomorphic to the geometric realization $|K|$. In fact the point $|(a_0,\ldots,a_n);(t_0,\ldots,t_{n+1})|$ of $|K|$ corresponds to the point of $|C|$ whose a-th barycentric coordinate, a being a vertex of C, is the sum, over all i for which $a_i = a$, of $t_{i+1}-t_i$. The proof is easily given.

2. Product complexes

Let $K \times K'$ be the cartesian product of two semi-simplicial complexes (that is $(K \times K')_n = K_n \times K'_n$). The projection maps $\rho : K \times K' \longrightarrow K$ and $\rho' : K \times K' \longrightarrow K'$ induce maps $|\rho|$ and $|\rho'|$ of the geometric realizations. A map $\eta : |K \times K'| \longrightarrow |K| \times |K'|$ is defined by $\eta = |\rho| \times |\rho'|$.

Theorem 2. η is a one-one map of $|K \times K'|$ onto $|K| \times |K'|$. If either (a) K and K' are countable, or (b) one of the two CW-complexes $|K|, |K'|$ is locally finite; then η is a homeomorphism.

The restrictions (a) or (b) are necessary in order to prove that $|K|x|K'|$ is a CW-complex. For the proof in case (b) see [8] and for case (a) see [6].

Proof (Compare [2] pg.68). If x'' is a point of $|K \times K'|$ with non-degenerate representative $(k_n \times k_n', \delta_n)$ we will first determine the non-degenerate representative of $|\rho|(x'') = |k_n, \delta_n|$. Since δ_n is an interior point of Δ_n, this representative has the form

$$(k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n) \quad \text{where} \quad k_n = s_{i_p} \cdots s_{i_1} k_{n-p}$$

(See proof of lemma 2). Similarly $|\rho'|(x'')$ is represented by $(k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n)$ where $k_n' = s_{j_q} \cdots s_{j_1} k'_{n-q}$. The induces i_α and j_β must be distinct; for if $i_\alpha = j_\beta$ for some α, β then $k_n \times k_n'$ would be an element of $s_{i_\alpha}(K_{n-1} \times K'_{n-1})$.

However the point x'' can be completely determined by its image.

$$|k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n| \times |k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n|.$$

In fact given any pair $(x,x') \in |K|x|K'|$ define $\eta(x,x') \in |K \times K'|$ as follows. Let (k_a, δ_a) and (k_b', δ_b') be the non-degenerate representatives; where $\delta_a = (t_0, \ldots, t_{a+1})$, $\delta_b' = (u_0, \ldots, u_{b+1})$. Let $0 = w_0 < \ldots < w_{n+1} = 1$ be the distant numbers t_i and u_j arranged in order. Set $\delta_n'' = (w_0, \ldots, w_{n+1})$. Then if $\mu_1 < \ldots < \mu_{n-a}$ are the indices μ such that $w_{\mu+1}$ is not one of the t_i, we have $\delta_n'' = s_{\mu_1} \cdots s_{\mu_{n-a}} \delta_a$. Similarly $\delta_n'' = s_{\nu_1} \cdots s_{\nu_{n-b}} \delta_b'$ where the sets $\{\mu_i\}$ and $\{\nu_j\}$ are disjoint.

Now define

$$\bar{\eta}(x,x') = |(s_{\mu_{n-a}} \ldots s_{\mu_1} k_a)x(s_{\nu_{n-b}} \ldots s_{\nu_1} k'_b), \delta_n|.$$

Clearly

$$|\rho|\bar{\eta}(x,x') = |s_{\mu_{n-a}} \ldots s_{\mu_1} k_a, \delta_n| = |k_a, s_{\mu_1} \ldots, s_{\mu_{n-a}} \delta_n|$$

$$= |k_a, \delta_a| = x$$

and $|\rho'|\bar{\eta}(x,x') = x'$, which proves that $\eta\bar{\eta}$ is the identity map of $|K| x |K'|$. On the other hand, taking x'' as above we have $\eta\bar{\eta}(x'') = \bar{\eta}(|k_{n-p}, s_{i_1} \ldots s_{i_p} \delta n|, |k'_{n-q}, s_{j_1} \ldots s_{j_q} \delta n|)$

$$= |(s_{i_p} \ldots s_{i_1} k_{n-p}) x (s_{j_q} \ldots s_{j_1} k'_{n-q}), \delta_n| = x''.$$

To complete the proof it is only necessary to show that $\bar{\eta}$ is continuous. However it is easily verified that $\bar{\eta}$ is continous on each product cell of $|K| x |K'|$. Since we are assuming that this product is a CW-complex, this completes the proof.

An important special case is the following. Let I denote the semi-simplicial complex consisting of a 1-simplex and its faces and degeneracies.

<u>Corollary.</u> A semi-simplicial homotopy $h: K x I \longrightarrow K'$ induces an ordinary homotopy $|K| x [0,1] \longrightarrow |K'|$.

In fact the interval $[0,1]$ may be identified with $|I|$. The homotopy is now given by the composition

$$|K| x |I| \xrightarrow{\bar{\eta}} |K x I| \xrightarrow{|h|} |K'|.$$

3. Product operations

Now let K be a countable complex. Any semi-simplicial map $p: K \times K \longrightarrow K$ induces by lemma 3 and theorem 2 a continuous product

$$|p|\overline{\eta} : |K| \times |K| \longrightarrow |K|.$$

If there is an element e_0 in K_0 such that $s_0^n e_0$ is a two-sided identity in each K_n, then it follows that $|e_0,\delta_0|$ is a two-sided identity in $|K|$; so that $|K|$ is an H-space. If the product operation p is associative or commutative then it is easily verified that $|p|\overline{\eta}$ is associative or commutative. Hence we have the following.

<u>Theorem 3</u>. If K is a countable group complex (countable abelian group complex), then $|K|$ is a topological group (abelian topological group).

Let $K(\pi,n)$ denote the Eilenberg MacLane semi-simplicial complex (see [1]).

<u>Corollary</u>. If π is a countable abelian group, then for $n \geq 0$ the geometric realization $|K(\pi,n)|$ is an abelian topological group.

It will be shown in the next section that $|K(\pi,n)|$ actually is a space with one non-vanishing homotopy group.

The above construction can also be applied to other algebraic operations. For example a pairing $K \times K' \longrightarrow K''$ between countable group complexes induces a pairing between their realizations. If K is a semi-simplicial complex of Λ-modules, where Λ is a discrete ring, then $|K|$ is a topological Λ-module.

4. The topology of $|K|$.

For any space X let S(X) be the total singular complex. For any semi-simplicial complex K a one-one semi-simplicial map $i : K \longrightarrow S(|K|)$ is defined by

$$i(k_n)(\delta_n) = |k_n, \delta_n|.$$

Let $H_*(K)$ denote homology with integer coefficients.

lemma 4. The inclusion $K \longrightarrow S(|K|)$ induces an isomorphism $H_*(K) \approx H_*(S|K|)$ of homology groups.

By the n-skeleton $K^{(n)}$ of K is meant the subcomplex consisting of all $K_i, i \leq n$ and their degeneracies. Thus $|K^{(n)}|$ is just the n-skeleton of $|K|$ considered as a CW-complex. The filtration

$$K^{(0)} \subset K^{(1)} \subset \ldots$$

gives rise to a spectral sequence $\{E^r_{pq}\}$; where E^{∞} is the graded group corresponding to $H_*(K)$ under the induced filtration; and

$$E^1_{pq} = H_{p+q}(K^{(p)} \bmod K^{(p-1)}).$$

It is easily verified that $E^1_{pq} = 0$ for $q \neq 0$, and that

E^1_{po} is the free abelian generated by the non-degenerate p-simplexes of K. From the first assertion it follows that $E^2_{po} = E^{\infty}_{po} = H_p(K)$.

On the other hand the filtration

$$S(|K^{(o)}|) \subset S(|K^{(1)}|) \subset \ldots.$$

gives rise to a spectral sequence $\{\bar{E}^r_{pq}\}$ where $\bar{E}^{\infty}$ is the graded group corresponding to $H_*(S(|K|))$. Since it is easily verified that the induced map $E^1_{pq} \longrightarrow \bar{E}^1_{pq}$ is an isomorphism, it follows that the rest of the spectral sequence is also mapped isomorphicly; which completes the proof.

Now suppose that K satisfies the Kan extension condition, so that $\pi_1(K,k_0)$ can be defined.

<u>lemma 5</u>. If K is a Kan complex then the inclusion i induces an isomorphism of $\pi_1(\ ,k_0)$ onto $\pi_1(S(|K|),i(k_0)) = \pi_1(|K|,|k_0,S_0|)$

Let K' be the subcomplex consisting of all simplices of K whose vertices are all at k_0. Then $\pi_1(K,k_0)$ can be considered as a group with one generator for each element of K'_1 and one relation for each element of K'_2.

The space $|K'|$ is a CW-complex with one vertex. For such a space the group π_1 is known to have one generator for each edge and one relation for each face. Thus the homomorphism $\pi_1(K) = \pi_1(K') \longrightarrow \pi_1(|K'|)$ is an isomorphism.

We may assume that K is connected. Then it is known (see [7]) that there is a semi-simplicial deformation retraction $r : K \times I \longrightarrow K$ of K onto K'. By the corollary to theorem 2 this proves that $|K'|$ is a deformation retract of $|K|$ which completes the proof.

Remark 1. From lemmas 4 and 5 it can be proved, using a relative Hurewicz theorem, that the homomorphisms

$$\pi_n(K,k_0) \longrightarrow \pi_n(|K|, |k_0,\delta_0|)$$

are isomorphisms for all n.

Remark 2. The space $|K(\pi,n)|$ has n-th homotopy group π, and other homotopy groups trivial. This clearly follows from the preceeding remark. Alternatively the proof given by Hu ([5]) may be used without essential change.

Now let X be any topological space. There is a canonical map

$$j : |S(x)| \longrightarrow X$$

defined by $j(|k_n,\delta_n|) = k_n(\delta_n)$.

<u>Theorem 4.</u> The map $j : |S(x)| \longrightarrow X$ induces isomorphisms of the singular homology and homotopy groups.

(This result is essentially due to Giever [4]).

The map j induces a semi-simplicial map $j_{\#}: S(|S(x)|) \longrightarrow S(X)$. A map i in the opposite direction was defined at the beginning of this section. The composition $j_{\#}i : S(X) \longrightarrow S(X)$ is the identity map. Together with lemma 4 this implies that $j_{\#}$ induces isomorphisms of the singular homology groups of $|S(x)|$ onto those of X. By lemma 5, the fundamental group is also mapped isomorphicly. Using the relative Hurewicz theorem, this completes the proof.

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Chapter II

Monoid Complexes and Production of Semi-Simplicial Complexes.

In this chapter we shall consider special properties of Kan complexes which have a multiplicative structure, and shall then begin the consideration of the problem of constructing new semi-simplicial complexes from such a complex.

Definition 1.1: A semi-simplicial complex Γ is a monoid complex if

1) Γ_q is monoid with identity for $q \in Z^+$,
2) $\partial_i : \Gamma_{q+1} \longrightarrow \Gamma_q$, and $s_i : \Gamma_q \longrightarrow \Gamma_{q+1}$

are homomorphisms which send identity elements into identity elements.

We will denote by e_q the identity of Γ_q.

Γ is a group complex if Γ is a monoid complex and each Γ_q is a group. When each Γ_q is abelian, Γ will be called an abelian monoid complex, or an abelian group complex, as the case may be. If $x \in \Gamma_q$, the inverse of x will be denoted by $\bar{x}$.

Example 1: Let G be a topological group, and let Γ be the total singular complex of G. If $u, v : \Delta_q \longrightarrow G$ are singular q-simplexes, define $(u.v) : \Delta_q \longrightarrow G$ by $(u.v)(t_0, \ldots, t_q) = u(t_0, \ldots, t_q)\, v(t_0, \ldots, t_q)$. It is easily verified that Γ is a group complex, and that Γ is abelian if and only if G is abelian.

Example 2: Let X be a topological space. A path in X is a pair (f,r) where r is a non-negative real number, and $f : [0,r] \longrightarrow X$ is a map ([0,r] denotes the closed interval from 0 to r). A loop is a path (f,r) such that $f(0) = f(r)$. Topologize the set of all paths in X by using as a subbasis for the topology the sets W(C,V,U) defined as follows:

1) C is a compact subset of [0,1]
2) V is an open subset of R^+ (the non-negative real number),
3) U is an open subset of X
4) $W(C,V,U) = \{(f,r) \mid (f,r) \text{ is a path in } X,$
$r \in V,\ f(rC) \subset U\}$.

Now let $x \in X$, and let E(X,x) be the space of paths in X which begin at x. Define $p : E(X,x) \longrightarrow X$ by $p(f,r) = f(r)$; Then (E(X,x),p,X) is a fibre space in the sense of Serre [1]. i.e. the covering homotopy theorem holds for finite complexes. The proof is the same as that of Serre, in which normalized paths $f : [0,1] \longrightarrow X$ are used. Further the space E(X,x) is contractible, and has as fibre $\Omega(X,x)$, the space of loops in X based at x. Define $(f,r)(g,s) = (h,r+s)$ where

$$h(t) = \begin{cases} f(t) & 0 \leq t \leq r \\ g(t-r) & r \leq t \leq r+s \end{cases} \quad \text{if } (f,r),(g,s) \in \Omega(X,x).$$

It is easily verified that $\Omega(X,x)$ is a monoid with identity, and that if Γ is the total singular complex of $\Omega(X,x)$, then Γ is a monoid complex when multiplication is defined as in the preceding examples by point-wise multiplication of q-simplexes.

Theorem 2.2: If Γ is a group complex, then Γ is a Kan complex.

Proof: To prove the proposition it suffices to show that Γ satisfies the extension condition. Suppose therefore that $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1} \in \Gamma_q$, and $\partial_i x_j = \partial_{j-1} x_j$ for $i < j$, $i, j \neq k$. We must find an $x \in \Gamma_{q+1}$ such that $\partial_i x = x_i$ for $i \neq k$.

We first show that there exists $u \in \Gamma_{q+1}$ such that $\partial_i u = x_i$ for $i < k$. This is trivial if $k = 0$; if $k > 0$ we define $u^r \in \Gamma_{q+1}$ by induction on r such that $\partial_i u^r = x_i$ for $i \leq r$. First let $u^0 = s_0 x_0$; then $\partial_0 u^0 = x_0$. Now if $r < k-1$, set $y^r = s_{r+1}((\partial_{r+1}\bar{u}^r)x_{r+1})$, $u^{r+1} = u^r y^r$. Now by an easy calculation it follows that $\partial_i y^r = e_q$ for $i \leq r$, and $\partial_{r+1} y^r = (\partial_{r+1}\bar{u}^r)x_{r+1}$, using the fact that $\partial_i u^r = x_i$ for $i \leq r$. Therefore we deduce that $\partial_i u^{r+1} = x_i$ for $i \leq r+1$. Finally let $u = u^{k-1}$, and we have $\partial_i u = x_i$ for $i < k$.

Now we shall show by induction on r that there exists an element $x^r \in \Gamma_{q+1}$ such that $\partial_i x^r = x_i$ for $i < k$ and for $i > q-r+1$. For $r = 0$ let $X^0 = u$. Suppose x^r is defined and $r \leq q-k$. Let $z^r = s_{q-r}((\partial_{q-r+1}\bar{x}^r)x_{q-r+1})$, $x^{r+1} = x^r z^r$. A simple calculation shows that $\partial_i z^r = e_q$ if $i < k$ and $i > q-r+1$ and $\partial_{q-r+1} z^r = (\partial_{q-r+1}\bar{x}^r)x_{q-r+1}$. It follows that $\partial_i x^{r+1} = x_i$ for $i < k$ and for $i > q-r$. Finally if we take for x the element x^{q-k+1}, we have $\partial_i x = x_i$ for $i \neq k$. Thus the proof of the theorem is complete.

<u>Definition 2.3</u>: The monoid complex Γ is a <u>monoid complex with homotopy</u> if it is a Kan complex.

We shall denote $\pi_q(\Gamma, e_0)$ by $\pi_q(\Gamma)$.

<u>Proposition 2.4</u>: If Γ is a monoid complex with homotopy and $x, y \in \Gamma_q$ are elements such that $\partial_i x = \partial_i y = e_{q-1}$ for $i = 0, \ldots, q$, then $[x], [y] \in \pi_q(\Gamma)$, and $[x][y] = [xy]$.

<u>Proof</u>: Consider the element $z = s_q x\, s_{q-1} y$ Now $\partial_i z = e_{q-1}$ for $i < q-1$, $\partial_{q-1} z = y$, $\partial_q z = xy$, and $\partial_{q+1} z = x$. In view of the definition of addition in the homotopy groups, the result is proved.

<u>Proposition 2.5</u>: If Γ is a monoid complex with homotopy, then $\pi_1(\Gamma)$ is abelian.

<u>Proof</u>: Let $x, y \in \Gamma_1$ be such that $\partial_i x = \partial_i y = e_0$, $i = 0, 1$. Let $w = s_0 y s_1 x$. Then $\partial_0 w = y$, $\partial_1 w = yx$, $\partial_2 w = x$. Therefore $[x][y] = [yx]$; but $[yx] = [y][x]$ by the preceding proposition, and the proof is complete.

The two preceding propositions are the analogues of the classical theorems that the group operations in the homotopy groups of a topological group comes from the group operation in group, and that the fundamental group of a topological group is abelian (cf. e.g. [2]).

If Γ is a group complex, we wish to define the homotopy group of Γ in an alternative fashion.

Definition 2:6 If Γ is a group complex, define

$$\tilde{\pi}_q(\Gamma) = \bigcap_{j=0}^{q-1} \text{kernel } \partial_i = \Gamma_q \longrightarrow \Gamma_{q-1}, \text{ and}$$

$$\tilde{\pi}(\Gamma) = \Sigma_q \tilde{\pi}_q(\Gamma).$$

Proposition 2:7 If Γ is a group complex, then

1) $\partial_{q+1}(\tilde{\pi}_{q+1}(\Gamma)) \subset \tilde{\pi}_q(\Gamma)$

2) $\partial_{q+1}(\tilde{\pi}_{q+1}(\Gamma))$ is a normal subgroup of Γ_q,

3) image $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is contained in kernel $\partial_q : \tilde{\pi}_q(\Gamma) \longrightarrow \tilde{\pi}_{q-1}(\Gamma)$ for $q > 0$.

Proof: Let $x \in \tilde{\pi}_{q+1}(\Gamma)$. Now $\partial_i \partial_{q+1} x = \partial_q \partial_i x = e_{q-1}$ for $i \leq q$, and this implies 1) and 3).

Suppose $z \in \Gamma_q$; consider $s_q z\, x\, s_q \bar{z}$.
Since $\partial_i(s_q z\, x\, s_q \bar{z}) = \partial_i(s_q z\, s_q \bar{z}) = e_q$ for $i \leq q$,
Therefore $s_q z\, x\, s_q \bar{z} \in \tilde{\pi}_{q+1}(\Gamma)$.
Since $\partial_{q+1}(s_q z\, x\, s_q \bar{z}) = z \partial_{q+1} x \bar{z}$, 2) follows.

The preceding proposition implies that $\tilde{\pi}(\Gamma)$ is a chain complex (not necessarily abelian) with respect to the last face operator.

Definition 2.8: If Γ is a group complex, define

$$\pi_q{}'(\Gamma) = H_q(\tilde{\pi}(\Gamma)).$$

<u>Proposition 2.9</u>: If Γ is a group complex,

$$\pi_q(\Gamma) = \pi'_q(\Gamma).$$

<u>Proof</u>: An element of $\pi_q(\Gamma)$ is represented by $x \in \Gamma_q$ such that $\partial_i x = e_{q-1}$ for $i = 0, \ldots, q$. However, such an element x also represents an element of $\pi'_q(\Gamma)$. Suppose $[x] = [y] \in \pi_q(\Gamma)$. Then there exists $z \in \Gamma_{q+1}$ such that $\partial_i z = e_q$ for $i < q$, $\partial_q z = x$, $\partial_{q+1} z = y$. Now $s_q \bar{x} \cdot z \in \tilde{\pi}_{q+1}(\Gamma)$, and $\partial_{q+1}(s_q \bar{x} \cdot z) = \bar{x}y$. Therefore $[x] = [y] \in \pi_q{}'(\Gamma)$, and there is a natural map of $\pi_q(\Gamma)$ into $\pi_q{}'(\Gamma)$. Further it is evident that this map is onto, and it is a homomorphism by proposition 2.4: Suppose now that $[x] = 0 \in \pi'_q(\Gamma)$. Then there exists $z \in \tilde{\pi}_{q+1}(\Gamma)$ such that $\partial_i z = e_q$ $i \leq q$ and $\partial_{q+1} z = x$. This means that $[x] = 0 \in \pi_q(\Gamma)$, and the proof is complete.

<u>Proposition 2.10</u>: A group complex Γ is minimal if and only if $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is zero for all q.

<u>Proof</u>: Suppose that Γ is minimal; then if $x, y \in \Gamma_{q+1}$, and $\partial_i x = \partial_i y$ for $i = 0, \ldots, q$, it follows that $\partial_{q+1} x = \partial_{q+1} y$. Now if $x \in \tilde{\pi}_{q+1}(\Gamma)$, then $\partial_i x = e_q = \partial_i e_{q+1}$ for $i \leq q$; hence, since Γ is minimal, $\partial_{q+1} x = \partial_{q+1} e_{q+1} = e_q$, and $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is zero.

Suppose now that $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is zero for all q, and that $x, y \in \Gamma_{q+1}$ are elements such that $\partial_i x = \partial_i y$ for $i \neq k$. Then $\partial_i x\bar{y} = e_q$ for $i \neq k$. If $k = q+1$, let $z = x\bar{y}$; if $k = q$, let $z = (s_q\partial_q x\bar{y})(y\bar{x})$; while if $k < q$, let $z = (s_q\partial_k x\bar{y})(s_{q-1}\partial_k y\bar{x})$. Then $\partial_i z = e_q$ for $i \neq q+1$, and $\partial_{q+1} z = \partial_k x\bar{y}$. But $z \in \pi_{q+1}(\Gamma)$; therefore by hypothesis $\partial_{q+1} z = e_q$, so that $\partial_k x = \partial_k y$, and the proof is complete.

In order to define the explicit complexes $K(\pi, n)$ of Eilenberg-MacLane ([3],[4],[5]) it is convenient to recall the definition of the standard alternating cochain complex for the q-simplex Δ_q with coefficients in the abelian group π. The n-dimensional cochain group $C^n(\Delta_q; \pi)$ is the group of functions u defined on (n+1)-tuples $(m_0, \ldots, m_n)$ of integers such that $0 \leq m_0 \leq \ldots \leq m_i \leq m_{i+1} \leq \ldots \leq m_n \leq q$ with values in π, such that $u(m_0, \ldots, m_n) = 0$ if $m_i = m_{i+1}$ for some $i \leq n$. $\delta : C^n(\Delta_q; \pi) \longrightarrow C^{n+1}(\Delta_q; \pi)$ is defined by $\delta u(m_0, \ldots, m_{n+1}) = \sum_{j=0}^{n+1} (-1)^j u(m_0, \ldots, m_{j-1}, m_{j+1}, \ldots, m_{n+1})$. Then $Z^n(\Delta_q; \pi)$ (the group of n-cocycles with coefficients in π) is the kernel of $\delta : C^n(\Delta_q; \pi) \longrightarrow C^{n+1}(\Delta_q; \pi)$.

<u>Notation</u>: Let $\lambda^i : \{0, \ldots, q\} \longrightarrow \{0, \ldots, q+1\}$ be defined by $\lambda^i(j) = j$ for $j < i$, and $\lambda^i(j) = j+1$ for $j \geq i$. Further, let $\eta^i : \{0, \ldots, q+1\} \longrightarrow \{0, \ldots, q\}$

be defined by $\eta^i(j) = j$ for $j \leq i$, $\eta^i(j) = j-1$ for $j > i$.

Definition 2.11: If π is an abelian group, define $K_q(\pi,n) = Z^n(\Delta_q;\pi)$. Further, define $\partial_i : K_{q+1}(\pi,n) \longrightarrow K_q(\pi,n)$ by $\partial_i u(m_0,\ldots,m_n) = u(\lambda^i(m_0),\ldots,\lambda^i(m_n))$, and $s_i : K_q(\pi,n) \longrightarrow K_{q+1}(\pi,n)$ by $s_i u(m_0,\ldots,m_n) = u(\eta^i(m_0),\ldots,\eta^i(m_n))$

Let $K(\pi,n) = \bigcup K_q(\pi,n)$

Theorem 2.12: If π is an abelian group, then

1) $K(\pi,n)$ is an abelian group complex,
2) $\pi_q(K(\pi,n)) = 0$ for $q \neq n$,
3) $\pi_n(K(\pi,n)) = \pi$.
4) $K(\pi,n)$ is minimal

Proof: The verification of u is routine, so that only 2),3) and 4) will be verified. First notice that $K_q(\pi,n) = 0$ for $q < n$. Therefore, $\tilde{\pi}_q(K(\pi,n)) = 0$ for $q < n$. Further since $Z^n(\Delta_n,\pi) = \pi$, we have that $\tilde{\pi}_n(K(\pi,n)) = K_n(\pi,n) = \pi$. Suppose now $u \in \tilde{\pi}_q(K(\pi,n))$ and $q > n$. Then $\partial_0 u = 0$; i.e. $u(m_{0+1},\ldots,m_{n+1}) = 0$ whenever $(m_0,\ldots,m_n)$ is a sequence of integers such that $0 \leq m_0 \leq \cdots \leq m_n \leq q-1$. This means $u(m_0,\ldots,m_n) = 0$ unless $m_0 = 0$. Therefore we only need consider sequences $(0,m_1,\ldots,m_n)$. However, $\partial_i u = 0$, or in other words

$u(0,m_1+1,\ldots,m_n+1) = 0$, but this implies that $u(0,m_1,\ldots,m_n) = 0$ unless $m_1 = 1$. Continuing in this fasion we see that $u(m_0,\ldots,m_n) = 0$ unless $m_i = 1$, for $i = 0,\ldots,n$. Then since u is a cocycle, $\delta u(0,\ldots,n+1) = \Sigma_{j=0}^{n+1}(-1)^j u(0,\ldots,j-1,j+1,\ldots,n+1) = 0$; thus $u = 0$, and $\tilde{\pi}_q(\Gamma) = 0$ for $q \neq n$. This implies 2) and 3). Statement 4) follows from Proposition 2.10, and the proof is complete.

Definition 2.13: A twisted Cartesian product is a triple (F,B,E) such that

1) F,B, and E are semi-simplicial complexes,
2) $E_q = \{(a,b) | a \in F_q, b \in B_q\}$, $q \geq 0$,
3) if $(a,b) \in E_{q+1}$, $\partial_i(a,b) = (\partial_i a, \partial_i b)$ for $i > 0$,
4) if $(a,b) \in E_q$, $s_i(a,b) = (s_i a, s_i b)$, and
5) if $p : E \longrightarrow B$ is the map defined by $p(a,b) = b$, then $p\partial_0 = \partial_0 p$.

F is called the fibre of the twisted Cartesian product, B the base, and E the total complex. Usually, but not always, the map p will be a fibre map.

E is the Cartesian product [6] of F and B if (F,B,E) is a twisted Cartesian product and $\partial_0(a,b) = (\partial_0 a, \partial_0 b)$ for $(a,b) \in E_{q+1}$, all q. In this case E is denoted by FxB. Also, the elements of E in any twisted Cartesian product will sometimes be written a x b.

If Γ is a monoid complex, and if (Γ,B,E) is a twisted Cartesian product, then Γ acts on the left of E according to the rule $a'.(a,b) = (a'a,b)$ for $a,a' \in \Gamma_q, b \in B_q$. The twisted Cartesian product is said to be <u>compatible with the left action of</u> Γ if $\partial_0(a,b) = \partial_0 a . \partial_0(e_{q+1},b)$ for $(a,b) \in E_{q+1}$. It will invariably be assumed that if a twisted Cartesian product has for fibre a monoid complex Γ, then the structure is compatible with the left action of Γ.

<u>Example 1</u>: Let A,B be topological spaces, $S(A),S(B)$ the total singular complexes of A and B respectively. Let $A \times B$ be the Cartesian product of A and B as topological spaces, and let $p_1 : A \times B \longrightarrow A, p_2 : A \times B \longrightarrow B$ be the projections. Then p_1 induces a semi-simplicial map which we shall still denote $p_1 : S(A \times B) \longrightarrow S(A)$, and p_2 induces $p_2 : S(A \times B) \longrightarrow S(B)$. It is easy to verify that the map $p' : S(A \times B) \longrightarrow S(A) \times S(B)$ defined by $p'(y) = (p_1(y), p_2(y))$ is an isomorphism of semi-simplicial complexes.

<u>Example 2</u>: Let E be the total space of a principal fibre bundle with fibre a topological group G and base space B. Assume that G acts on the left of E. Denote the total singular complexes of E,B, and G by $S(E),S(B)$, and $S(G)$ respectively. Since G acts on the left of E, $S(G)$ acts on the left of $S(E)$. Let $\phi : S(B) \longrightarrow S(E)$ be a pseudo-cross section, i.e. $\phi(\partial_i) = \partial_i \phi$ for $i > 0$, and $\phi s_i = s_i \phi$. Define $\psi : S(G) \times S(B) \longrightarrow S(E)$ by $\psi(a,b) = a \cdot \phi(b)$ for $a \in S(G)_q, b \in S(B)_q$. Now ψ is a 1:1 correspondance, is

compatible with ∂_i for $i > 0$, and with s_i for all i. Consequently if $S(E)$ is identified with $S(G) \times S(B)$ as a set by means of ψ we see that $(S(G), S(B), S(E))$ is a twisted Cartesian product. In other words, to make the total singular complex of a principal fibre bundle into the total complex of a twisted Cartesian product it suffices to choose a pseudo-cross section, and this can be done for any fibre map.

<u>Definition 2.14:</u> If Γ is a monoid complex, a twisted Cartesian product (Γ, B, E) is said to <u>satisfy the condition (W)</u> if

1) B_0 has one element, and

2) the map ϕ of B_{q+1} into E_q defined by $\phi(b) = \partial_0(e_{q+1}, b)$ is a 1:1 correspondance.

<u>Theorem 2.15:</u> If

1) Γ, Γ' are monoid complexes,

2) $f : \Gamma \longrightarrow \Gamma'$ is a map of monoid complexes,

3) (Γ, B, E) and (Γ', B', E') are twisted Cartesian products, the latter satisfying the condition (W), then there is a unique map $g : E \longrightarrow E'$ such that

4) $g(e_q \times B_q) \subset e_q \times B'_q$, and

5) $g(a,b) = f(a) \cdot g(e_q, b)$ for $(a,b) \in E_q$.

<u>Proof</u>: Suppose that we have such a map g. Denote by g_q the induced map of E_q into E'_q. Then

$g_0(e_0,b) \in e_0 \times B_0'$; but B_0' has one element, so that g_0 is uniquely determined. Let $S: E_q' \longrightarrow e_{q+1} \times B_{q+1}'$ denote the inverse of ∂_0. Since $g: e_q \times B_q \longrightarrow e_q \times B_q'$, we have $g_{q+1}(e_{q+1},b) = S\partial_0 g_{q+1}(e_{q+1},b) = Sg_q\partial_0(e_{q+1},b)$. Consequently there is at most one such map g; but the above formulas have defined a function g such that $\partial_0 g = g\partial_0$ and $g(e_q \times B_q) \subset e_q \times B_q'$. It remains to verify that $\partial_{i+1} g = g\partial_{i+1}$ and that $s_i g = g s_i$.

If $b \in B_1$, we observe that $\partial_1 g(e_1,b) = (e_0,b')$, where b' is the unique element of B_0'. Further $g\partial_1(e_1,b) = g(e_0,\partial_1 b) = (e_0,b')$. Suppose now that $\partial_i g = g\partial_i$ for $i \leq j$. Then for $b \in B_{q+2}$, $\partial_{j+1} g(e_{q+2},b) = \partial_{j+1} Sg\partial_0(e,b) = S\partial_j g\partial_0(e,b) = Sg\partial_j\partial_0(e,b) = S\partial_0 g\partial_{j+1}(e,b) = g\partial_{j+1}(e,b)$.

Now $S: e_q \times B_q' \longrightarrow e_{q+1} \times B_{q+1}'$ is 1:1 into; but since $\partial_0 s_0 = \text{identity}$, S is equal to s_0. Therefore $s_0 g(e,b) = s_0 Sg\partial_0(e,b) = SSg\partial_0(e,b) = Sg(e,b) = Sg\partial_0 s_0(e,b) = gs_0(e,b)$.

Finally, $s_{i+1} Sg\partial_0(e,b) = Ss_i g\partial_0(e,b) = Sgs_i\partial_0(e,b)$ by inductive hypothesis, and $Sgs_i\partial_0(e,b) = Sg\partial_0 s_{i+1}(e,b) = gs_{i+1}(e,b)$.

This completes the proof.

<u>Corollary 2.16:</u> If (Γ,B,E) and (Γ,A,D) are twisted Cartesian products satisfying the condition (W), and $g: E \longrightarrow D$,

$g': D \longrightarrow E$ are the maps of the preceding theorem, induced by the identity map of Γ, then $g'g$ and gg' are the identity maps of E and D.

We have now shown the essential uniqueness of twisted Cartesian products satisfying the condition (W), but it remains to prove existence. This will be done after the manner of MacLane [7].

<u>Definition 2.17</u>: Let Γ be a monoid complex.
Let $W_0(\Gamma) = \Gamma_0$, $W_{q+1}(\Gamma) = \Gamma_{q+1} + W_q(\Gamma)$, $\bar{W}_0(\Gamma)$ a set consisting of one element, and $\bar{W}_{q+1}(\Gamma) = \Gamma_q + \bar{W}_q(\Gamma)$.

Now in $W(\Gamma) = \bigcup_q W_q(\Gamma)$ define

1) $\partial_0(a,b) = \partial_0 a.b, \partial_1(a,b) = a$, where $a \in \Gamma_1, b \in \Gamma_0$;
2) $\partial_0(a,b) = \partial_0 a.b$ where $a \in \Gamma_{q+1}, b \in W_q(\Gamma)$, for $q > 0$;
3) $\partial_{i+1}(a,b) = (\partial_{i+1}a, \partial_i b)$;
4) $s_0(a,b) = (s_0 a, e_{q+1}, b)$, noting that $W_{q+2}(\Gamma) = \Gamma_{q+2} + \Gamma_{q+1} + W_q(\Gamma)$;
5) $s_{i+1}(a,b) = (s_{i+1}a, s_i b)$.

<u>Theorem 2.18</u>: If Γ is a monoid complex, then $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ is a twisted Cartesian product satisfying the condition (W).

The proof of this theorem is straightforward, and is left to the reader.

We remark that the notation here is somewhat different from that of [5], in that we consider only semi-simplicial complexes and not FD complexes, and that $\bar{W}$ corresponds to the W of [5].

If X is a Kan complex, and x is a point of X, it was shown in chapter 1 that there is a fibre space $(E(X,x),p,X)$ with fibre $\Omega(X,x)$ such that $\partial^{\#}: \pi_q(X,x) \longrightarrow \pi_{q-1}(\Omega(X,x),s_0(x))$ is an isomorphism for $q > 0$. If Γ is a monoid complex with homotopy, we shall always choose the base point to be $e_0 \in \Gamma_0$, and we shall denote $E(\Gamma,e_0)$ by $E(\Gamma)$, and $\Omega(\Gamma,e_0)$ by $\Omega(\Gamma)$.

Suppose now that Γ is a group complex such that $\tilde{\pi}_0(\Gamma) = \Gamma_0 = 0$. Then $E_q(\Gamma) = \Gamma_{q+1}$, and $0 \longrightarrow \Omega(\Gamma)_q \xrightarrow{i} E_q(\Gamma) \xrightarrow{p} \Gamma_q \longrightarrow 0$ is exact; but the homomorphism $s_0:\Gamma_q \longrightarrow \Gamma_{q+1}$ induces a homomorphism $u:\Gamma_q \longrightarrow E_q(\Gamma)$ such that pu is the identity. Therefore $E_q(\Gamma)$ is a split extension of Γ_q by $\Omega(\Gamma)_q$. this means that we may identify the set $E(\Gamma)$ with the set $\Omega(\Gamma) \times \Gamma$, the identification being compatible with the degeneracy operators s_i, and also with the face operators $\partial_i, i > 0$. Consequently we have the following

<u>Theorem 2.19:</u> If Γ is a group complex such that $\tilde{\pi}_0(\Gamma) = 0$, then $(\Omega(\Gamma),\Gamma,E(\Gamma))$ is a twisted Cartesian product satisfying the condition (W).

<u>Proof</u>: We need only verify that the twisted Cartesian product satisfies the condition (W). We have, however, that $\Gamma_0 = \tilde{\pi}_0(\Gamma)$ has one element. Further if $S:E_q(\Gamma) \longrightarrow E_{q+1}(\Gamma)$ is the homomorphism induced by $s_0:\Gamma_{q+1} \longrightarrow \Gamma_{q+2}$, then $\partial_0 S$ is the identity; but the image of S is just the subgroup

identified with $e_{q+1} \times \Gamma_{q+1}$, and the result is proved.

By the preceding theorem we have, therefore, that Γ is in a natural 1:1 correspondence with $\bar{W}(\Omega(\Gamma))$. However Γ is a group complex, and therefore in general has more structure than $\bar{W}(\Omega(\Gamma))$.

Suppose now that Γ is a commutative monoid complex. Then the multiplication map of $\Gamma \times \Gamma \longrightarrow \Gamma$ is a map of monoid complexes. This induces by the preceding theorem a map $W(\Gamma \times \Gamma) \longrightarrow W(\Gamma)$. However $W(\Gamma \times \Gamma)$ may be identified in a natural manner with $W(\Gamma) \times W(\Gamma)$. Now $W_q(\Gamma) = \Gamma_q + \cdots + \Gamma_0$, and the map $W(\Gamma) \times W(\Gamma) \longrightarrow W(\Gamma)$ is given by $(x_q, \ldots, x_0) \times (y_q, \ldots, y_0) \longrightarrow (x_q y_q, \ldots, x_0 y_0)$. Thus $W(\Gamma)$ is a commutative monoid complex. Further, $\bar{W}(\Gamma)$ is also a commutative monoid complex, and as a monoid, $W_q(\Gamma) = \Gamma_q + \bar{W}_q(\Gamma)$. Therefore if Γ is a commutative monoid complex, we shall always mean by $W(\Gamma)$ and $\bar{W}(\Gamma)$ the commutative monoid complexes whose structure has just been described. Notice that if Γ is an abelian group complex, then $W(\Gamma)$ and $\bar{W}(\Gamma)$ are abelian group complexes.

Now if Γ is an abelian group complex and $\tilde{\pi}_0(\Gamma) = 0$, then $E_q(\Gamma)$ is the direct sum $E_q(\Gamma) = \Omega_q(\Gamma) + \Gamma_q$. Further, the map $E(\Gamma) \times E(\Gamma) \longrightarrow E(\Gamma)$ given by the multiplication is just the map induced by $\Omega(\Gamma) \times \Omega(\Gamma) \longrightarrow \Omega(\Gamma)$. Therefore, in this case we may identify $E(\Gamma)$ and $W(\Omega(\Gamma))$, and Γ and $\bar{W}(\Omega(\Gamma))$, not only as semi-simplicial complexes, but as abelian group complexes.

<u>Theorem 2.20:</u> If Γ is a minimal abelian group complex such that $\pi_q(\Gamma) = 0$ for $q \neq n$, and $\pi_n(\Gamma) = \pi$, then Γ is naturally isomorphic to $K(\pi,n)$.

<u>Proof</u>: Since $\Omega^n(\Gamma)$ is an abelian group complex, $\overline{W}(\Omega^n(\Gamma))$ is also; we may thus iterate the $\overline{W}$ construction, setting $\overline{W}^1 = \overline{W}$, $\overline{W}^n = \overline{W}(\overline{W}^{n-1})$. Then since $\Gamma_q = 0$ for $q < n$,

$$\Gamma = \overline{W}^n(\Omega^n(\Gamma)).$$

Now $\Omega^n(\Gamma)$ is a minimal abelian group complex with one homotopy group π in dimension 0. Therefore if we prove the theorem for dimension 0, it will follow for dimension n by the above formula, since $K(\pi,n) = \overline{W}^n(K(\pi,0))$.

Suppose that $n = 0$. Then since Γ is minimal, $\Gamma_0 = \pi$, and $\tilde{\pi}_q(\Gamma) = 0$ for $q > 0$. Further $\Omega(\Gamma)$ is minimal, and $\pi_q(\Omega(\Gamma)) = 0$ for all q. Therefore $\Omega(\Gamma)_q = 0$ for all q. This means that if $x \in \Gamma_{q+1}$, $\partial_0 x = e_q$ and $\partial_1 \cdots \partial_{q+1} x = e_0$, then $x = e_{q+1}$. Suppose then that $x \in \Gamma_{q+1}$, and let $y = x\, s_0 \partial_0 \bar{x}$, $z = s_0 \partial_0 x$. Then $yz = x$, $\partial_0 y = e_q$, $\partial_1 \cdots \partial_{q+1} y_1 = (\partial_1 \cdots \partial_{q+1} x)(\partial_1 s_0 \partial_1 \cdots \partial_q \partial_0 \bar{x})$ $= (\partial_1 \cdots \partial_{q+1} x)(\partial_1 \cdots \partial_{q+1} \bar{x}) = e_0$. Therefore $y = e_{q+1}$, and $x = z$. In other words if $x \in \Gamma_{q+1}$, then $x = s_0 \partial_0 x$, and therefore $\partial_0 : \Gamma_{q+1} \longrightarrow \Gamma_q$ is an isomorphism. Consequently $\Gamma_q \approx \pi$ for all q and the mappings $\pi \longrightarrow \pi$ induced by either

s_0 or ∂_0 are the identity. However $\partial_1 s_0$ is the identity, and thus the mapping $\pi \longrightarrow \pi$ induced by ∂_1 is the identity. Continuing in this manner we see that the mappings $\pi \longrightarrow \pi$ determined by either $\partial_i : \Gamma_{q+1} \longrightarrow \Gamma_q$ or $s_i : \Gamma_q \longrightarrow \Gamma_{q+1}$ are the identity. This proves the theorem.

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Chapter 2, Appendix A

Abelian group complexes.

Abelian group complexes have very special properties; we have already seen in the first part of this chapter that there is a unique minimal abelian group complex with the abelian group π for its n-th homotopy group, and with all other homotopy groups zero. Essentially all other abelian group complexes are products of such complexes. This will be proved here only for minimal abelian group complexes, but it will be proved later in studying cohomology operations that this is true in general.

Before dealing with minimal abelian group complexes, it will be convenient to clear up a small point. In chapter I, appendix C, it was shown that there was, up to isomorphism, a unique minimal complex with a single non zero homotopy group π in dimension n. We know therefore that such a complex is isomorphic as a semi-simplicial complex with the explicit complex $K(\pi, n)$. We now see that the multiplication in $K(\pi, n)$ is determined by the fact that it has a single homotopy group π in dimension n, and that it is minimal.

<u>Theorem:</u> If X is a minimal complex, π an abelian group, $n \in Z^+$, and $\pi_q(X) = 0$ for $q \neq n$, $\pi_n(X) = \pi$, then there is a unique multiplication in X such that $X_n \cong \pi_n(X)$, and X is a group complex.

<u>Proof</u>: X_q has only one element if $q < n$. Therefore

the multiplication is determined in dimension k, where $k \leq n$. Suppose now that the multiplication is given in X_q for $q \leq k$, $k \geq n$, and we want to define a multiplication in X_{k+1}. Let $x, y \in X_{k+1}$; we want the product of x and y to be an element $z \in X_{k+1}$ such that $\partial_i z = \partial_i x \cdot \partial_i y$, $i = 0, \ldots, k+1$. There is a unique such z since $\pi_{k+1}(X) = 0$ and X is minimal. Therefore, we define $x \cdot y = z$. It is now easy to verify the group axioms using the uniqueness of z.

Now let us turn to the decomposition of minimal abelian group complexes.

<u>Theorem:</u> If Γ is a minimal abelian group complex, then $\Gamma = \overset{\infty}{\underset{n=0}{X}} K(\pi_n(\Gamma), n)$.

<u>Proof:</u> Since Γ is minimal, we have $\Gamma_0 = \pi_0(\Gamma)$. Further recall that $K_q(\pi_0(\Gamma), 0) \cong \pi_0(\Gamma)$, and that under this isomorphism all face and degeneracy operators correspond to the identity homomorphism. Now define $\phi : \Gamma \longrightarrow K(\pi_0(\Gamma), 0)$ by $\phi_q : \Gamma_q \longrightarrow K_q(\pi_0(\Gamma), 0)$ is the composite of $\partial_0^q : \Gamma_q \longrightarrow \Gamma_0$, and $s_0^q : \Gamma_0 = K_0(\pi_0(\Gamma), 0) \longrightarrow K_q(\pi_0(\Gamma), 0)$. ϕ is a homomorphism, since ∂_0 and s_0 are such, and we need only show that it commutes with ∂_i and s_i. We have $\partial_i s_0^q \partial_0^q = s_0^{q-1} \partial_0^q$ for $i \leq q$, and $s_0^{q-1} \partial_0^{q-1} \partial_i = s_0^{q-1} \partial_0^q$ for $i \leq q-1$, so that $\phi_{q-1} \partial_i x = \partial_i \phi_q x$, $i \leq q-1$. Further $s_0^{q-1} \partial_0^{q-1} \partial_q = s_0^{q-1} \partial_1 \partial_0^{q-1}$. Now since Γ is minimal, if $x, x' \in \Gamma_1$, and $\partial_0 x = \partial_0 x'$, then $\partial_1 x = \partial_1 x'$. This means

however that $\partial_1 x = \partial_1 s_0 \partial_0 x = \partial_0 x$ for $x \in \Gamma_1$, and that for $x \in \Gamma_q$, $s_0^{q-1}\partial_0^{q-1}\partial_q x = s_0^{q-1}\partial_0^q x$. Hence we also have $\phi_{q-1}\partial_q x = \partial_q \phi_q x$, and ϕ is a map of group complexes.

Let $\lambda : K(\pi_0(\Gamma), 0) \longrightarrow \Gamma$ be defined by $\lambda_0 : K_0(\pi_0(\Gamma), 0) \longrightarrow \Gamma_0$ is the identity, and $\lambda_q = s_0^q \lambda_0 \partial_0^q$. It is easily verified that λ is a map of group complexes, and $\phi\lambda$ is the identity. Consequently, letting $\Gamma' =$ kernel ϕ, we have $\Gamma \simeq K(\pi_0(\Gamma))\times\Gamma'$.

Now we are in a position to proceed by induction. First, $\tilde{\pi}_0(\Gamma') = 0$. Therefore $\Gamma' = \bar{W}(\Omega(\Gamma'))$ by theorem 2.19. However, by what we have already proved $\Omega(\Gamma') \simeq K(\pi_0(\Omega), 0) \times \Omega'$, and $\Gamma' \simeq K(\pi_1(\Gamma), 1) \times \bar{W}(\Omega')$ since $\pi_0(\Omega) = \pi_1(\Gamma)$, and $\bar{W}(K(\pi_0(\Gamma), 0)) = K(\pi_1(\Gamma), 1)$. The remaining details of the induction will be left to the reader, and the theorem is now considered proved.

Although we are not yet ready to prove that every abelian group complex has the same homotopy type as a product of $K(\pi, n)$'s, we will prove a key fact in this proof, namely that for abelian group complexes there is a natural map of <u>homology</u> into <u>homotopy</u>.

<u>Lemma:</u> If Γ is an abelian group complex, $x \in \tilde{\pi}_q(\Gamma)$, $\partial_q x = e_{q-1}$, $y \in \Gamma_{q+1}$, and $\prod_{j=0}^{q+1}(\partial_j y)^{\sigma(j)} = x$ where $\sigma(j) = (-1)^j$, then there exists $z \in \tilde{\pi}_{q+1}(\Gamma)$ such that $\partial_{q+1} z = x$.

<u>Proof:</u> Let $y^0 = y s_0 \partial_0 \bar{y}$. Then $\prod_{j=0}^{q+1}(\partial_j y^0)^{\sigma(j)} =$

$$x s_0 \partial_0 (\prod_{j=2}^{q+1} \partial_j y^{\sigma(j+1)}) = x s_0 \partial_0 (\bar{x} \partial_0 y \partial_1 \bar{y}) = x.$$

Suppose now that $r < q$, and we have defined y^r so that $\partial_i y^r = e_q$ for $i \leq r$, and $\prod_{j=0}^{q+1} (\partial_j y^r)^{\sigma(j)} = x$. Let $y^{r+1} = y^r s_{r+1} \partial_{r+1} \bar{y}^r$. It is not difficult to verify that $\partial_i y^{r+1} = e_q$ for $i \leq r+1$, and $\prod_{j=0}^{q+1} (\partial_j y^{r+1})^{\sigma(j)} = x$. Let $z = (y^q)^{\sigma(q+1)}$, and the result follows.

<u>Definition:</u> If Γ is an abelian group complex, define $\partial : \Gamma_q \longrightarrow \Gamma_{q+1}$ by $\partial x = \prod_{j=0}^{q+1} (\partial_j x)^{\sigma(j)}$.

Define $\pi_q^{\#}(\Gamma)$ to be kernel $\partial : \Gamma_q \longrightarrow \Gamma_{q-1}$ modulo image $\partial : \Gamma_{q+1} \longrightarrow \Gamma_q$.

Let $\phi : \pi_q(\Gamma) \longrightarrow \pi_q^{\#}(\Gamma)$ be the natural map.

<u>Proposition:</u> If Γ is an abelian group complex, then

$$\phi : \pi_q(\Gamma) \xrightarrow{\approx} \pi_q^{\#}(\Gamma).$$

<u>Proof:</u> By the preceding lemma ϕ is monomorphism. To prove that ϕ is an epimorphism suppose that $x \in \Gamma_q$, and $\prod_{j=0}^{q} (\partial_j x)^{\sigma(j)} = e_{q-1}$. Let $y^0 = x s_0 \partial_0 \bar{x}$.

Now $\prod_{j=0}^{q} (\partial_j y^0)^{\sigma(j)} = \prod_{j=2}^{q} (s_0 \partial_0 \partial_j \bar{x})^{\sigma(j)}$, and $e_{q-2} = \partial_0 e_{q-1} = \prod_{j=2}^{q} \partial_0 (\partial_j x)^{\sigma(j)}$. Consequently $\prod_{j=0}^{q} (\partial_j y^0)^{\sigma(j)} = e_{q-1}$.

Notice that $\prod_{j=0}^{q+1} \partial_j (s_0 x)^{\sigma(j)} =$ $s_0 \prod_{j=2}^{q+1} \partial_{j-1} x^{\sigma(j)} = s_0 \partial_0 x = x \bar{y}^0.$

Therefore, $[x] = [y^0] \in \pi_q^{\#}(\Gamma)$, and $\partial_0 y^0 = e_{q-1}$.

Now proceed inductively to find y^q such that $\partial_i y^q = e_{q-1}$ $i \leq q$, and $[x] = [y^q]$. Then y^q represents an element of $\pi_q(\Gamma)$, and the proof is complete.

<u>Theorem:</u> If Γ is an abelian group complex, then there is a map

$\lambda : H_q(\Gamma) \longrightarrow \pi_q(\Gamma)$ such that if $\mu : \pi_q(\Gamma) \longrightarrow H_q(\Gamma)$ is the natural map of homology into homotopy, then $\lambda\mu$ is the identity.

<u>Proof:</u> There is a natural map of $c_q(\Gamma) \longrightarrow \Gamma_q$ which sends $r \circ x$ into x^r for $x \in \Gamma_q$. This gives rise to a chain map of $C(\Gamma) \longrightarrow \Gamma$ or a homomorphism $\lambda^{\#} : H(\Gamma) \longrightarrow \pi^{\#}(\Gamma)$. We now have a commutative diagram

$$\begin{array}{ccc} \pi_q(\Gamma) & \xrightarrow{\mu} & H_q(\Gamma) \\ & \searrow^{\phi}_{\approx} & \downarrow \lambda^{\#} \\ & & \pi_q^{\#}(\Gamma) \end{array}$$

Letting $\lambda = \phi^{-1}\lambda^{\#}$, the proof is complete.

<u>Errata:</u> p. 1C-7, <u>Theorem (Poincaré)</u>:
isomorphism $\phi' : \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)] \longrightarrow H_1(X)$.

The construction FK

John Milnor

§1. Introduction

The reduced product construction of Ioan James [5] assigns to each CW-complex a new CW-complex having the same homotopy type as the loops in the suspension of the original. This paper will describe an analougous construction proceding from the category of semi-simplicial complexes to the category of group complexes. The properties of this construction FK are studied in §2.

A theorem of Peter Hilton [4] asserts that the space of loops in a union $S_1 \vee \dots \vee S_r$ of spheres splits into an infinite direct product of loops spaces of spheres. In §3 the construction of FK is applied to prove a generalization (Theorem 4) of Hilton's theorem in which the spheres may be replaced by the suspensions of arbitrary connected (semi-simplicial) complexes.

The author is indebted to many helpful discussions with John Moore.

§2. The construction.

It will be understood that with every semi-simplicial complex there is to be associated a specified base point.

Let K be a semi-simplicial complex with base point b_o. Denote $S_o^n b_o$ by b_n. Let FK_n denote the free group generated by the elements of K_n with the single relation $b_n = 1$. Let the face and degeneracy operations ∂_i, s_i in $FK = \cup FK_n$ be the unique homomorphisms which carry the generators k_n into $\partial_i k_n$, $s_i k_n$ respectively. Thus each complex K determines a group complex FK.

It will be shown that FK is a loop space for EK, the suspension of K. (Definitions will be given presently.)

Alternatively let $F^+K_n \subset FK_n$ be the free monoid (=associative semi-group with unit) generated by K_n, with the same relation $b_n = 1$. Then the monoid complex F^+K is also a loop space for EK. This construction is the direct generalization of James' construction. (See Lemma 4.)

The suspension EK of the semi-simplicial complex K is defined as follows. For each simplex k_n, other than b_n, of K there is to be a sequence (Ek_n), $(s_o Ek_n)$, $(s_o^2 Ek_n)$, ... of simplexes of EK having dimensions $n+1$, $n+2$, ... In addition there is to be a base point (b_o) and its degeneracies (b_n). The symbols $(s_o^i Eb_n)$ will be identified with (b_{n+i+1}). The face and degeneracy operations in EK are given by

$$\partial_j(Ek_n) = (E\,\partial_{j-1} k_n) \qquad (j > 0, n > 0)$$

$$s_j(Ek_n) = (E s_{j-1} k_n) \qquad (j > 0)$$

$$\partial_0(Ek_n) = (b_n), \qquad \partial_1(Ek_0) = (b_0)$$

$$s_0(Ek_n) = (s_0 Ek_n).$$

The face and degeneracy operations on the remaining simplexes $(s_0^i Ek_n) = s_0^i(Ek_n)$ are now determined by the identities

$$\partial_j s_0^i = \begin{cases} s_0^i \partial_{j-1} & (j > i) \\ s_0^{i-1} & (j \leq i \neq 0) \end{cases}$$

$$s_j s_0^i = \begin{cases} s_0^i s_{j-1} & (j > i) \\ s_0^{i+1} & (j \leq i). \end{cases}$$

It is not hard to show that this defines a semi-simplicial complex. The following lemma will justify calling it the suspension of K. Recall that the suspension of a topological space A with base point a_0 is the identification space of $A \times I$ obtaining by collapsing $(A \times \dot{I}) \cup (a_0 \times I)$ to a point.

Lemma 1. <u>The geometric realization $|EK|$ is canonically homeomorphic to the suspension of $|K|$.</u>

(For the definition of realization see [6.] In fact the required homeomorphism is obtained by mapping the point $(|k_n, \delta_n|, 1-t)$ of the suspension of $|K|$, where δ_n

has barycentric coordinates $(t_0,\dots,t_n)$ into the point $|(Ek_n), \delta_{n+1}| \in |EK|$, where δ_{n+1} has barycentric coordinates $(1-t, t\,t_0,\dots,t\,t_n)$.

Next the space of loops on a semi-simplicial complex K will be discussed. If K satisfies the Kan extension condition then ΩK can be defined as in [7]. This definition has two disadvantages:

(1) Many interesting complexes do not satisfy the extension condition. In particular EK does not.

(2) There is no natural way (and in some cases[1] no possible way) of defining a group structure in ΩK.

The following will be more convenient. A group complex G, or more generally a monoid complex, will be called a loop space for K if there exists a (semi-simplicial) principal bundle with base space K, fibre G, and with contractible total space T.

(By a principal bundle is meant a projection p of T onto K together with a left translation $G \times T \longrightarrow T$ satisfying

$$(g_n \cdot g_n') \cdot t_n = g_n \cdot (g_n' \cdot t_n)$$

where $g_n \cdot t_n = t_n$ if and only if $g_n = 1_n$, and where $g_n \cdot t_n = t_n'$ for some g_n if and only if $p(t_n) = p(t_n')$. A complex is called contractible if its geometric realization is contractible. This is equivalent to requiring that the integral homology groups and the fundamental group be trivial.)

[1] Let K be the minimal complex of the n-sphere, $n > 2$. Then it can be shown that there is no group complex structure in ΩK having the correct Pontrjagin ring.

The existence of such a loop space for any connected complex K has been shown in recent work of Kan, which generalizes the present paper. The following Lemma is given to help justify the definition.

Lemma 2. *If* K *satisfies the extension condition, and the group complex* G *is a loop space for* K, *then there is a homotopy equivalence* $\Omega K \longrightarrow G$.

The proof is based on the following easily proven fact (compare [7] p. 2-10): Every principal bundle can be given the structure of a twisted cartesian product. That is one can find a one-one function

$$\eta : G \times K \longrightarrow T$$

satisfying $\partial_i \eta = \eta \partial_i$ for $i > 0$ and $s_i \eta = \eta s_i$ for all i, where $\partial_0 \eta$ is given by an expression of the form

$$\partial_0 \eta (g_n k_n) = \eta((\partial_0 g_n)\cdot(\tau k_n), \partial_0 k_n).$$

(For this assertion the fibre must be a monoid complex satisfying the extension condition.) Thus the bundle is completely described by G and K together with the "twisting function" $\tau : K_n \longrightarrow G_{n-1}$; where τ satisfies the identities

$$s_i \tau = \tau s_{i+1} \quad (i \geq 0), \qquad \partial_i \tau = \tau \partial_{i+1} \quad i \geq 1,$$
$$\tau s_0 k_n = 1_n, \qquad (\partial_0 \tau k_n)\cdot(\tau \partial_0 k_n) = \tau \partial_1 k_n.$$

Now a map $\bar{\tau} : \Omega K_{n-1} \longrightarrow G_{n-1}$ is defined by $\bar{\tau}(k_n) = \tau(k_n)$. From the definition of ΩK and the above identities it follows that $\bar{\tau}$ is a map. From the homotopy sequence of the bundle it is easily verified that $\bar{\tau}$ induces isomorphisms of the homotopy groups, which proves Lemma 2.

To define a principal bundle with fibre FK and base space EK it is sufficient to define twisting functions $\tau : EK_{n+1} \longrightarrow FK_n$. These will be given by

$$\tau(Ek_n) = k_n, \qquad \tau(s_0^i Ek_{n-i}) = 1_n \qquad (i > 0).$$

Theorem 1. FK <u>is a loop space for</u> EK. <u>In fact the twisted cartesian product</u> $\{FK, EK, \tau\}$ has a con-<u>tractible total space</u>.

It is easy to verify that τ satisfies the conditions for a twisting function. Hence we have defined a twisted cartesian product, and therefore a principal bundle. Let T denote its total space. Note that T may be identified with FK x EK except that ∂_0 is given by

$$\partial_0(g_n, (Ek_{n-1})) = (\partial_0 g_n \cdot k_{n-1}, (b_{n-1}))$$

$$\partial_0(g_n, (s_0^i Ek_{n-i-1})) = (\partial_0 g_n, (s_0^{i-1}(Ek_{n-i-1})) \qquad (i \geq 1).$$

It will first be shown that the homology groups of T are trivial. This will be done by giving a contracting homotopy S for the chain complex C(T).

Lemma 3. <u>Let</u> G <u>be the free group on generators</u> x_α. <u>Then the integral group ring</u> ZG <u>has as basis</u>

<u>(over Z) the elements</u> $gx_\alpha - g$, <u>where</u> g <u>ranges over all elements of</u> G; <u>together with the element</u> 1.

The proof is not difficult. Now define S by the rules

$$S(1_n,(b_n)) = \begin{cases} 0 & (n \text{ even}) \\ (1_{n+1},(b_{n+1})) & (n \text{ odd}) \end{cases}$$

$$S[(g_n\cdot k_n,(b_n)) - (g_n,(b_n))]$$

$$= \sum_{i=0}^{n} (-1)^i [(s_i g_n,(s_o^i E\,\partial_o^i k_n)) - (s_i g_n,(b_{n+1}))]$$

$$S[(g_n,(s_o^{r-1} E\, k_{n-r})) - (g_n,(b_n))]$$

$$= \sum_{j=r}^{n} (-1)^j [(s_j g_n,(s_o^j E\,\partial_o^{j-r} k_{n-r})) - (s_j g_n,(b_{n+1}))]$$

where g_n ranges over all elements of the group FK_n.

It follows easily from Lemma 3 that the elements for which S has been defined form a basis for $C(T)$, providing that k_n, k_{n-r} are restricted to elements other than b_n, b_{n-r}. However the above rules reduce to the identity $0 = 0$ if we substitute $k_n = b_n$ or $k_{n-r} = b_{n-r}$. This shows that S is well defined.

The necessary identity $Sd + dS = 1 - \varepsilon$, where $dx_n = \sum_{i=0}^{n} (-1)^i \partial_i x_n$ and where $\varepsilon : C(T) \longrightarrow C(T)$ is the augmentation $(\varepsilon \sum \alpha_i (g_o, b_o) = \sum \alpha_i (1_o, b_o))$ can now be verified by direct computation. Since this computation is rather long it will not be given here.

Proof that $|T|$ is simply connected. A maximal tree in the CW-complex $|T|$ will be chosen. Then $\pi_1(|T|)$ can be considered as the group with one generator corresponding to each 1-simplex not in the tree, and one relation corresponding to each 2-simplex.

As maximal tree take all 1-simplexes of the form $(s_0 g_0, (Ek_0))$. Then as generators of $\pi_1(|T|)$ we have all elements $(g_1, (Ek_0))$ such that g_1 is non-degenerate. The relation $\partial_1 x = (\partial_2 x) \cdot (\partial_0 x)$ where $x = (s_1 g_1, (s_0 Ek_0))$ asserts that

$$(g_1, (Ek_0)) = (g_1, (b_1)) \cdot (s_0 \partial_0 g_1, (Ek_0))$$
$$= (g_1, (b_1)).$$

From the 2-simplex $(s_0 g_1, (Ek_1))$ we obtain

$$(g_1, (E\partial_0 k_1)) = (s_0 \partial_1 g_1, (E\partial_1 k_1)) \cdot (g_1 k_1, (b_1))$$
$$= (g_1 k_1, (b_1)).$$

Combining these two relations we have

$$(g_1, (b_1)) = (g_1 k_1, (b_1)),$$

from which it follows easily that

$$(g_1, (b_1)) = 1$$

for all g_1. In view of the first relation, this shows that $|T|$ is simply connected, and completes the proof of theorem 1.

The following theorem shows that FK is essentially unique.

Theorem 2. *Any principal bundle over* EK *with any group complex* G *as fibre is induced from the above bundle by a homomorphism* $FK \longrightarrow G$.

Proof: We may assume that this bundle is a twisted cartesian product with twisting function $\tau : (EK)_{n+1} \longrightarrow G_n$. Define the homomorphism $\bar{\tau} : FK \longrightarrow G$ by $\bar{\tau}(k_n) = \tau(Ek_n)$. Since $\bar{\tau}(b_n) = \tau(Eb_n) = \tau(s_o(b_n)) = 1_n$ this defines a homomorphism. It is easy to verify that $\bar{\tau}$ commutes with the face and degeneracy operations, and induces a map between the two twisted cartesian products.

Corollary. *If* G *is also a loop space for* EK *then there is a homomorphism* $FK \longrightarrow G$ *inducing an isomorphism between the Pontrjagin rings.*

This follows easily using [7], IV Theorem B.

Analogues of theorems 1 and 2 for the construction $F^+(K)$ can be proved using exactly the same formulas. The following shows the relationship between $F^+(K)$ and the construction of James.

Lemma 4. *If* K *is countable then the realization* $|F^+K|$ *is homeomorphic to the reduced product of* $|K|$.

In fact the product $(k_n, k'_n, k''_n, \ldots) \longrightarrow k_n \cdot k' \cdot k'' \cdot \ldots$ maps $K \times \ldots \times K$ into F^+K. Taking realizations we obtain a map $|K| \times \ldots \times |K| \longrightarrow |F^+K|$. From these maps it is easy to define a map of the reduced product of $|K|$ into $|F^+K|$, and to show that it is a homeomorphism.

§3. A theorem of Hilton

If A, B are semi-simplicial complexes with base points a_0, b_0 let $A \vee B$ denote the subcomplex $A \times [b_0] \cup [a_0] \times B$ of $A \times B$. Let $A \mathbin{\#} B$ denote the complex obtained from $A \times B$ by collapsing $A \vee B$ to a point. The notation $A^{(k)}$ will be used for the k-fold "collapsed product" $A \mathbin{\#} \cdots \mathbin{\#} A$.

The free product $G * H$ of two group complexes is defined by $(G * H)_n = G_n * H_n$. There is clearly a canonical isomorphism between the group complexes $F(A \vee B)$ and $FA * FB$.

Lemma 5. <u>The complex</u> $F(A \vee B)$ <u>is isomorphic</u> (<u>ignoring group structure</u>) <u>to</u> $FA \times F(B \vee (B \mathbin{\#} FA))$.

In fact we will show that $F(A \vee B)$ is a split extension:

$$I \longrightarrow F(B \vee (B \mathbin{\#} FA)) \longrightarrow F(A \vee B) \longrightarrow FA \longrightarrow I.$$

The collapsing map $A \vee B \xrightarrow{c} A$ induces a homomorphism c' of $F(A \vee B)$ onto FA. Denote the kernel of c' by F'. The inclusion $A \xrightarrow{i} A \vee B$ induces a homomorphism $i' : FA \longrightarrow F(A \vee B)$. Since $c'i'$ is the identity it follows that $F(A \vee B)$ is a split extension of F' by FA.

We will determine this kernel F'_n for some fixed dimension n. Let a, b, ϕ range over the n-simplexes other than the base point of A, B, and FA respectively. Then $F(A \vee B)_n$ is the free group $\{a, b\}$ and F'_n is the normal subgroup generated by the b. By the Reidemeister-Schreier theorem (see [8]) F'_n is freely generated by the elements $w(a)bw(a)^{-1}$ where $w(a)$ ranges over all elements of the free group $\{a\} = FA_n$. Thus

$$F'_n = \{b, \phi b \phi^{-1}\}.$$

Now setting $[b,\phi] = b\phi b^{-1}\phi^{-1}$ and making a simple Tietze transformation (see for example [1]) we obtain

$$F'_n = \{b, [b,\phi]\}.$$

Identify $[b,\phi]$ with the simplex $b \times \phi$ of $B \times F(A)$. Then we can identify F'_n with $F(B \vee (B \times FA))$. Since this identification commutes with face and degeneracy operations, this proves Lemma 5.

Lemma 6. <u>The group complex</u> $F(B \times FA)$ <u>is isomorphic to</u>

$$F((B \times A) \vee (B \times A \times FA)).$$

The inclusion $A \longrightarrow FA$ induces a homomorphism

$$F(B \times A) \longrightarrow F(B \times FA).$$

A homomorphism

$$F(B \times A \times FA) \longrightarrow F(B \times FA)$$

is defined by

$$b \times a \times \phi \longrightarrow (b \times a)(b \times \phi a)^{-1}(b \times \phi).$$

(This is motivated by the group identity $[[b,a],\phi] = [b,a][b,\phi a]^{-1}[b,\phi]$).

Combining these we obtain a homomorphism

$$F(B \times A) * F(B \times A \times FA) \longrightarrow F(B \times FA)$$

which is asserted to be an isomorphism.

Using the same notation as in Lemma 5 and identifying $b \ast a \ast \phi$ with $[[b,a],\phi]$ it is evidently sufficient to prove the following.

Lemma 7. <u>In the free group</u> $\{a,b\}$ <u>the subgroup freely generated by the elements</u> $[b,\phi]$ <u>is also freely generated by the elements</u> $[b,a]$ <u>and</u> $[[b,a],\phi]$.

The proof consists of a series of Tietze transformations. Details will not be given.

As a consequence of Lemma 6 we have:

Lemma 8. <u>For each</u> m <u>the group complex</u> $F(B \ast FA)$ <u>is isomorphic to</u>

$$F(B \ast A) \ast F(B \ast A \ast A) \ast \cdots \ast F(B \ast A^{(m)}) \ast F(B \ast A^{(m)} \ast FA).$$

Proof by induction on m. For $m=1$ this is just Lemma 6. Given this assertion for the integer $m-1$ it is only necessary to show that $F(B \ast A^{(m-1)} \ast FA)$ is isomorphic to $F(B \ast A^{(m)}) \ast F(B \ast A^{(m)} \ast FA)$. But this follows immediately from Lemma 6 by substituting $B \ast A^{(m-1)}$ in place of B.

Theorem 3. <u>If</u> A <u>and</u> B <u>are semi-simplicial complexes with</u> A <u>connected, then there is an inclusion homomorphism</u>

$$F(\bigvee_{i=1}^{\infty} B \ast A^{(i)}) \longrightarrow F(B \ast F(A))$$

<u>which is a homotopy equivalence.</u>

Proof. Every element of $F(\bigvee_{i=1}^{\infty} B \ast A^{(i)})$

is already contained in

$$F(\bigvee_{i=1}^{m} B \ast\!\!\ast A^{(i)}) = F(B \ast\!\!\ast A) \ast \cdots \ast F(B \ast\!\!\ast A^{(m)})$$

for some m. Hence by Lemma 8 it may be identified with an element of $F(B \ast\!\!\ast FA)$. Since A is connected, the "remainder term" $B \ast\!\!\ast A^{(m)} \ast\!\!\ast FA$ has trivial homology groups in dimensions less than m. From this it follows easily that the above inclusion induces isomorphisms of the homotopy groups in all dimensions.

Remark. The complex B may be eliminated from Theorem 3 by taking B as the sphere S^o, and noting the identity $S^o \ast\!\!\ast K = K$.

Combining theorem 3 with Lemma 5 we obtain the following

Corollary. *If A is connected then there is a homotopy equivalence*

$$F(A) \times F(\bigvee_{i=0}^{\infty} B \ast\!\!\ast A^{(i)}) \subset F(A \vee B).$$

This corollary will be the basis for the following.

Theorem 4. Let $A_1, \ldots, A_r$ *be connected complexes. Then* $F(A_1 \vee \cdots \vee A_r)$ *has the same homotopy type as a weak infinite cartesian product* $\prod_{i=1}^{\infty} F(A_i)$ *where each* $A_i, i > r,$ *has the form*

$$A_1^{(n_1)} \ast\!\!\ast \cdots \ast\!\!\ast A_r^{(n_r)} .$$

The number of factors of a given form is equal to the Witt number

$$\phi(n_1,\ldots,n_r) = \frac{1}{n}\sum_{d|\delta} \frac{\mu(d)(n/d)!}{(n_1/d)!\cdots(n_r/d)!}$$

where $n = n_1+\cdots+n_r$, $\delta = GCD(n_1,\ldots,n_r)$.

Proof. For $n=1,2,3,\ldots$ define complexes A_i, to be called "basic products of weight n" as follows, by induction on n. The given complexes $A_1,\ldots A_r$ are the basic products of weight 1. Suppose that

$$A_1,\ldots,A_r,\ldots,A_\alpha$$

are the basic products of weight less than n. To each $i=1,\ldots,r,\ldots,\alpha$ assume we have defined a number $e(i)<i$, where $e(1) = \cdots = e(r) = 0$. Then as basic products of weight n take all expressions $A_i \divideontimes A_j$ where weight A_i + weight $A_j = n$ and $e(i)\leq j<i$. Call these new complexes $A_{\alpha+1},\ldots,A_\beta$ in any order. If $A_h = A_i \divideontimes A_j$ define $e(h) = j$. (For this disussion we must consider complexes such as $(A\divideontimes B)\divideontimes C$ and $A\divideontimes(B\divideontimes C)$ to be distinct!) This completes the construction of the A_i.

For each $m\geq 1$ define

$$R_m = F(\bigvee_{\substack{h\geq m\\ e(h)<m}} A_h).$$

Thus $R_1 = F(A_1\vee\cdots\vee A_r)$.

Lemma 9. <u>There is a homotopy equivalence</u>

$$F(A_m)\times R_{m+1} \subset R_m .$$

Note that $R_m = F(A_m \vee B)$, where $B = \bigvee_{\substack{h>m \\ e(h)<m}} A_h$.

By the corollary to theorem 3 there is a homotopy equivalence

$$F(A_m) \times F(\bigvee_{i=0}^{\infty} B \mathbin{⋇} A_m^{(i)}) \subset F(A_m \vee B) = R_m .$$

Substituting in the definition of B and using the distributive law

$$(A \vee B) \mathbin{⋇} C = (A \mathbin{⋇} C) \vee (B \mathbin{⋇} C),$$

the second factor of the first expression becomes

$$F(\bigvee_{i=0}^{\infty} \bigvee_{\substack{h>m \\ e(h)<m}} A_h \mathbin{⋇} A_m^{(i)}).$$

But (filling in parentheses correctly) this is just

$$F(\bigvee_{\substack{h>m \\ e(h)\leq m}} A_h) = R_{m+1} ,$$

which proves Lemma 9.

Now it follows by induction that there is a homotopy equivalence

$$F(A_1) \times F(A_2) \times \cdots \times F(A_m) \times R_{m+1} \subset R_1 = F(A_1 \vee \cdots \vee A_r).$$

This defines an inclusion of the weak infinite cartesian product $\prod_{i=1}^{\infty} F(A_i)$ into R_1. Since $A_1, \ldots, A_r$ are connected, it follows easily that the "remainder terms" R_m are k-connected where $k \longrightarrow \infty$ as $m \longrightarrow \infty$. From this it follows that the above inclusion map induces isomorphisms of the homotopy groups in all dimensions. This proves the first part of theorem 4.

Let $\phi(n_1,\ldots,n_r)$ denote the number of A_h having the form $A_1^{(n_1)} \divideontimes \cdots \divideontimes A_r^{(n_r)}$. To compute these numbers consider the free Lie ring L on generators $\alpha_1,\ldots,\alpha_r$. Corresponding to each "basic product" $A_h = A_i \divideontimes A_j$ define an element $\alpha_h = [\alpha_i, \alpha_j]$ of L, for $h = r+1, r+2, \ldots$. Then the elements α_h obtained in this way are exactly the standard monomials of M. Hall [2] and P. Hall [3]. M.Hall has proved that these elements form an additive basis for L.

The number of linearly independent elements of L which involve each of the generators $\alpha_1,\ldots,\alpha_r$ a given number $n_1,\ldots,n_r$ of times has been computed by Witt [9]. Since his formula is the same as that in theorem 4, this completes the proof.

In conclusion we mention one more interesting consequence of theorem 3.

Theorem 5. <u>If</u> A <u>is connected then the complex</u> EFA <u>has the same homotopy type as</u> $\bigvee_{i=1}^{\infty} E A^{(i)}$.

The proof is based on the following lemma, which depends on Theorem 1.

Lemma 10. <u>If</u> A <u>is connected, there is a homotopy equivalence</u>

$$EA \subset \overline{W}FA .$$

In fact the inclusion is defined by

$(s_0^1 E a_n) \longrightarrow s_0^1(a_n, 1_{n-1}, \ldots, 1_0)$. It is easily verified that this is a map, and that it induces a map of the twisted cartesian product T into the twisted cartesian product W. Since both total spaces are acyclic, it follows from [7], IV Theorem A that the homology groups of EA map isomorphicly into those of $\bar{W}FA$. Since both spaces are simply connected, this completes the proof of Lemma 10.

Now from Theorem 3 we have a homotopy equivalence

$$\bar{W}F(\vee_{i=1}^{\infty} A^{(i)}) \subset \bar{W}FFA.$$

In view of Lemma 10, and the identity

$$E(A \vee B) = EA \vee EB,$$

this completes the proof.

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Chapter 3. Acyclic Models

Acyclic models.[1]

If $\mathcal{A}$ is a category and $\mathfrak{M}$ a subset of the objects of $\mathcal{A}$, we shall denote by $\mathcal{A}^{\mathfrak{M}}$ the set of mappings in $\mathcal{A}$ with domain in $\mathfrak{M}$.

Definition 3.1: The quadruple$(\mathcal{A}, \mathfrak{M}, \alpha, \beta)$ will be called a category with models if $\mathcal{A}$ is a category, $\mathfrak{M}$ a certain subset of the objects of $\mathcal{A}$, called the set of models, and α, β are functions of $\mathcal{A}^{\mathfrak{M}}$ into itself such that

0) $\alpha(1(M)) = \beta(1(M)) = 1(M)$, $M \in \mathfrak{M}$

1) $\beta(u)\alpha(u) = u$.

2) $\alpha(\beta(u)) = \beta(\alpha(u)) = 1(M)$ where $M =$ domain $\beta(u)$ $=$ range $\alpha(u)$.

3) $\beta(f u) = \beta(f\beta(u))$ where f is a mapping of $\mathcal{A}$ such that domain $f =$ range u.

4) $\alpha(f u) = \alpha(f\beta(u))\alpha(u)$, where f means the same as in 3);

where $u \in \mathcal{A}^{\mathfrak{M}}$ throughout.

Notice that 3) implies $\beta(\beta(u)) = \beta(u)$ and 1) and 2) imply $\alpha(\alpha(u)) = \alpha(u)$.

Assumption: For the rest of this section, $(\mathcal{A}, \mathfrak{M}, \alpha, \beta)$ is a fixed category with models; it will usually be denoted by $\mathcal{A}$; "object" will mean "object of $\mathcal{A}$", and "mapping", "mapping of $\mathcal{A}$".

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1)The theory of acylic models was introduced by Eilenberg and MacLane [1]. The version given here is a part of [2].

Definition 3.2: For any object A, S(A) will denote the set of mappings $u : M \longrightarrow A$ with $M \in \mathcal{M}$, such that $\alpha(u) = 1(M)$.

Notation 3.3: For the rest of this paper, Λ will denote a fixed commutative ring with unit element; $\mathcal{G}_\Lambda$ the category of Λ-modules and Λ-homomorphisms.

Definitions 3.4: If $K : \mathcal{A} \longrightarrow G_\Lambda$ is a covariant functor, and $u : M \longrightarrow A$ an element of $\mathcal{A}^{\mathcal{M}}$, we shall denote by $K(M,u)$ the module $K(M)$ with u added as an indexing symbol; the elements of $K(M, u)$ will be denoted by (k, u), where $k \in K(M)$; $(k, u) + (k', u) = (k + k', u)$, $\lambda(k, u) = (\lambda k, u)$ if $\lambda \in \Lambda$. We define the natural isomorphisms

$$K(M) \underset{j(u)}{\overset{i(u)}{\rightleftarrows}} K(M, u)$$

by $i(u)k = (k, u)$; $j(u)(k, u) = k$.

We now define a new functor $\hat{K} : \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ as follows:

$\hat{K}(A) = \Sigma_{u \in S(A)} K(M, u)$ for any object A.

$\hat{K}(f) \mid K(M,u) = i(\beta(fu))\, K(\alpha(fu))\, j(u)$ for any map $f : A \longrightarrow B$; thus $\hat{K}(f) \mid K(M, u) : K(M, u) \longrightarrow K(M', \beta(fu))$ where $M' = \text{domain } \beta(fu)$; clearly $\beta(fu) \in S(B)$, as required.

Next, we define a natural transformation of functors $\Gamma_K : \hat{K} \longrightarrow K$ by

$\Gamma_K(A) \mid K(M, u) = K(u)\, j(u)$ for any object A; the necessary naturality condition is easily verified.

The functor K is said to be <u>representable</u> if there is a natural transformation of functors $\chi_K : K \longrightarrow \hat{K}$ such that $\Gamma_K \chi_K : K \longrightarrow K$ is the identity.

<u>Notations and Conventions 3.5</u>: Let $d\mathcal{G}_\Lambda$ denote the category of differential Λ-modules and admissible homomorphisms; in other words, an object of $d\mathcal{G}_\Lambda$ is a pair (G, d_G) such that $G = \Sigma_{n \geq 0} G_n$, a direct sum of Λ-modules, d_G is a Λ-endomorphism of G such that $d_G d_G = 0$, $d\, G_n \subset G_{n-1}$ for $n > 0$ and $d_G G_0 = 0$. A mapping $f : (G, d_G) \longrightarrow (F, d_F)$ of $d\mathcal{G}_\Lambda$ is a Λ-homomorphism $f : G \longrightarrow F$ such that $d_F f = f d_G$. Usually we shall denote (G, d_G) simply by G, and d_G, indiscriminately, by d. The elements of G_n will be called <u>n-dimensional.</u> For every object (G, d) we define the k-skeleton (G^k, d), itself an object of $d\mathcal{G}_\Lambda$, by setting $G^k_n = G_n$ for $n \leq k$ and $G^k_n = 0$ for $n > k$, and using for d the natural restriction. In the category $d\mathcal{G}_\Lambda$, homology is defined as usual; we write $Z(G) =$ kernel d_G, $B(G) =$ image d_G, $H(G) = Z(G)/B(G)$, $Z_n(G) = Z(G) \cap G_n$, $B_n(G) = B(G) \cap G_n$, $H_n(G) = Z_n(G)/B_n(G)$ so that $H(G) = \Sigma_{n \geq 0} H_n(G)$. Note that H and H_n can be regarded as covariant functors $d\mathcal{G}_\Lambda \longrightarrow \mathcal{G}_\Lambda$; the definition of $H(f)$, $H_n(f)$ being evident. The natural transformation $G_0 \longrightarrow H_0$ will be indiscriminately denoted by ε.

<u>Definition 3.6:</u> If $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ is a covariant functor, define $K^n : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ by $K^n(A) = (K(A))^n$ for any object

A and $K^n(f) = K(f) \mid K^n(A)$ for any map $f : A \longrightarrow B$. Further, define $K_n : \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ by $K_n(A) = (K(A))_n, K_n(F) \mid K_n(A) = K(F) \mid K_n(A)$. We say that K is <u>representable</u> if K_n is representable for every $n \geq 0$; this is the same as saying that K is representable when regarded as a functor $K : A \longrightarrow \mathcal{G}_\Lambda$.

<u>Notations 3.7:</u> By $\hat{\mathfrak{M}}$ we denote the subcategory of $\mathcal{A}$ the objects of which are those of $\mathfrak{M}$, and the maps all maps of the type $\alpha(u)$, or compositions of such maps.

Let $K, L : \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ be two functors and $U : K|\hat{\mathfrak{M}} \longrightarrow L|\hat{\mathfrak{M}}$ a natural transformation; then U determines a natural transformation $\hat{U} : \hat{K} \longrightarrow \hat{L}$ by $\hat{U}|K(M, u) = i(u)\ U(M)\ j(u)$ [1] (cf 1.4); so that $\hat{U} \mid K(M, u) : K(M, u) \longrightarrow L(M, u)$. If U is the restriction of $T : K \longrightarrow L$, i.e. $U = T\ |\hat{\mathfrak{M}}$, we shall write $\hat{U} = \hat{T}$; and in this case we have $T\Gamma_K = \Gamma_L \hat{T}$.

This last remark is applied, for a functor $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$, to $d : K \longrightarrow K$; we thus obtain $\hat{d} : \hat{K} \longrightarrow \hat{K}$ such that $\hat{d}^2 = 0$, $d\Gamma_K = \Gamma_K \hat{d}$; and accordingly we can (and shall) regard $\hat{K}$ as a functor $\mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$.

<u>Definition 3.8:</u> A covariant functor $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ will be said to be a cyclic on models if there exist natural transformations[2] of functors.

$$\eta : H_0 K\,|\hat{\mathfrak{M}} \longrightarrow K_0\,|\hat{\mathfrak{M}}, \qquad U : K\,|\hat{\mathfrak{M}} \longrightarrow K\,|\hat{\mathfrak{M}}$$

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1) Note that we use $i(u)$, $j(u)$ indiscriminately. In this formula $j(u)$ is related to K, $i(u)$ to L.

2) Here $d\mathcal{G}_\Lambda$ is considered only as a category of Λ-modules; i.e. $U(M)$ is a homomorphism of Λ-modules, but does not preserve gradation nor commute with d.

such that $U K_n | \hat{\mathfrak{M}} \subset K_{n+1} | \hat{\mathfrak{M}}$ and, writing $U_n = U | (K_n | \hat{\mathfrak{M}})$, the following are satisfied:

(1) $$d U_0 = 1 - \eta \varepsilon$$

(2) $$d U_n + U_{n-1} d = 1 \ (K_n | \hat{\mathfrak{M}}) \quad \text{for} \quad n > 0$$

(3) $$U_0 \eta = 0$$

where $\varepsilon : K_0 \longrightarrow H_0 K$ is the natural transformation. Notice that for $M \in \mathfrak{M}$, any element $h \in H_0 K(M)$ is of the form εk where $k \in K_0(M)$. Now, by the above

$$\varepsilon \eta \varepsilon k = \varepsilon (1 - d U_0) k = \varepsilon k$$

so that condition (1) implies

(4) $$\varepsilon \eta = 1$$

<u>Lemma 3.9:</u> If $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ is acyclic on models, there are natural transformations of functors $\hat{\eta} : H_0 \hat{K} \longrightarrow \hat{K}_0$, $U : \hat{K} \longrightarrow \hat{K}$ such that $\hat{U} \hat{K}_n \subset \hat{K}_{n+1}$ and, writing

$\hat{U}_n = U | \hat{K}_n$,

(1) $$\hat{d} \hat{U}_0 = 1 - \hat{\eta} \hat{\varepsilon}$$

(2) $$\hat{d} \hat{U}_n + \hat{U}_{n-1} \hat{d} = 1 \quad \text{if} \quad n > 0$$

(3) $$\hat{U}_0 \hat{\eta} = 0$$

(4) $$\hat{\varepsilon} \hat{\eta} = 1.$$

This is immediate from 1.8.

<u>Notation 3.10:</u> By $\overline{\mathfrak{M}}$ denote the sub-category of $\mathcal{A}$ the objects of which are all those of $\mathfrak{M}$, and the mappings all mappings having models as domain and range. $\hat{\mathfrak{M}}$ and $\overline{\mathfrak{M}}$ have the same objects; but $\overline{\mathfrak{M}}$ has more mappings.

Theorem 3.11: Let $K, L : \mathcal{Q} \longrightarrow d\mathcal{G}_\Lambda$ be covariant functors and let $T : H_0K|\mathfrak{M} \longrightarrow H_0L|\mathfrak{M}$ be a natural transformation of functors; let K be representable and L acyclic on models. Then there is a natural transformation of functors $\Phi : K \longrightarrow L$ such that $\Phi \,|\, (K_0|\mathfrak{M})$ induces T; Φ will be called "an extension of T".

Proof: T induces $\hat{T} : H_0\hat{K} \longrightarrow H_0\hat{L}$. Since L is acyclic, we have transformations $\hat{U} : \hat{L} \longrightarrow \hat{L}$, $\hat{\eta} : H_0\hat{L} \longrightarrow \hat{L}_0$ satisfying the conditions of 1.9. We define $\Phi_0 : K_0 \longrightarrow L_0$ by $\Phi_0 = \Gamma_L \hat{\eta} \hat{T} \hat{\varepsilon} \chi_K$ and $\Phi_1 : K_1 \longrightarrow L_1$ by $\Phi_1 = \Gamma_L \hat{U}_0 \hat{\Phi}_0 \hat{d} \chi_K$. (cf. 1.7). Then $d\,\Phi_1 = d\,\Gamma_L \hat{U}_0 \hat{\Phi}_0 \hat{d} \chi_K = \Gamma_L \hat{d} \hat{U}_0 \hat{\Phi}_0 \hat{d} \chi_K = \Gamma_L (1 - \hat{\eta}\hat{\varepsilon}) \hat{\Phi}_0 \hat{d} \chi_K = \Gamma_L \hat{\Phi}_0 \hat{d} \chi_K = \Phi_0 \Gamma_K \hat{d} \chi_K = \Phi_0\, d$, since $\hat{\varepsilon} \hat{\Phi}_0 \hat{d} = 0$; in fact $\varepsilon\, \Phi_0\, d\, |\mathfrak{M} = 0$. For restricting everything to the category $\mathfrak{M}$, we have $\varepsilon\, \Phi_0\, d = \varepsilon \Gamma_L \hat{\eta} \hat{T} \hat{\varepsilon} \chi_K\, d = \Gamma_{HL} \hat{\varepsilon} \hat{\eta} \hat{T} \hat{\varepsilon} \chi_K\, d = \Gamma_{HL} \hat{T} \hat{\varepsilon} \chi_K\, d = T\, \Gamma_{HK} \hat{\varepsilon} \chi_K\, d = T \varepsilon \Gamma_K \chi_K d = T\, \varepsilon\, d = 0.$

We proceed by induction: if Φ_k is defined, so is $\hat{\Phi}_k$, and we write $\Phi_{k+1} = \Gamma_L \hat{U}_K \hat{\Phi}_k \hat{d} \chi_K$; and verify

$$\begin{aligned} d\,\Phi_{k+1} &= d\Gamma_L \hat{U}_k \hat{\Phi}_k \hat{d} \chi_K \\ &= \Gamma_L \hat{d} \hat{U}_k \hat{\Phi}_k \hat{d} \chi_K \\ &= \Gamma_L (1 - \hat{U}_{k-1} \hat{d}) \hat{\Phi}_k \hat{d} \chi_K \\ &= \Gamma_L \hat{\Phi}_k \hat{d} \chi_K \\ &= \Phi_k \Gamma_K \hat{d} \chi_K \\ &= \Phi_k d, \text{ as required.} \end{aligned}$$

Further notice that on $\widetilde{\mathfrak{M}}$ we have

$$\varepsilon\Phi_0 = \varepsilon\Gamma_L\hat{\eta}\hat{T}\hat{\varepsilon}\lambda_K = \Gamma_{HL}\hat{\varepsilon}\hat{\eta}\hat{T}\hat{\varepsilon}\lambda_K = \Gamma_{HL}\hat{T}\hat{\varepsilon}\lambda_K$$

$$= T\Gamma_{HK}\hat{\varepsilon}\lambda_K = T\varepsilon\Gamma_K\lambda_K = T\varepsilon,$$ and so Φ is an extension of T.

<u>Definition 3.12:</u> Let $K, L : \mathcal{a} \longrightarrow d\mathcal{g}_\Lambda$ be covariant functors and let $\Phi, \Phi' : K \longrightarrow L$ be natural transformations. A <u>homotopy</u> V between Φ and Φ' is a natural transformation of functors $V : K \longrightarrow L$ such that $VK_n \subset L_{n+1}$ and $dV + Vd = \Phi - \Phi'$.

<u>Theorem 3.13:</u> If $K, L : \mathcal{a} \longrightarrow d\mathcal{g}_\Lambda$ are covariant functors, $T : H_0K \mid \widetilde{\mathfrak{M}} \longrightarrow H_0L \mid \widetilde{\mathfrak{M}}$ is a natural transformation of functors, K is representable and L acyclic on models, and if Φ, Φ' are extensions of T (cf. 1.7), then there is a homotopy V between Φ and Φ'.

Proof: Since Φ, Φ' are both extensions of T, we must have $\hat{\varepsilon}\hat{\Phi}_0 = \hat{\varepsilon}\hat{\Phi}'_0 = \hat{T}\hat{\varepsilon}$. We define

$$V_0 = \Gamma_L\hat{U}_0(\Phi_0 - \Phi'_0)\lambda_K$$

where U, η again are the functors appropriate to L. Then

$$\begin{aligned} dV_0 &= \Gamma_L\hat{d}\hat{U}_0(\hat{\Phi}_0 - \hat{\Phi}'_0)\lambda_K \\ &= \Gamma_L(1 - \hat{\eta}\hat{\varepsilon})(\hat{\Phi}_0 - \hat{\Phi}'_0)\lambda_K \\ &= \Gamma_L(\hat{\Phi}_0 - \hat{\Phi}'_0)\lambda_K \\ &= \Phi_0 - \Phi'_0 \end{aligned}$$

as required. Now, we proceed inductively. Let $V_0, \ldots, V_k$ with all the necessary properties be defined. Then, in particular

$$d(\phi_{k+1}-\phi'_{k+1}-V_k d) = d\phi_{k+1} - d\phi'_{k+1} - dV_k d$$
$$= (\phi_k - \phi'_k - dV_k)d$$
$$= V_{k-1}\, d\, d = 0$$

whence $\hat{d}(\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d}) = 0$. Now we define

$$V_{k+1} = \Gamma_L \hat{U}_{k+1}\, (\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K.$$

Then

$$dV_{k+1} = \Gamma_L\, \hat{d}\, \hat{U}_{k+1}(\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K.$$
$$= \Gamma_L(1 - \hat{U}_k\, \hat{d})\, (\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K.$$
$$= \Gamma_L\, (\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K$$
$$= \phi_{k+1} - \phi'_{k+1} - V_k\, d\ ,$$

as required.

Combining 1.11 and 1.13 we get

<u>Theorem 3.14:</u> If K, L.: $\mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ are covariant representable functors which are acyclic on models, and if $T : H_0K|\mathfrak{M} \longrightarrow H_0L|\mathfrak{M}$ is a natural equivalence, then there is a unique natural equivalence $\phi_* : HK \longrightarrow HL$ such that $\phi_*|(H_0K|\mathfrak{M}) = T$, and such that ϕ_* is induced by an extension of T

Now let $\mathcal{A}$ be the category of semi-simplicial complexes and maps. The model objects are to be the semi-simplicial complexes Δ_q (cf. appendix 1A), and α and β are defined as follows. If $u: \Delta_q \longrightarrow X$, let $x = u(0,\ldots,q) \in X_q$. If x is non-degenerate, define $\alpha(u): \Delta_q \longrightarrow \Delta_q$ to be the identity, and $\beta(u) = u: \Delta_q \longrightarrow X$. Suppose that x is degenerate; then $x = s_{i_r} \cdots s_{i_1} y$, where y is non-degenerate and $i_r > \cdots > i_1$.

(1) Define $\beta(u): \Delta_{q-r} \longrightarrow X$ to be the map determined by $\beta(u)(0,\ldots,q-r) = y$. Then $\beta(u)\ (s_{i_r} \cdots s_{i_1}(0,\ldots,q-r)\ ) = x$.

(2) Define $\alpha(u): \Delta_q \longrightarrow \Delta_{q-r}$ to be the map determined by $\alpha(u)(0,\ldots,q) = s_{i_r} \cdots s_{i_1}(0,\ldots,q-r)$.

It is easily verified that α and β satisfy the axioms and are uniquely defined, so that $\mathcal{A}$ is a category with models.

Let $d\mathcal{G}$ be the category of differential modules over the integers (taking Λ as the ring of integers in 3.5). We define functors $C, C_N: \mathcal{A} \longrightarrow d\mathcal{G}$ as follows. Let $C_q(X)$ be the free abelian group having the elements of X_q as generators; and set $C(X) = \sum_q C_q(X)$. The homomorphism $\partial: C_{q+1}(X) \longrightarrow C_q(X)$ is determined by $\partial x = \sum_0^i (-1)^i \partial_i x$, $x \in X_{q+1}$. Let $D_q(X)$ be the free abelian group having the degenerate elements of X_q as generators, and set $C_q(X)_N = C_q(X)/D_q(X)$, $C(X)_N = \sum C_q(X)_N$. Now $\partial(D_q(X)) \subset D_{q-1}(X)$; for

$$\partial s_i x = \sum_{j<i} (-1)^j \partial_j s_i x + (-1)^i \partial_i s_i x + (-1)^{i+1} \partial_{i+1} s_i x + \sum_{j>i+1} (-1)^j \partial_j s_i x$$

$$= \sum_{j<i} (-1)^j s_{i-1} \partial_j x + \sum_{j>i+1} s_i \partial_{j-1} x,$$

since the two middle terms are equal. Therefore ∂ induces a homomorphism $\partial: C_{q+1}(X)_N \longrightarrow C_q(X)_N$. It follows in the usual manner that $\partial\partial = 0$ in both cases, which completes the definition of C and C_N. C is called the <u>chain functor.</u>

C_N the <u>normalized chain functor</u>.

We now wish to show that C and C_N give the same homology. There is a natural transformation of functors $\Phi: C \longrightarrow C_N$ such that $\Phi(X): C(X) \longrightarrow C(X)_N$ is the projection onto the factor group. In order to obtain a homotopy inverse for Φ, we shall show that both C and C_N are representable and acyclic on models, and shall then apply theorems 3.11 and 3.13.

To show that C is representable, we define a natural transformation $\chi_c: C \longrightarrow \hat{C}$ as follows. Recall that $C_q(X)$ is free abelian, and let $x \in X_q$ be a generator. There is a unique map $u: \Delta_q \longrightarrow X$ such that $u(0,\ldots,q) = x$. Let Δ_{q-r} be the domain of $\beta(u)$. Then $\chi_c(X)(x) = (\alpha(u)(0,\ldots,q), \beta(u)) \in (C(\Delta_{q-r}), \beta(u)) \subset \hat{C}(X)$. Since $\Gamma\chi_c = \text{identity}$, C is representable.

Now the homomorphism $\chi(X): C(X) \longrightarrow \hat{C}(X)$ carries $D(X)$ into the subgroup generated by degenerate simplexes, and hence induces a homomorphism $\chi'(X): C(X)_N \longrightarrow (\widehat{C(X)_N})$. It is easy to verify that $\chi': C_N \longrightarrow (\hat{C}_N)$ is a natural transformation of functors, and that $\Gamma\chi' = \text{identity}$, so that C_N is also representable. To show that C and C_N are acyclic on models, define

$S: (\Delta_q)_r \longrightarrow (\Delta_q)_{r+1}$ by $S(m_0,\ldots,m_r) = (0,m_0,\ldots,m_r)$.

Then S has the properties

$$\partial_0 S(m_0,\ldots,m_r) = (m_0,\ldots,m_r)$$

$$\partial_{i+1} S = S\,\partial_i$$

$$s_{i+1} S = S\, s_i$$

$$s_0 S = S^2$$

Let $x \in (\Delta_q)_r, r > 0$. Then

$$\partial S x = \sum_{0}^{r+1} (-1)^i \partial_i S x = x + \sum_{1}^{r+1} (-1)^i S \partial_{i-1} x,$$ so that

$\partial S x + S \partial x = x$. If $x \in (\Delta_q)_0$, then $\partial S x = x - (0)$.

Now suppose that $h: \Delta_{q+1} \longrightarrow \Delta_q$ is a map in the category $\hat{\mathcal{M}}$. Since h is a simplicial map onto Δ_q, we need only define it on the vertices, and it has the form

$$h(j) = \begin{Bmatrix} j & \text{for } j \leq i \\ j-1 & \text{for } j > i \end{Bmatrix}$$ for some $i \leq q$. Then clearly

$S h = h S$. Since any map in $\hat{\mathcal{M}}$ is a composition of maps of the form of h, S commutes with the maps of $\hat{\mathcal{M}}$.

We define a natural transformation of functors $U : C | \hat{\mathcal{M}} \longrightarrow C | \hat{\mathcal{M}}$ as follows. The homomorphism[1] $U(\Delta_q): C(\Delta_q) \longrightarrow C(\Delta_q)$ is determined by $U(\Delta_q)(x) = S(x)$ for $x \in X_q, x \neq (0)$; $U(\Delta_q)(0) = 0$. The fact that S commutes with the maps of $\hat{\mathcal{M}}$ implies that U is a natural transformation of functors. Define $\eta : H_0 C / \hat{\mathcal{M}} \longrightarrow C_0 / \hat{\mathcal{M}}$ as follows: $H_0(\Delta_q)$ may be considered in a natural manner as a free group on the generator (0), and $\eta(\Delta_q) : H_0(\Delta_q) \longrightarrow C_0(\Delta_q)$ is determined by $\eta(\Delta_q)(0) = (0) \in C_0(\Delta_q)$. η is clearly a natural transformation of functors.

The conditions satisfied by S insure that U satisfies the conditions of (3.8), and hence C is acyclic on models.

Since S carries degenerate simplexes into degenerate simplexes, it induces a homomorphism $S: C_r(\Delta_q)_N \longrightarrow C_{r+1}(\Delta_q)_N$,

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[1] of modules; $U(\Delta_q)$ does not preserve gradation nor commute with d. Cf. footnote on p. 3-4 .

and the transformation $U': C_N/\widehat{\mathfrak{M}} \longrightarrow C_N/\widehat{\mathfrak{M}}$ in which $U'(\Delta_q) = S: C(\Delta_q)_N \longrightarrow C(\Delta_q)_N$ is a natural transformation of functors. The conditions on S insure that U' satisfies the conditions of (3.8), and hence C_N is also acyclic on models. Let H denote the homology functor obtained from the chain functor C, H_N that obtained from C_N.

Theorem 3.15 : $\Phi: C \longrightarrow C_N$ induces a natural equivalence $\Phi': H \longrightarrow H_N$.

Proof: $H_0|\overline{\mathfrak{M}} = (H_N)_0|\overline{\mathfrak{M}}$, so that in theorems (3.11) and (3.13) we may take T to be the identity. By 3.11 we have natural transformations of functors

$$C \underset{\psi}{\overset{\phi}{\rightleftarrows}} C_N$$

which induce the identity on $H_0|\overline{\mathfrak{M}} = (H_N)_0\overline{\mathfrak{M}}$. The composition $\psi\phi$ is a natural transformation of C into itself which induces the identity on $H_0|\overline{\mathfrak{M}}$; therefore by (3.13), $\psi\phi$ is homotopic to the identity transformation of C. Similarly $\phi\psi$ is homotopic to the identity transformation of C_N. Hence ϕ induces a natural equivalence $\Phi': H \longrightarrow H_N$. But by (3.13) Φ is homotopic to ϕ, and hence also induces Φ'. This completes the proof of the theorem.

Consider the category $\mathcal{Q} \times \mathcal{Q}$, having as objects pairs (K,L) of semi-simplicial complexes, and as maps pairs

$(f,g):(K,L) \longrightarrow (P,Q)$, where $f:K \longrightarrow P, g:L \longrightarrow Q$ are maps. The models are to be pairs (Δ_p, Δ_q) of models from $\mathfrak{a}$. We give three methods for defining degeneracy in $\mathfrak{a} \times \mathfrak{a}$, and thus turning it into a category with models. Let $(u,v):(\Delta_p, \Delta_q) \longrightarrow (K,L)$ be a map in $\mathfrak{a} \times \mathfrak{a}$:

(i) ("Tensor product"): $\alpha(u,v) = (\alpha u, \alpha v)$; $\beta(u,v) = (\beta u, \beta v)$.

(ii) ("Cartesian product") $\alpha(u,v) = (1,1)$ $\beta(u,v) = (u,v)$, unless $p = q$; in this case, let $u(0,\dots,p) = a \in K$, $v(0,\dots,p) = b \in L$. Then $a \times b = s_{i_r} \dots s_{i_1}(a' \times b')$, where $i_r > \dots > i_1$ and $a' \times b'$ is non-degenerate in $K \times L$; furthermore, this decomposition is unique. Define $\alpha(u,v) = (\bar{u}, \bar{v}): (\Delta_p, \Delta_p) \longrightarrow (\Delta_{p-r}, \Delta_{p-r})$, where $\bar{u} = \bar{v}$ is determined by $\bar{u}(0,\dots,p) = s_{i_r} \dots s_{i_1}(0,\dots,p-r)$, and $\beta(u,v) = (u',v'):(\Delta_{p-r}, \Delta_{p-r}) \longrightarrow (K,L)$, where u' and v' are determined by $u'(0,\dots,p-r) = a'$, $v'(0,\dots,p-r) = b'$.

(iii) If neither of the above systems of degeneracy is postulated, we assume that $\mathfrak{a} \times \mathfrak{a}$ has no degeneracy; i.e. $\alpha(u,v) = (1,1)$, $\beta(u,v) = (u,v)$.

We wish to determine the relation between the two functors $C_N^{\otimes}, C_N^{\times}: \mathfrak{a} \times \mathfrak{a} \longrightarrow dg$ defined by

$$C_N^{\otimes}(K,L) = C(K)_N \otimes C(L)_N$$

$$C_N^{\times}(K, L) = C(K \times L)_N$$

(1) $C_N^{\otimes}$ is representable using tensor product degeneracies. For, $C(K)_N \otimes C(L)_N$ is free abelian, and a typical generator is $\sigma \otimes \tau$, where $\sigma \in K_p$, $\tau \in L_q$ are non-degenerate. Let $u: \Delta_p \longrightarrow K, v: \Delta_q \longrightarrow L$ be the unique maps determined by $u(0,\ldots,p) = \sigma, v(0,\ldots,q) = \tau$. Define a natural transformation of functors $\chi : C_N^{\otimes} \longrightarrow \hat{C}_N^{\otimes}$ by
$\chi(K,L)(\sigma \otimes \tau) = ((0,\ldots,p) \otimes (0,\ldots,q), (u,v) \in (C(\Delta_p)_N \otimes C(\Delta_q)_N, (u,v)) \subset \hat{C}_N^{\otimes}(K,L)$.
Clearly $\Gamma\chi$ = identity, so that χ is a representation.

(2) C_N^{x} is representable using Cartesian product degeneracies : $C(K,L)_N$ is a free abelian group, and a typical generator is a non-degenerate simplex $\sigma x \rho$, where $\sigma \in K_p$, $\rho \in L_p$. Let u,w be the maps corresponding to σ, ρ respectively. Define a natural transformation of functors $\chi : C_N^{x} \longrightarrow \hat{C}_N^{x}$ by $\chi(K,L)(\sigma x \rho)$ =
$((0,\ldots,p)x(0,\ldots,p),(u,w)) \in (C(\Delta_p x \Delta_p)_N, (u,w)) \subset \hat{C}_n^{x}(K, L)$
Then $\Gamma\chi$ = identity, so that χ is a representation.

(3) $C_N^{\otimes}$ is acyclic on models, using either system of degeneracy. Consider first the tensor degeneracies. $H_0(C(\Delta_p)_N \otimes C(\Delta_q)_N$ is an infinite cyclic group cyclic group, for which we may take as generator the class of $(0) \otimes (0)$.
$\eta : H_0 C_N^{\otimes} | \hat{\mathfrak{m}} \longrightarrow (C_N^{\otimes})_0 | \hat{\mathfrak{m}}$ is then defined by
$\eta(\Delta_p, \Delta_q)((0) \otimes (0)) = (0) \otimes (0)$. Recall that we defined a contracting homotopy $U' : C(\Delta_q)_N \longrightarrow C(\Delta_q)_N$; we may also define a contracting homotopy $U : C(\Delta_p)_N \otimes C(\Delta_q)_N \longrightarrow C(\Delta_p)_N \otimes C(\Delta_q)_N$

by

$$U(\sigma \otimes \tau) = U'\sigma \otimes \tau + \eta\varepsilon(\sigma) \otimes U'\tau .$$

Then $\partial U + U\partial = 1 - \eta\varepsilon$, and $U\eta = 0$. U commutes with the homomorphisms induced by maps of $\hat{\mathcal{M}}$, and thus defines a natural transformation of functors. Hence, by definition (3.8), $C_N^{\otimes}$ is acyclic on models.

Using cartesian product degeneracies, the corresponding category $\hat{\mathcal{M}}$ is a subcategory of that obtained from tensor product degeneracies; hence U commutes with the induces homomorphisms in this case also, and $C_N^{\otimes}$ is again acyclic on models.

(4) C_N^{x} is acyclic on models, using either system of degeneracy. $H_0(C(\Delta_p x \Delta_q)_N)$ is cyclic infinite, generated by the class of $((0)x(0))$, and $\eta_x : H_0 C_N^x |\hat{\mathcal{M}} \longrightarrow (C_N^x)_0 | \hat{\mathcal{M}}$ is defined by $\eta_x(\Delta_p, \Delta_q)((0)x(0)) = ((0)x(0))$. Define $S_x : (\Delta_p x \Delta_q)_r \longrightarrow (\Delta_p x \Delta_q)_{r+1}$ by $S_x((m_0, \ldots, m_r)x(\ell_0 \ldots \ell_r)) = (0, m_0, \ldots, m_r)x(0, \ell_0, \ldots, \ell_r)$. S_x induces $U_x : C_r(\Delta_p x \Delta_q)_N \longrightarrow C_{r+1}(\Delta_p x \Delta_q)_N$ such that $\partial_{r+1} U_x = U_x \partial_r$ for $r \geq 1$. Hence $\partial U_x + U_x \partial = 1 - \eta_x \varepsilon_x$, and $U_x \eta_x = 0$. Using tensor product degeneracies, it is clear that U_x commutes with the homomorphisms induced by maps of $\hat{\mathcal{M}}$; by the argument of the previous paragraph, the same holds true using Cartesian product degeneracies. Hence C_N^x is acyclic on models in either case.

Now, using tensor product degeneracies so that $C_N^{\otimes}$ is representable, we apply theorem 3.11 with

$T: H_0C_N^{\otimes}|\overline{\mathcal{M}} \longrightarrow H_0C_N^{x}|\overline{\mathcal{M}}$ the natural equivalence defined by $T(\Delta_p,\Delta_q)((0)\times(0)) = ((0)\times(0))$, to obtain a natural transformation of functors

$$\nabla : C_N^{\otimes} \longrightarrow C_N^{x}$$

Similarly, using Cartesian product degeneracies and the equivalence $T':H_0C_N^{x}|\overline{\mathcal{M}} \longrightarrow H_0C_N^{\otimes}|\overline{\mathcal{M}}$ defined by $T'(\Delta_p,\Delta_q)((0)\times(0)) = ((0)\otimes(0))$, we obtain a natural transformation of functors

$$f:C_N^{x} \longrightarrow C_N^{\otimes} \quad .$$

Thus ∇f is a natural transformation of the functor C_N^{x} into itself. If we use the system of Cartesian product degeneracies, then C_N^{x} is representable; and since ∇f induces the transformation $TT' = 1$ in $H_0C_N^{x}|\overline{\mathcal{M}}$, by theorem 3.13 there is a homotopy between ∇f and the identity transformation of C_N^{x}. The fact that such a homotopy is (by definition) natural will be used in later proofs. By a completely similar argument, using tensor product degeneracies, we see that $f\nabla$ is homotopic to the identity transformation of $C_N^{\otimes}$, so that ∇ and f are equivalences.

We now wish to find the explicit formulae for ∇ and f, as determined by (3.11). Throughout let u be the map corresponding to $a \in k_r$, v the map corresponding to $b \in L_s$. We first consider ∇ .

Dimension 0: Let $a \in K_0$, $b \in L_0$. Then

$$\nabla(a\otimes b) = \Gamma_x \hat{\eta}_x \hat{T} \hat{\varepsilon} \chi(a\otimes b) = \Gamma_x \hat{\eta}_x \hat{T} \hat{\varepsilon}(\,(0)\otimes(0),(u,v)\,)$$
$$= \Gamma_x(\,(0)\times(0),\ (u,v)) = a \times b \ .$$

<u>Dimension 1:</u> <u>case 1</u>: Let $a \in K_1$ be non-degenerate, and let $b \in L_0$. Then

$$\nabla(a \otimes b) = \Gamma_X \hat{U}_X \hat{\nabla} \hat{\partial} \chi(a \times b) = \Gamma_X \hat{U}_X \hat{\nabla} \hat{\partial}(\ (0,1) \otimes (0), (u,v)\ )$$
$$= \Gamma_X \hat{U}_X \hat{\nabla}((1) \otimes (0) - (0) \otimes (0), (u,v)) = \Gamma_X \hat{U}_X((1) x (0) - (0) x (0), (u,v))$$
$$= \Gamma_X(\ (0,1) \times (0,0), (u,v)\ ) = a \times s_0 b.$$

<u>case 2</u>: Let $a \in K_0$, and let $b \in L_1$ be non degenerate. Then in a similar fashion

$$\nabla(a \otimes b) = s_0 a \times b.$$

<u>Dimension 2</u>: <u>case 1</u>: Let $a \in K_1, b \in L_1$ be non-degenerate. Then

$$\nabla(a \otimes b) = \Gamma_X \hat{U}_X \hat{\nabla} \hat{\partial} \chi(a \otimes b) = \Gamma_X \hat{U}_X \hat{\nabla} \hat{\partial}(\ (0,1) \otimes (0,1), (u,v)\ )$$
$$= \Gamma_X \hat{U}_X \hat{\nabla}(\ (1) \otimes (0,1) - (0) \otimes (0,1) - (0,1) \otimes (1) + (0,1) \otimes (0), (u,v))$$
$$= \Gamma_X \hat{U}_X(\ (1,1) x (0,1) - (0,0) x (0,1) - (0,1) x (1,1) + (0,1) x (0,0), (u,v))$$
$$= \Gamma_X(\ (0,1,1) x (0,0,1) - (0,0,1) x (0,1,1), (u,v)\ )$$
$$= s_1 a \times s_0 b - s_0 a \times s_1 b.$$

Similarly we have

<u>case 2</u>: Let $a \in K_0, b \in L_2$ be non-degenerate. Then

$$\nabla(a \otimes b) = s_1 s_0 a \times b.$$

<u>case 3:</u> Let $a \in K_2, b \in L_0$ be non-degenerate. Then

$$\nabla(a \otimes b) = a \times s_1 s_0 b.$$

The general formula, which we shall not prove, is the following. If (μ,ν) is a (p,q)-shuffle (cf. appendix 1A), let $\sigma(\mu,\nu)$ be the sign of the permutation $(\mu_1,\ldots,\mu_p,\nu_1,\ldots,\nu_q)$ of the integers $(0,1,\ldots,p+q-1)$.

Then for $a \in K_p, b \in K_q$ both non-degenerate,

$$(3.16)\qquad \nabla(a \otimes b) = \sum_{(\mu,\nu)} \sigma(\mu,\nu) s_{\nu_q} \ldots s_{\nu_1} a \times s_{\mu_1} \ldots s_{\mu_1} b,$$

the sum being taken over all (p,q)-shuffles.

We now consider $f: C(K \times L)_N \longrightarrow C(K)_N \otimes C(L)_N$

Dimension 0: Let $a \in K_0, b \in L_0$. Then, with the appropriate meanings of the functors in this case,

$$f(a \times b) = \Gamma \hat{\eta} \hat{T} \hat{\xi}_x \chi(a \times b) = \Gamma \hat{\eta} \hat{T} \hat{\xi}_x((0)\times(0),(u,v)) = \Gamma((0)\otimes(0),(u,v)) = a \otimes b$$

Dimension 1: Let $a \times b$ $(K \times L)$, be non-degenerate. Then

$$\begin{aligned} f(a \times b) &= \Gamma \hat{U} \hat{f} \hat{\partial}_x \chi_x(a \times b) = \Gamma \hat{U} \hat{f} \partial_x((0,1)\times(0,0),(u,v)) \\ &= \Gamma \hat{U} \hat{f}((1)\times(1) - (0)\times(0),(u,v)) = \Gamma \hat{U}((1)\otimes(1) - (0)\otimes(0),(u,v)) \\ &= \Gamma\ (\ (0,1)\otimes(1) + (0)\otimes(0,1),(u,v)\ ) \\ &= \Gamma\ (\ (0,1)\otimes\partial_0(0,1) + (\partial_1(0,1))\otimes(0,1),(u,v)\ ) \\ &= a \otimes \partial_0 b + (\partial_1 a)\otimes b. \end{aligned}$$

Dimension 2: Let $a \times b \in (K \times L)_2$ be non-degenerate. Then

$$\begin{aligned} f(a \times b) &= \Gamma \hat{U} \hat{f} \hat{\partial}_x \chi_x(a \times b) = \Gamma \hat{U} \hat{f} \hat{\partial}_x(\ (0,1,2)\times(0,1,2),(u,v)) \\ &= \Gamma \hat{U} \hat{f}(\ (1,2)\times(1,2) - (0,2)\times(0,2) + (0,1)\times(0,1),(u,v)\ ) \\ &= \Gamma \hat{U}(\ (1,2)\otimes(2) + (1)\otimes(1,2) - (0,2)\otimes(2) - (0)\otimes(0,2) + (0,1)\otimes(1) + \\ &\qquad (0)\otimes(0,1),(u,v)) \\ &= \Gamma\ (\ (0,1,2)\otimes(2) + (0,1)\otimes(1,2) + (0)\otimes(0,1,2),(u,v)) \\ &= \Gamma((0,1,2)\otimes\partial_0^2(0,1,2) + \partial_2(0,1,2)\otimes\partial_0(0,1,2) + \partial_2\partial_1(0,1,2)\otimes(0,1,2),(u,v)) \\ &= a\otimes\partial_0^2 b + \partial_2 a\otimes\partial_0 b + \partial_2\partial_1 a\otimes b. \end{aligned}$$

The general formula for f, which we shall not prove is the following, where $\tilde{\partial}$ denotes the last face operator in any situation: let $a \times b \in (K \times L)_p$; then

$$(3.17) \qquad f(a \times b) = \sum_{i=0}^{p} (\tilde{\partial})^i a \otimes (\partial_0)^{p-i} b.$$

Note that this is the formula for the Alexander-Čech-Whitney cup product; it is not symmetric with respect to permuting K and L. It is routine to verify that

$$(3.18) \qquad f\nabla = \text{identity}$$

<u>Lemma 3.19:</u> ∇ is associative; i.e. the following diagram commutes, where the isomorphism is the natural one:

$$\begin{array}{ccccc}
(C(K)_N \times C(L)_N) \times C(M)_N & \xrightarrow{\nabla \otimes 1} & C(K \times L)_N \otimes C(M)_N & \overset{\nabla}{\searrow} & \\
\downarrow \approx & & & & C(K \times L \times M)_N \\
C(K)_N \otimes (C(L)_N \otimes C(M)_N) & \xrightarrow{1 \otimes \nabla} & C(K)_N \otimes C(L \times M)_N & \underset{\nabla}{\nearrow} &
\end{array}$$

References

[1] S. Eilenberg and S. MacLane, <u>Acyclic models</u>, Am. J. Math. 75(1953), 189-199.

[2] V. Gugenheim and J. C. Moore, <u>Acyclic models and fibre spaces</u>, to appear.

[3] S. Eilenberg and J. C. Zilber, <u>On products of complexes</u>, Am. J. Math. 75 (1953), 200-204.

The theory of spectral sequences was introduced by J. Leray [1]. Leray obtained spectral sequences from differential filtered modules (see below). A more general procedure of obtaining spectral sequences was introduced by W. S. Massey in his theory of exact couples [2]. Yet another way of obtaining spectral sequences was introduced by S. Eilenberg, and is expounded in his forthcoming book with H. Cartan [3]. This method has the advantage that there is both an inductive and a direct definition of the term E^r in the spectral sequence, and consequently will be followed here.

Notation and Conventions: Let $\tilde{Z}$ be the set $Z \cup \{-\infty, \infty\}$. Order $\tilde{Z}$ by $-\infty < r < \infty$ for $r \in Z$.

Definition 4.1: Let $\mathcal{A}$ be the category such that

1) objects of $\mathcal{A}$ are pairs (p,q) of elements of $\tilde{Z}$ such that $p \geq q$, and

2) a map in $\mathcal{A}$ is an assignment to an object (p,q) in $\mathcal{A}$ another object (p',q') in $\mathcal{A}$ such that $p' \geq p$, $q' \geq q$.

If $\alpha : (p,q) \longrightarrow (p',q')$ and $\beta: (p',q') \longrightarrow (p'',q'')$ are maps in $\mathcal{A}$ we say that (α,β) is a couple if $q = q'$, $p' = p''$, and $q'' = p$ (see [4], p. 114). In other words there is a correspondence between couples and triples (p,q,r) of elements of $\tilde{Z}$ such that $p \geq q \geq r$, the correspondence being that which assigns to the triple (p,q,r) the couple (α, β), where $\alpha : (q,r) \longrightarrow (p,r)$, and $\beta : (p,r) \longrightarrow (p,q)$.

<u>Notation:</u> Let Λ be a commutative ring with unit. Denote by $\mathcal{G}_\Lambda$ the category of Λ-modules and Λ-homomorphisms, and by $\mathcal{G}_\Lambda'$ the category of graded Λ-modules and graded Λ-homomorphisms.

<u>Definition 4.2:</u> A <u>covariant ∂-functor</u> on the category with couples $\mathcal{A}$ consists of a covariant functor $H : \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ together with a homomorphism $\partial_{(\alpha,\beta)} : H(C) \longrightarrow H(A)$ for each couple (α, β) in $\mathcal{A}$, $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$, satisfying the following condition:

1) if

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} \\ A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 \end{array}$$

is a commutative diagram in $\mathcal{A}$, where (α, β) and (α_1, β_1) are couples, then

$$\begin{array}{ccc} H(C) & \xrightarrow{\partial(\alpha,\beta)} & H(A) \\ \downarrow{\scriptstyle H(\gamma_3)} & & \downarrow{\scriptstyle H(\gamma_1)} \\ H(C_1) & \xrightarrow{\partial(\alpha_1,\beta_1)} & H(A_1) \end{array}$$

is a commutative diagram.

2) For every couple (α, β) in $\mathcal{A}$, $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$,

the sequence

$$\cdots \longrightarrow H(A) \xrightarrow{H(\alpha)} H(B) \xrightarrow{H(\beta)} H(C) \xrightarrow{\partial(\alpha,\beta)} H(A) \longrightarrow \cdots$$

is exact.

If $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$ satisfies 1) and 2) above, and if in addition $\partial_{(\alpha,\beta)}: H_{n+1}(C) \longrightarrow H_n(A)$ for every couple $(\alpha,\beta): A \longrightarrow B \longrightarrow C$, then H will be said to be a <u>graded covariant ∂-functor</u> on $\mathcal{A}$ ([4], p. 115).

<u>Definition 4.3:</u> Let M be a differential Λ-module. A <u>filtration</u> on M is a set of submodules $\{F_pM\}_{p\in\tilde{Z}}$ such that

1) $F_pM \subset F_{p+1}M$,

2) $dF_pM \subset F_pM$,

3) $F_{-\infty}M = 0$

4) $F_{\infty}M = M$

If M is a graded differential Λ-module, the filtration will be assumed to be compatible with the gradation, i.e. $F_pM = \Sigma_n(F_pM)\cap M_n$ for all $p\in\tilde{Z}$.

The module M together with its differential operator and filtration is called a <u>differential filtered Λ-module</u>, and if it is graded it is called a <u>differential graded filtered Λ-module.</u>

<u>Definition 4.4:</u> Let $\{M, F_pM\}$ be a differential filtered Λ-module. If (p,q) is an object of $\mathcal{A}$, let $H(p,q) = H(F_pM/F_qM)$, and if $\alpha: (p,q) \longrightarrow (p',q')$ is a

map, let $H(\alpha):H(p,q) \longrightarrow H(p',q')$ be the natural map. If $\alpha:(p,r) \longrightarrow (p,r)$, $\beta:(p,r) \longrightarrow (p,q)$ is a couple in $\mathcal{A}$, then there is an exact sequence

$$0 \longrightarrow F_qM/F_rM \longrightarrow F_pM/F_rM \longrightarrow F_pM/F_qM \longrightarrow 0$$

and a resulting exact sequence

$$\cdots \longrightarrow H(q,r) \longrightarrow H(p,r) \longrightarrow H(p,q) \xrightarrow{\partial} H(q,r) \longrightarrow \cdots.$$

Let $\partial_{(\alpha,\beta)}:H(p,q) \longrightarrow H(q,r)$ be the homomorphism denoted by ∂ in this exact sequence (Henceforth $\partial_{(\alpha,\beta)}$ will be denoted merely by ∂.)

It is evident that the functor H just defined and the homomorphisms $\partial:H(p,q) \longrightarrow H(q,r)$ form a covariant ∂-functor on $\mathcal{A}$, and that this functor is graded if M is graded. This covariant ∂-functor is said to be the one associated with the differential filtered Λ-module $\{M,F_pM\}$.

Definition 4.5: If $H:\mathcal{A} \longrightarrow \mathcal{G}$ is a covariant ∂-functor, define

$$Z_p^r = \text{Image } H(p,p-r) \longrightarrow H(p,p-1)$$

$$B_p^r = \text{Image } \partial:H(p+r-1,p) \longrightarrow H(p,p-1)$$

for $r,p \in \tilde{Z}, r \geq 2$. If H is graded, define

$$Z_{p,q}^r = \text{Image } H_{p+q}(p,p-r) \longrightarrow H_{p+q}(p,p-1)$$

$$B_{p,q}^r = \text{Image } H_{p+q+1}(p+r-1,p) \longrightarrow H_{p+q}(p,p-1)$$

Lemma: $\cdots Z_p^r \supset Z_p^{r+1} \supset \cdots \supset Z_p^\infty \supset B_p^\infty \supset \cdots \supset B_p^{r+1} \supset B_p^r \supset \cdots$, and $\cdots Z_{p,q}^r \supset Z_{p,q}^{r+1} \supset \cdots \supset Z_{p,q}^\infty \supset B_{p,q}^\infty \supset \cdots \supset B_{p,q}^{r+1} \supset B_{p,q}^r \supset \cdots$

The proof of this lemma is straightforward, and will be omitted.

Definition 4.6: If $H: \mathcal{a} \longrightarrow \mathcal{g}_\Lambda$ is a covariant ∂-functor, define $E^r_p = Z^r_p/B^r_p$ for $r, p \in \tilde{Z}$, $r \geq 2$. Define $E^1_p = H(p, p-1)$, and set $E^r = \Sigma_p E^r_p$. If H is graded, set $E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}$, $E^r_p = \Sigma_q E^r_{p,q}$, $E^r = \Sigma_{p,q} E^r_{p,q}$. $\{E^r\}_{r \geq 2}$ is the spectral sequence of H. If H is the covariant ∂-functor associated with a differential filtered Λ-module $\{M, F_p, M\}$, the spectral sequence will sometimes be denoted by $\{E^r(M)\}$. Further in this case $E^0_p(M) = F_pM/F_{p-1}M$, and $E^0(M) = \Sigma_p E^0_p(M)$.

We now have spectral sequences defined, but we have not as yet proved two of their basic properties. First, E^{r+1} should be the homology of E^r with respect to some differential operator. Second, if M is a filtered Λ-module, $E^\infty(M)$ should approximate $H(M)$ in a certain sense. We now proceed to define $d^r: E^r \longrightarrow E^r$ so that E^{r+1} will be isomorphic to $H(E^r)$.

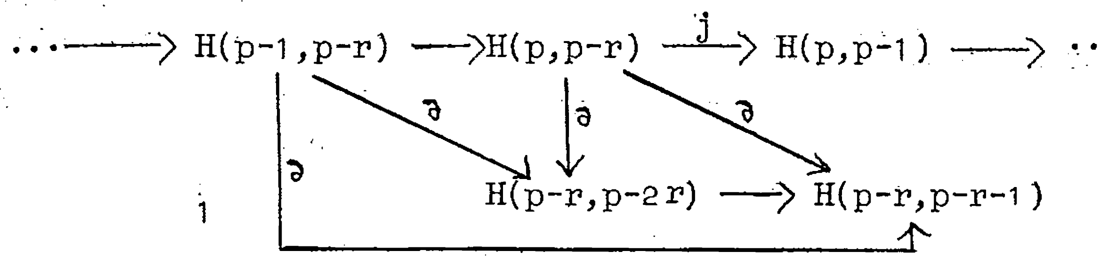
Lemma: If $p \geq q \geq r \geq s$ then
$$H(p,q) \xrightarrow{\partial} H(q,r) \xrightarrow{\partial} H(r,s), \text{ and } \partial\partial = 0.$$

Proof: This follows immediately from the commutativity of the following diagram

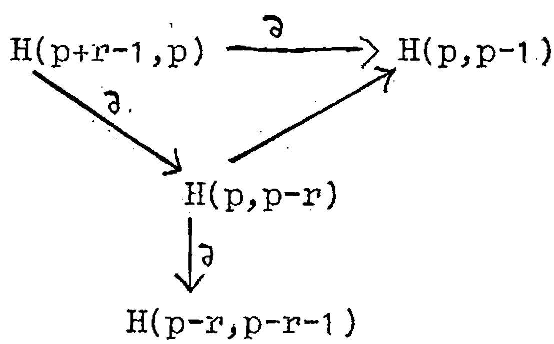
$$\begin{array}{ccccccccc} \cdots \longrightarrow H(p,r) & \longrightarrow & H(p,q) & \xrightarrow{\partial} & H(q,r) & \longrightarrow & H(p,r) & \longrightarrow & \cdots \\ & & & & \downarrow \partial & \swarrow \partial & & & \\ & & & & H(r,s) & & & & \end{array}$$

and the fact that the horizontal sequence is exact.

<u>Definition 4.7:</u> Notice that the diagram



is commutative. Consequently there is a natural map $\partial^r : Z^r_p \longrightarrow E^r_{p-r}$ such that $\partial^r(z)$ is the equivalence class of $\partial j^{-1} z \in H(p-r,p-r-1)$. Further it follows from the commutativity of the diagram



and the fact that $\partial\partial = 0$, that $\partial^r(B^r_p) = 0$. Define $d^r : E^r_p \longrightarrow E^r_{p-r}$ to be the homomorphism induced by $\partial^r : Z^r_p \longrightarrow E^r_{p-r}$. Further denote by d^r the induced endomorphism of E^r.

<u>Proposition 4.8:</u> $d^r \circ d^r = 0$, and $H(E^r)$ is naturally isomorphic with E^{r+1}.

<u>Proof:</u> The fact that $d^r \circ d^r = 0$ follows from the diagram

$$\begin{array}{ccc}
H(p,p-r) & \longrightarrow & H(p,p-1) \\
\Big|\partial & \overset{\partial}{\searrow} & \\
H(p-r,p-2r) & \longrightarrow & H(p-r,p-r-1) \\
\Big|\partial & \overset{\partial}{\searrow} & \\
H(p-2r,p-3r) & \longrightarrow & H(p-2r,p-2r-1)\ ,
\end{array}$$

and the fact that $\partial\partial = 0$.

Further it follows from the diagram

$$\begin{array}{ccccc}
H(p,p-r-1) & \longrightarrow & H(p,p-r) & \longrightarrow & H(p,p-1) \\
\downarrow & & \downarrow\partial & & \\
0 = H(p-r-1,p-r-1) & \longrightarrow & H(p-r.p-r-1) & &
\end{array}$$

that the sequence

$$0 \longrightarrow Z_p^{r+1}/B_p^r \longrightarrow E_p^r \xrightarrow{d^r} E_{p-r}^r$$

is exact, or that the sequence

$$Z_p^{r+1} \longrightarrow H(E^r) \longrightarrow 0$$

is exact. From the diagram

$$\begin{array}{ccc}
H(p+r,p) & \longrightarrow & H(p+r,p+r-1) \\
\downarrow\partial & \overset{\partial}{\searrow} & \\
H(p,p-r) & \longrightarrow & H(p,p-1) \\
 & \overset{\partial}{\searrow} & \\
 & & H(p-r,p-r-1)
\end{array}$$

and the fact that $\partial\partial = 0$, it follows that B_p^{r+1} = kernel $Z_p^{r+1} \longrightarrow H(E^r)$, or that $E_p^{r+1} \xrightarrow{\approx} H(E^r)$

as was to be proved.

Notice that if $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$, then $d^r: E^r_{p-q} \longrightarrow E^r_{p-r,q+r-1}$

<u>Definition 4.9:</u> We define a filtration on $H(\infty, -\infty)$ by setting $F_p(H(\infty, -\infty)) = \text{Image } H(p, -\infty) \longrightarrow H(\infty, -\infty)$

<u>Proposition 4.10:</u> $E^0_p(H(\infty, -\infty))$ is naturally isomorphic to E^∞_p for all $p \in \tilde{Z}$.

<u>Proof:</u> Recall that $Z^\infty_p = \text{Image } H(p, -\infty) \longrightarrow H(p, p-1)$, $B^\infty_p = \text{Image } H(\infty, p) \xrightarrow{\partial} H(p, p-1)$. Further the sequence

$$\cdots \longrightarrow H(p-1, -\infty) \longrightarrow H(p, -\infty) \longrightarrow H(p, p-1) \longrightarrow \cdots$$

is exact, and $\text{Image } H(p-1, -\infty) \longrightarrow H(\infty, -\infty) = F_{p-1} H(\infty, -\infty)$. Therefore there is a natural map $E^\infty_p \longrightarrow E^0_p H(\infty, -\infty)$, and this map is clearly an epimorphism. However, it follows from a similar argument that it is a monomorphism, and the result follows.

<u>Proposition 4.11:</u> Suppose that $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$ is a graded covariant ∂-functor, and

1) $H(p,q) = 0$ if $p < 0$, and
2) $H_n(p,q) = 0$ if $n \leq q$, then

$E^r_{p,q}$ is naturally isomorphic with $E^\infty_{p,q}$ for $r > \sup\{p, q+1\}$.

Proof: Suppose that $r > \sup\{p, q+1\}$. The horizontal sequence

$$\begin{array}{ccccccc} & & & H(p,p-1) & & & \\ & & \nearrow & & \nwarrow & & \\ \cdots \longrightarrow H(p-r,-\infty) & \longrightarrow & H(p,-\infty) & \longrightarrow & H(p,p-r) & \longrightarrow & \cdots \end{array}$$

is exact, and $H(p-r,-\infty) = 0$. Therefore, $H(p,-\infty) \approx H(p,p-r)$, and $Z_p^r = Z_p^\infty$.

Further, the horizontal sequence in the diagram

$$\begin{array}{ccccccc} & & & H_{p+q}(p,p-1) & & & \\ & & \partial\nearrow & & \nwarrow\partial & & \\ \cdots \to H_{p+q+2}(\infty,p+r-1) & \to & H_{p+q+1}(p+r-1,p) & \to & H_{p+q+1}(\infty,p) & \to & H_{p+q+1}(\infty,p+r-1) \end{array}$$

is exact, $H_n(\infty, p+r-1) = 0$ for $n = p+q+2, p+q+1$, and hence $B^r_{p,q} = B^\infty_{p,q}$. Then the proof is complete.

Definition: If $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$ is a graded covariant ∂-functor, then H is regular if $H(p,q) = 0$ if $p < 0$, and $H_n(p,q) = 0$ if $n \leq q$; in other words if the hypotheses of the preceding proposition are fulfilled.

If $\{M, F_p M\}$ is a differential graded filtered Λ-module, then $\{F_p M\}$ is a regular filtration if

1) $F_p M = 0$ for $p < 0$, and

2) $M_p \subset F_p M$.

Notice that this definition assures that the covariant

∂ - functor associated with $\{M, F_p M\}$ is regular. Almost all of the filtration in which we shall be interested have this property.

We are now in a position to prove the exact sequence theorem of Serre [5], which will be used extensively later in the notes.

Theorem 4.12: Suppose that $H: \mathcal{A} \longrightarrow \mathcal{G}'_{\Lambda}$ is a regular covariant ∂- functor, and that i, j, r are positive integers with $i < j, r \geq 2$. Suppose further that if $i \leq n \leq j$ then

1) (a_n, b_n) and (c_n, d_n) are pairs of integers such that $n = a_n + b_n = c_n + d_n$, and $a_n < c_n$,
2) $E^r_{p,q} = 0$ if $p+q = n-1$, $p \leq a_n - r$, and
3) $E^r_{p,q} = 0$ if $p+q = n$, $(p,q) \notin \{(a_n,b_n),(c_n,d_n)\}$
4) $E^r_{p,q} = 0$ if $p+q = n+1$, $p \geq c_n + r$.

Under these hypotheses there is an exact sequence

$$E^r_{a_j,b_j} \longrightarrow H_j(\infty, -\infty) \longrightarrow E^r_{c_j,d_j} \longrightarrow E^r_{a_{j-1},b_{j-1}} \longrightarrow \cdots$$
$$\cdots \longrightarrow E^r_{a_i,b_i} \longrightarrow H_i(\infty, -\infty) \longrightarrow E^r_{c_i,d_i} .$$

Proof: It follows immediately from the hypotheses of the theorem that $E^{\infty}_{p,q} = 0$ if $p+q = n$, $(p,q) \notin \{(a_n,b_n), (c_n,d_n)\}$, where $i \leq n \leq j$. From this fact and proposition 4.10, with gradation considered, it follows that there is an exact sequence

$$0 \longrightarrow E^{\infty}_{a_n,b_n} \longrightarrow H_n(\infty,-\infty) \longrightarrow E^{\infty}_{c_n,d_n} \longrightarrow 0.$$

However it follows from 2) above that if $n > 1$, then either

a) $r \leq s = c_n - a_{n-1}$ and $E^{\infty}_{c_n,d_n}$ is the kernel of $d^s: E^s_{c_n,d_n} \longrightarrow E^s_{a_{n-1},b_{n-1}}$, or

b) $r > c_n - a_{n-1}$ and $E^{\infty}_{c_n,d_n} = E^r_{c_n,d_n}$.

Consequently there is an exact sequence

$$0 \longrightarrow E^{\infty}_{a_n,b_n} \longrightarrow H_n(\infty,-\infty) \longrightarrow E^s_{c_n,d_n} \longrightarrow E^s_{a_{n-1},b_{n-1}} .$$

However $E^s_{c_n,d_n} = E^r_{c_n,d_n}$, $E^s_{a_{n-1},b_{n-1}} = E^r_{a_{n-1},b_{n-1}}$, and $E^{\infty}_{a_{n-1},b_{n-1}}$ is the cokernel of $d^s: E^s_{c_n,d_n} \longrightarrow E^s_{a_{n-1},b_{n-1}}$ in case a, or $E^s_{a_{n-1},b_{n-1}}$ in case b. This follows from 2) and 4) in the hypotheses of the theorem. These facts combine to imply that there is an exact sequence

$$0 \longrightarrow E^{\infty}_{a_n,b_n} \longrightarrow H_n(\infty,-\infty) \longrightarrow E^r_{c_n,b_n} \longrightarrow$$
$$E^r_{a_{n-1},b_{n-1}} \longrightarrow H_{n-1}(\infty,-\infty) \longrightarrow E^{\infty}_{c_{n-1},d_{n-1}} .$$

To complete the proof it is necessary only to continue in this manner.

<u>Definiton 4.13:</u> If M, M' are filtered Λ- modules, then $f: M \longrightarrow M'$ is <u>filtration preserving</u>, or is a <u>map of filtered modules</u> if $f(F_pM) \subset F_pM'$ for $p \in \widetilde{Z}$.

If $f,g: M \longrightarrow M'$ are maps of differential filtered modules, a <u>homotopy of degree s</u> between f,g is a Λ- homomorphism $D: M \longrightarrow M'$ such that

1) $dD + Dd = f - g$, and

2) $D(F_pM) \subset F_{p+s}M'$.

If M and M' are graded, it will be assumed that $D(M_n) \subset M'_{n+1}$.

Proposition 4.14: 1) If $f: M \longrightarrow M'$ is a map of differential filtered Λ- modules, then f induces

$$f^r: E^r(M) \longrightarrow E^r(M')$$

a map of differential Λ-modules for $r \geq 0$, and further if M and M' are graded, then $f^r(E^r_{p,q}(M)) \subset E^r_{p,q}(M')$).

2) If $f, g: M \longrightarrow M'$ are maps of differential filtered Λ- modules which are homotopic by a homotopy of degree s, then $f^r = g^r$ for $r > s$.

Proof: The first part of the proposition is obvious, and its proof will be omitted.

To prove the second part, it suffices to show that if D is a homotopy of degree s between f and g, then D induces a homotopy D^s between f^s and g^s. If $x \in F_pM$ represents $[x] \in E^s_p(M)$, define $D^s[x] = [Dx] \in E^s_{p+s}(M')$. It will be left to the reader to verify that the definition is independent of the choice of representatives, and that $d^sD^s + D^sd^s = f^s - g^s$.

The preceding definition and proposition could have been extended to include maps of covariant ∂- functors on $\mathcal{A}$. However, to avoid complications we now abandon covariant

∂- functors, and for the remainder of this chapter consider only spectral sequences which arise from filtered modules.

Before proceeding to the proof of some comparison theorems, we first study coefficient sequences.

Definition 4.15: If N is a differential graded Λ- module, and $\mathcal{G}$ is a Λ - module, then $\mathcal{G}\otimes_\Lambda N$ is the differential graded Λ-module such that $(\mathcal{G}\otimes_\Lambda N)_q \models \mathcal{G}\otimes_\Lambda N_q$, and $d(a\otimes b) = a\otimes db$ for $a \in G$, $b \in N$. The homology of $\mathcal{G}\otimes_\Lambda N$ is denoted by $H(N; \mathcal{G})$.

If $\mathcal{G}$ is graded, then $\mathcal{G}_p\otimes_\Lambda N_q$ is the submodule of gradation (p,q) of the bigraded differential module $\mathcal{G}\otimes_\Lambda N$, and $d(a\otimes b) = (-1)^p a\otimes db$ if $a \in \mathcal{G}_p, b \in N_q$. Thus $\mathcal{G}\otimes_\Lambda N = \Sigma_p \mathcal{G}_p \otimes_\Lambda N$. The elements of total degree (or gradation) n in $\mathcal{G}\otimes_\Lambda N$ are those of $\Sigma_{p+q=n}\mathcal{G}_p\otimes N_q$.

Definition 4.16: Let $f: M \longrightarrow M'$ be a map of differential graded modules. The mapping cylinder of f is the differential graded Λ - module M'' such that

1) $M''_q = M_{q-1} + M_q + M'_q$, and

2) $d(a,b,c) = (-da, db-a, dc + f(a))$.

Let $i: M \longrightarrow M''$ be the map defined by $i(b) = (0,b,0)$ $j: M'' \longrightarrow M'$ the map defined by $j(a,b,c) = f(b) + c$, and $\lambda : M' \longrightarrow M''$ by $\lambda(c) = (0,0,c)$. Let $D: M'' \longrightarrow M''$ be defined by $D(a,c,c) = (b,0,0)$.

If M, M' are filtered and f is filtration preserving, define $F_pM'' = F_{p-1}M + F_pM + F_pM'$.

<u>Proposition 4.17:</u> Under the conditions of the preceding definition we have

1) $f = ji$,

2) $j\lambda$ is the identity,

3) $dD + Dd = \lambda j$ - identity, and

4) if f is filtration preserving, D is a homotopy of degreee 1.

<u>Corollary:</u> $j_x : H(M'') \longrightarrow H(M')$, and if f is a map of filtered modules, then $j^2 : E^2(M'') \xrightarrow{\approx} E^2(M')$.

<u>Definition 4.18</u> Let N be a differential graded Λ- module, $f: \mathfrak{g} \longrightarrow \mathfrak{g}'$ a map of Λ- modules. Then $f \otimes 1 : \mathfrak{g} \otimes N \longrightarrow \mathfrak{g}' \otimes N$. Let M be the mapping cylinder of $f \otimes 1$. Then M is the <u>mapping cylinder for N of the coefficient homomorphism</u> f. Define

$$F_pM = \mathfrak{g} \otimes \Sigma_{q \leq p} N_p + \mathfrak{g} \otimes \Sigma_{q \leq p} N_p + \mathfrak{g}' \otimes \Sigma_{q \leq p} N_p$$

Further let A be the kernel of f, and C the cokernel.

Note that the filtration $\{F_pM\}$ induces a filtration $\{F_pM'\}$ on $M' = M/\mathfrak{g} \otimes N$.

<u>Proposition 4.19:</u> If in addition to the hypotheses of the preceding definition N is a free Λ - module, then there is an exact sequence

$$\cdots \longrightarrow H_{p-1}(N;A) \longrightarrow H_p(M') \longrightarrow H_p(N;C) \longrightarrow H_{p-2}(N;A) \longrightarrow \cdots$$

Proof: We have $E^0_{p,q}(M') = 0$ if $q \neq 0,1$, and $E^0_{p,1} = \mathcal{G} \otimes N_p$, $E^0_{p,0} = \mathcal{G}' \otimes N_p$. By an easy calculation $E'_{p,1} = A \otimes N_p$, $E'_{p,0} = C \otimes N_p$. The proposition now follows from Theorem 4.12.

Collary: If in addition $H_0(N) \simeq \Lambda + H'_0(N)$, then

1) $H_0(M') = 0$ implies $C = 0$, and
2) $H_0(M') = H_1(M') = 0$ implies $A = C = 0$, and $f: G \xrightarrow{\approx} G'$.

Proof: The last term of the exact sequence of 4.19 are

$$\cdots \longrightarrow H_0(N;A) \longrightarrow H_1(M') \longrightarrow H_1(N;C) \longrightarrow H_{-1}(N;A) \longrightarrow H_0(M') \xrightarrow{\approx} H_0(N;C)$$

Therefore if $H_0(M') = 0$, we have $H_0(N;C) = 0$, and since $H_0(N) = \Lambda + H'_0(N)$ it follows that $C = 0$. Now if $C = 0$, $H_0(N;A) \xrightarrow{\approx} H_1(M')$ and the result follows.

It is not difficult to prove that if Λ is a principal ideal domain, then the exact sequence of 4.19 reduces to

$$0 \longrightarrow H_q(N;A) \longrightarrow H_{q+1}(M') \longrightarrow H_{q+1}(N;C) \longrightarrow 0$$

Further, even in the general case, there is an exact sequence

$$\cdots \longrightarrow H_q(N;\mathcal{G}) \longrightarrow H_q(N;\mathcal{G}') \longrightarrow H_q(M') \longrightarrow H_{q-1}(N;\mathcal{G}) \longrightarrow \cdots$$

since M' is the relative mapping cylinder of $\mathcal{G} \otimes N \longrightarrow \mathcal{G}' \otimes N$. If $A = 0$, then $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow C \longrightarrow 0$, and $H_q(M') \simeq H_q(N;C)$. Thus the preceding exact sequence reduces to the usual one coming from the exact sequence of coefficients $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow C \longrightarrow 0$. Similarly if $C = 0$, then $H_{q-1}(N;A) \simeq H_q(M')$, and our exact sequence reduces to the usual one corresponding to the exact sequence of coefficients $0 \longrightarrow \mathcal{a} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}' \longrightarrow 0$.

Proposition 4.20: Let $f: M \longrightarrow M'$ be a map of differential filtered Λ- modules, and let M'' denote the relative mapping cylinder of f. Then there is an exact sequence

$$\cdots \longrightarrow E^2_p(M) \longrightarrow E^2_p(M') \longrightarrow E^2_p(M'') \longrightarrow E^2_{p-1}(M) \longrightarrow \cdots ,$$

and further if f is a map of graded Λ- modules, there are exact sequences

$$\cdots \longrightarrow E^2_{p,q}(M) \longrightarrow E^2_{p,q}(M') \longrightarrow E^2_{p,q}(M'') \longrightarrow E^2_{p-1,q}(M) \longrightarrow \cdots$$

for each q.

Proof: Let $M^{\#}$ be the mapping cylinder of f. Then there is an exact sequence

$$0 \longrightarrow M \xrightarrow{i} M^{\#} \xrightarrow{j} M'' \longrightarrow 0 .$$

Further there is a map $\lambda : M^{\#} \longrightarrow M$ such that λ^i is the identity defined by $\lambda(a,b,c) = b$. The map λ is only a map of Λ- modules, and is not compatible with d. However it induces a map $\lambda^0 : E^0(M^{\#}) \longrightarrow E^0(M)$, and for this map we have $\lambda^0 d^0 = d^0 \lambda^0$. It now follows easily that there is an exact sequence

$$0 \longrightarrow E'(M) \xrightarrow{i} E'(M^{\#}) \longrightarrow E'(M'') \longrightarrow 0 .$$

On passing to homology this gives rise to an exact sequence

$$\cdots \longrightarrow E^2_p(M) \longrightarrow E^2_p(M^{\#}) \longrightarrow E^2_p(M'') \longrightarrow E^2_{p-1}(M) \longrightarrow \cdots$$

Now noting that $E^2(M^{\#})$ is naturally isomorphic with

$E^2(M')$ by 4.17 and 4.14, the result follows.

We now wish to prove a comparison theorem for spectral sequences of differential graded Λ - modules. Since the hypotheses of this theorem are somewhat complicated, they will be listed first in a section of their own preceding the theorem.

<u>Hypotheses of the theorem:</u> Let $g:M \longrightarrow M'$ be a map of differential graded filtered Λ- modules, $h:U \longrightarrow U'$ a map of graded Λ- modules, $\overline{g}: N \longrightarrow N'$ a map of differential graded Λ- modules, and suppose that N, N' are free Λ - modules. Finally, suppose there is given a commutative diagram

$$\begin{array}{ccc} E'(M) & \xrightarrow{g'} & E'(M') \\ \psi \downarrow & & \downarrow \psi' \\ U \otimes_{\Lambda} N & \xrightarrow{h \times \overline{g}} & U' \otimes_{\Lambda} N' \end{array}$$

where $\psi(E'_{p,q}(M)) \subset U_q \otimes N_p$, $\psi'(E'_{p,q}(M')) \subset U'_q \otimes N'_p$, such that ψ and ψ' are maps of differential Λ - modules, and induce isomorphisms

$\psi_*:E^2(M) \xrightarrow{\approx} H(N;U)$ and $\psi'_*: E^2(M') \xrightarrow{\approx} H(N';U')$.

Under all the preceding hypotheses, one has the following two theorems:

<u>Theorem A:</u> If $g_*: H(M) \longrightarrow H(M')$ is an isomorphism, $h:U \longrightarrow U'$ is an isomorphism, and if $U_0 \simeq \Lambda + U'_0$, then $\overline{g}_*:H(N) \longrightarrow H(N')$ is an isomorphism.

<u>Theorem B:</u> If $g_*: H(M) \longrightarrow H(M')$ is an isomorphism, $\bar{g}_*: H(N) \longrightarrow H(N')$ is an isomorphism, and $H_0(N) \simeq \Lambda + H_0'(N)$, then $h: U \longrightarrow U'$ is an isomorphism.

<u>Proof of Theorem A:</u> We may as well assume that h is the identity map. Let $M^{\#}$ be the mapping cylinder of g, M'' the relative mapping cylinder. Further let $N^{\#}$ be the mapping cylinder of $\bar{g}$, N'' the relative mapping cylinder. Since $E_p'(M^{\#}) = E_{p-1}'(M) + E_p'(M) + E_p'(M')$, we now have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'(M) & \longrightarrow & E'(M^{\#}) & \longrightarrow & E'(M'') & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \psi^{\#} & & \downarrow \psi'' & & \\ 0 & \longrightarrow & U \otimes N & \longrightarrow & U \otimes N^{\#} & \longrightarrow & U \otimes N'' & \longrightarrow & 0 \end{array}$$

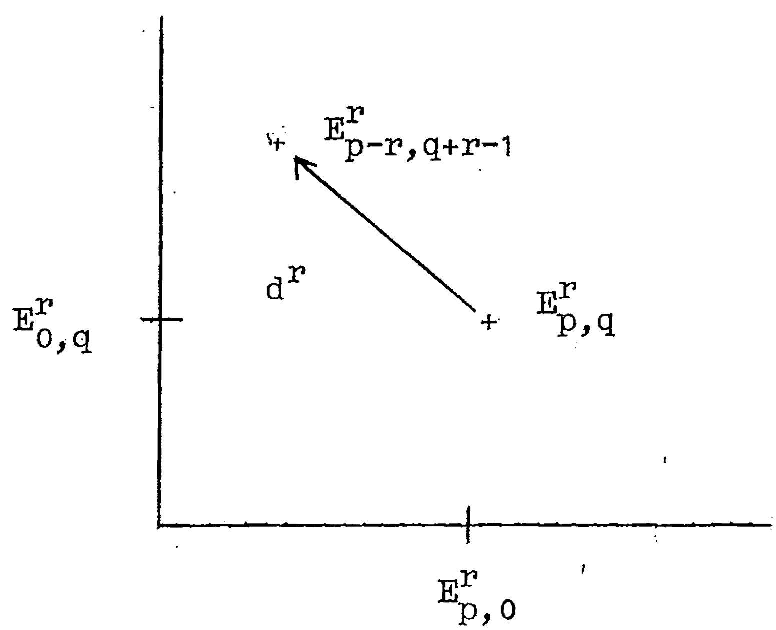
of differential modules such that the horizontal lines are exact. Passing to homology, we have the commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & E^2_{p,q}(M) & \longrightarrow & E^2_{p,q}(M^{\#}) & \longrightarrow & E^2_{p,q}(M'') & \longrightarrow & E^2_{p-1,q}(M) & \longrightarrow \cdots \\ & \downarrow \approx & & \downarrow \approx & & \downarrow & & \downarrow \approx & \\ \longrightarrow & H_p(N;U_q) & \longrightarrow & H_p(N^{\#};U_q) & \longrightarrow & H_p(N'';U_q) & \longrightarrow & H_{p-1}(N;U_q) & \longrightarrow \cdots \end{array}$$

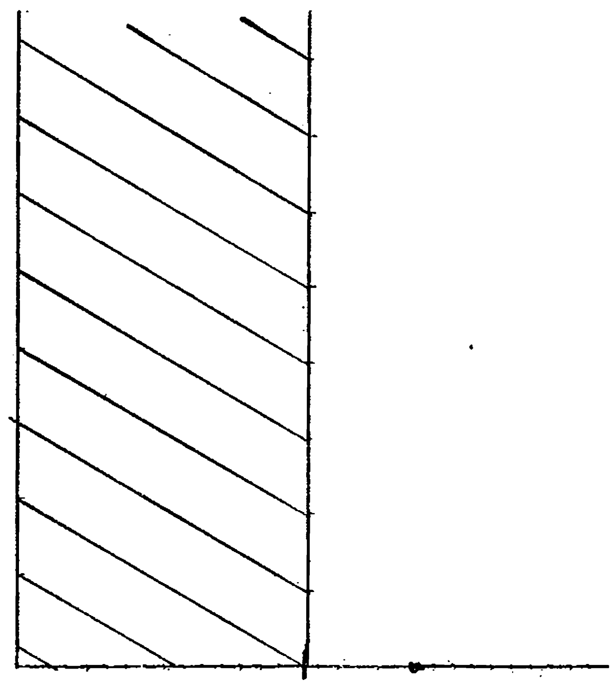
with exact horizontal lines. Therefore, by the 5 -lemma, we have $E^2_{p,q}(M'') \simeq H_p(N'';U_q)$. Now since $g_*: H(M) \xrightarrow{\approx} H(M')$ we have $H(M'') = 0$, and hence $E^{\infty}_{p,q}(M'') = 0$ for all p,q. Assume that $H_p(N'') = 0$ for $p < p_0$. This means that

$H_p(N''; U_q) = 0$ for $p < p_0$, or that $E^2_{p,q}(M'') = 0$ for $p < p_0$. However $d^r: E^2_{p_0,0} \longrightarrow E^2_{p_0-r,r-1}$ and therefore we have $E^2_{p_0,0} = E^{\infty}_{p_0,0} = 0$, or $H_{p_0}(N;U_0) = 0$. Now since $U_0 = \Lambda + U'_0$ this means that $H_{p_0}(N'') = 0$, and proceeding inductively we have $H_p(N'') = 0$ for all p. Then because N'' was the relative mapping cylinder of $\bar{g} : N \longrightarrow N'$, $\bar{g}_* : H(N) \longrightarrow H(N')$ is an isomorphism.

The basis for the preceding argument may be found by making a diagram for $E^r(M'')$ by plotting $E^r_{p,q}$ at the point (p,q) in the first quadrant of the plane.



Now in this diagram d^r is represented by an arrow going up and to the left. In the preceding argument the assertion that $H_0(N'') = 0$ for $p < p_0$ meant that $E^2_{p,q}(M'') = 0$ for $p < p_0$, or that only 0 groups appear in the shaded portion of the diagram

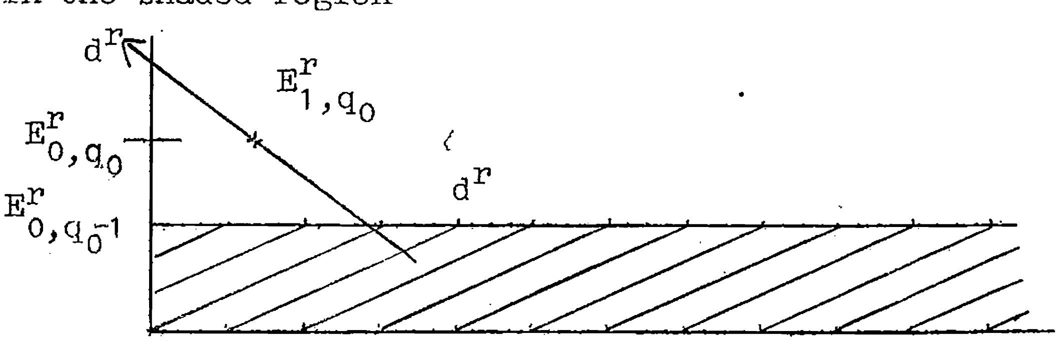


Consider now $E^r_{p_0,0}$: it sits on the horizontal axis, and therefore contains no boundaries. Further since d^r slopes up and to the left, it is mapped into zero. In other words we have the well-known principle that a spectral sequence with $E^{\infty} = 0$ identically has no corners.

<u>Proof of Theorem B:</u> In this case we may assume that $\bar{g}$ is the identity. Let $M^{\#}$ be the mapping cylinder of g, M'' the relative mapping cylinder, and let N'' be the mapping cylinder of $h \otimes 1 : U \otimes N \longrightarrow U' \otimes N$. We then show as before that $E^2(M'') \simeq H(N'')$, and recall that N'' is just the mapping cylinder associated with a coefficient homomorphism which we have already studied (4.19 and the corollary to 4.19).

Let A_q be the kernel of $h: U_q \longrightarrow U'_q$, and let C_q be the cokernel. Now since $g_* : H(M) \longrightarrow H(M')$ is an isomorphism, $H(M") = 0$, and $E^{\infty}_{p,q}(M") = 0$. Therefore $E^2_{0,0}(M") = 0$; but $E^2_{0,0}(M") = H_0(N;C_0)$. Therefore $C_0 = 0$. Now we also have $E^2_{1,0}(M") = H_0(N; A_0) = 0$ from the corollary to 4.19. Therefore $A_0 = 0$. Suppose now that $A_q = C_q = 0$ for $q < q_0$. Then $H_{p,q}(N") = 0$ for $q < q_0$, or $E^2_{p,q}(M") = 0$ for $q < q_0$. This means that $E^r_{p,q}(M") = 0$ for $q < q_0$, $r \geq 2$. Consider E^r_{0,q_0}. It consists entirely of d^r cycles for $r \geq 2$, and since $d^r: E^r_{r,q_0+1-r} \longrightarrow E^r_{0,q_0}$, it contains no boundaries. Therefore $E^2_{0,q_0} = E^{\infty}_{0,q_0} = 0$. However, $E^2_{0,q_0} = H_0(N; C_{q_0})$ and this means that $C_{q_0} = 0$. Now consider E^r_{1,q_0}. Again it consists entirely of d^r cycles for $r \geq 2$, and contains no boundaries. Therefore $E^2_{1,q_0} = E^{\infty}_{1,q_0} = 0$; but $E^2_{1,q_0} = H_0(N;A_{q_0})$, and therefore $A_{q_0} = 0$. Proceeding by induction we have $A_q = C_q = 0$ for all q, so that $h : U_q \longrightarrow U'_q$ is an isomorphism for all q. Thus the proof is complete.

The idea of the preceding proof is again that there can be no "corners" in a spectral sequence with $E^{\infty} = 0$. For $E^2_{p,q} = 0$ for $q < q_0$ means there are only 0 - groups in the shaded region



A version of theorem A involving only spectral sequences was proved by Borel, and by Serre, but is unpublished. However, theorem A as it stands will suffice for what we need here. For completeness we now state a well known theorem of Leray.

<u>Theorem C:</u> If $h: U \longrightarrow U'$ is an isomorphism, and $\bar{g}_*: H(N) \longrightarrow H(N')$ is an isomorphism, then $g_*: H(M) \longrightarrow H(M')$ is an isomorphism.

This theorem may be proved by the usual procedure of observing that since $g^{\infty}: E^{\infty}(M) \longrightarrow E^{\infty}(M')$ is an isomorphism, $g_*^0: E^0(H(M)) \longrightarrow E^0(H(M'))$ is also an isomorphism.

Chapter V

DGA Algebras and the Construction of Cartan

We shall now prepare to make Cartan's calculation of $H_q(X)$, where X is an Eilenberg-MacLane space; i.e. $\pi_q(X) = 0$ for $q \neq n$, $\pi_n(X) = \pi$. A number of preliminary notions are necessary before we can actually do this, and we shall present these in a manner similar to that of [1]. In the course of this work we shall obtain a special case of a theorem of Borel [2] which is useful in the study of the topology of Lie groups.

Conventions: In this chapter Λ will denote a fixed commutative ring with unit. If N and N' are graded Λ-modules, $N = \Sigma_{n \geq 0} N_n$, $N' = \Sigma_{n \geq 0} N'_n$, then $N \otimes_\Lambda N'$ is the graded Λ-module such that $(N \otimes_\Lambda N')_n = \Sigma_{r+s=n} N_r \otimes_\Lambda N'_s$. If N, N' are differential graded Λ-modules, then $N \otimes_\Lambda N'$ is a differential graded Λ-module with

$$d(x \otimes y) = dx \otimes y + (-1)^r x \otimes dy$$

for $x \in N_r$, $y \in N'$.

Definitions: A graded Λ-algebra is a pair (A, ϕ) where A is a graded Λ-module, and $\phi: A \otimes_\Lambda A \longrightarrow A$ is a homomorphism of graded Λ-modules such that if we denote $\phi(x \otimes y)$ by $x \cdot y$, then $(x \cdot y) z = (x \cdot y) \cdot z$.

If, in addition to the preceding, A is a differential graded Λ- module, and ϕ is a homomorphism of differential graded Λ- modules, then (A,ϕ) is a <u>differential graded Λ - algebra</u>.

Usually either a graded Λ - algebra or a differential graded Λ-algebra will be denoted merely by the symbol for its underlying module .

The graded Λ- algebra A has a <u>unit</u> if there exists an element $1 \in A_0$ such that $1 \cdot x = x \cdot 1 = x$ for $x \in A$, and it is <u>anti-commutative</u> if $x \cdot y = (-1)^{rs} y \cdot x$ for $x \in A_r, y \in A_s$.

The ring Λ itself will be considered as either

1) a Λ- module,
2) a graded Λ- module N such that $N_n = 0$ for $n > 0$, and $N_0 = \Lambda$
3) a differential graded Λ - module with $d = 0$,
4) a graded Λ- algebra, or
5) a differential graded Λ- algebra.

If A, A' are (differential) graded Λ-algebras, then $A \otimes_\Lambda A'$ is the (differential) graded Λ- algebra such that $(x \otimes y) \cdot (x' \otimes y') = (-1)^{rs} xx' \otimes yy'$ for $x' \in A_r, y \in A'_s$.

Notice that if A is a graded Λ-algebra, then the multiplication $\phi : A \otimes_\Lambda A \longrightarrow A$ is a homomorphism of graded Λ- algebras if and only if A is anti-commutative.

<u>Definitions:</u> An <u>augmentation</u> of a (differential) graded Λ- module N is a homomorphism $\varepsilon : N \longrightarrow \Lambda$ of (differential)

graded Λ-modules. A DGA-module is a differential graded Λ-module N together with an augmentation $\varepsilon : N \longrightarrow \Lambda$.

If N, N' are DGA-modules, then $N \otimes N'$ is a DGA-module with $\varepsilon(n \otimes n') = \varepsilon(n)\varepsilon(n')$.

An augmentation of a (differential) graded Λ-algebra A is a homomorphism $\varepsilon : A \longrightarrow \Lambda$ of (differential) graded Λ algebras with unit. Note that this implies that ε is an epimorphism. A DGA-algebra is a differential graded Λ-algebra together with an augmentation $\varepsilon : A \longrightarrow \Lambda$.

Example 1: Let X be a semi-simplicial complex. Then $C(X)_N \otimes \Lambda$ is in a natural way a DGA-module. It already has a differential operator and a gradation, so it suffices to define an augmentation. This is done by setting $\varepsilon = 0$ on positive dimensional elements, and $\varepsilon(x \otimes \lambda) = \lambda$ for $x \in X_0$, $\lambda \in \Lambda$.

Example 2: It was pointed out in Chapter III that if X, X', X'' are semi-simplicial complexes, then the diagram

$$\begin{array}{ccccc} (C(X)_N \otimes C(X')_N) \otimes C(X'')_N & \xrightarrow{\nabla \otimes 1} & C(X \times X')_N \otimes (C(X'')_N & \searrow^{\nabla} & \\ \downarrow \approx & & & & C(X \times X' \times X'')_N \\ C(X)_N \otimes (C(X')_N \otimes C(X'')_N) & \xrightarrow{1 \otimes \nabla} & C(X)_N \otimes C(X' \times X'')_N & \nearrow_{\nabla} & \end{array}$$

is commutative.

This means that if Γ is a monoid complex, and a multiplication is defined in $C(\Gamma)_N$ by the diagram

$$C(\Gamma)_N \otimes C(\Gamma)_N \xrightarrow{\nabla} C(\Gamma \times \Gamma)_N \longrightarrow C(\Gamma)_N$$

where $C(\Gamma \times \Gamma)_N \longrightarrow C(\Gamma)_N$ is the homomorphism induced by the multiplication in Γ, then $C(\Gamma)_N$ is a differential graded algebra over the ring of integers. Further it is not difficult to see that the unit of Γ_0 gives rise to a unit in the algebra $C(\Gamma)_N$. Consequently $C(\Gamma)_N \otimes \Lambda$ is in a natural way a DGA-algebra. Finally if Γ is commutative we have a commutative diagram

$$\begin{array}{ccccc}
C(\Gamma)_N \otimes C(\Gamma)_N & \xrightarrow{\nabla} & C(\Gamma \times \Gamma)_N & \searrow & \\
\downarrow T & & \downarrow T' & & C(\Gamma)_N \\
C(\Gamma)_N \otimes C(\Gamma)_N & \xrightarrow{\nabla} & C(\Gamma \times \Gamma)_N & \nearrow &
\end{array}$$

where $T(x \otimes y) = (-1)^{rs}\, y \otimes x$ for y of dim s, x of dim r, and T' is the map induced by the map of $\Gamma \times \Gamma$ into itself which interchanges factors. Therefore, if Γ is commutative, then $C(\Gamma)_N$ is an anti-commutative DGA-algebra.

Example 3: If A is a DGA-algebra, then $H_*(A) = \Sigma H_n(A)$ is a DGA-algebra with d identically zero.

Definition: If A is a DGA-algebra, then a graded augmented (left) A-module is a graded augmented module M and a homomorphism $\phi : A \otimes_\Lambda M \longrightarrow M$ of graded augmented modules such that if we write $\phi(a \otimes m) = a \cdot m$ for $a \in A, m \in M$, then $a \cdot (a' \cdot m) = (a \cdot a') \cdot m$ for $a, a' \in A$, and $1 \cdot m = m$.

M is a <u>DGA-module over A</u> if, in addition to the preceding, ϕ is a homomorphism of DGA-modules.

<u>Definition</u>: If A, A' are DGA-algebras and $f : A \longrightarrow A'$ is a DGA homomorphism, M a DGA-module on A, and M' a DGA-module on A', then $g : M \longrightarrow M'$ is a <u>DGA-homomorphism compatible with f</u> if the diagram

$$\begin{array}{ccc} A \otimes_{\Lambda} M & \xrightarrow{\ f \otimes g\ } & A' \otimes_{\Lambda} M' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ g\ } & M' \end{array}$$

is a commutative diagram of maps of DGA-modules.

<u>Definition</u>: If A is a DGA-algebra, then a <u>construction</u> on A consists of

1) a filtered DGA-module M on A such that if $m \in F_p M$, $a \in A$, then $a \cdot m \in F_p M$
2) a DGA-module N,
3) a homomorphism of DGA modules $p : M \longrightarrow N$ which is compatible with $\varepsilon : A \longrightarrow \Lambda$, and
4) a homomorphism of graded augmented left A-modules $\nabla : A \otimes_{\Lambda} N \longrightarrow M$ subject to the following conditions:

a) $p \nabla (1 \otimes n) = n$,

b) $p F_r M \subset \Sigma_{q \leq r} N_r$,

c) if $F_r(A \otimes_{\Lambda} N) = \Sigma_{q \leq r} A \otimes_{\Lambda} N_r$, then $\nabla(F_r(A \otimes_{\Lambda} N)) \subset F_r M$, and

d) $\nabla^0 : E^0(A \otimes_\Lambda N) \longrightarrow E^0(M)$ is a homomorphism of DGA-modules such that $\nabla^1 : E^1(A \otimes_\Lambda N) \xrightarrow{\approx} E^1(M)$.

A construction on A will be denoted by (A, N, M).

<u>Definition:</u> A construction (A,N,M) is <u>free</u> if $\nabla : A \otimes N \longrightarrow M$ is an isomorphism of filtered Λ-modules, and N is a free Λ-module. In this case we will frequently identify $A \otimes N$ and M as Λ-modules. Note, however, that the differential operator in M is not necessarily the natural one of $A \otimes N$; in fact it is usually twisted.

<u>Definition:</u> A DGA module M is <u>acyclic</u> if $\varepsilon : M \longrightarrow \Lambda$ induces an isomorphism $\varepsilon_\lambda : H(M) \longrightarrow \Lambda$, or in other words if $\cdots \longrightarrow M_n \xrightarrow{d} M_{n-1} \longrightarrow \cdots \xrightarrow{d} M_0 \xrightarrow{\varepsilon} \Lambda$ is an exact sequence.

A construction (A,N,M) is acyclic if M is acyclic.

<u>Theorem 1</u>: Let (A,N,M) be a free construction, (A',N',M') an acyclic construction, and $f : A \longrightarrow A'$ a DGA homomorphism. Under these conditions there exists a DGA homomorphism $g : M \longrightarrow M'$ which is compatible with f. If g' is another such homomorphism, then there is a homotopy $D : M \longrightarrow M'$ such that

$dD + Dd = g - g'$, and $D\, a . m = (-1)^r f(a) Dm$ for $a \in A_r$. Further if the filtration on M is regular then **g** is filtration preserving, and D is a homotopy of degree 1.

Proof: Let C_1 be a basis for N_1 over Λ. For this proof, identify $x \in C_1$ with $1 \otimes x \in M$. Now if $x \in C_0$, define $g(x)$ to be any element of M'_0 such that $\varepsilon g(x) = \varepsilon(x)$. If $y \in A \otimes N_0$ then y may be written uniquely as $\sum a_j \otimes x_j$ where $x_j \in C_0$ and $g(y)$ is defined to be $\sum f(a_j)\, g(x_j)$.

For $x \in C_1$, we have $dx \in A \otimes N_0$ and $\varepsilon(dx) = 0$. Therefore $g(dx)$ is defined and $\varepsilon g(dx) = 0$. Define $g(x)$ to be some element of M'_1 such that $dg(x) = g(dx)$. Now if $y \in A \otimes N_1$, $y = \sum a_j \otimes x_j$ where $x_j \in C_1$ and we define $g(y)$ to be $\sum f(a_j)\, g(x_j)$.

Suppose now that g is defined on $A \otimes \sum_{q < r} N_q = F_{r-1} M$. For $x \in C_r$; we have $dx \in F_{r-1}M$, $g(dx)$ is defined and $dg(dx) = 0$. Therefore we may define $g(x)$ to be any element of M'_r such that $dg(x) = g(dx)$. Consequently the existence of g is proved.

Let g' be another map compatible with f. Then for $x \in C_0$, $\varepsilon g(x) = \varepsilon(x) = \varepsilon g'(x)$, and $\varepsilon(g(x) - g'(x)) = 0$. Define Dx to be any element of M'_1 such that $dDx = g(x) - g'(x)$. Now extend D to F_0M by defining $D a \otimes x = (-1)^r f(a) Dx$ for $x \in A_r$.

Suppose that D is defined on $F_{r-1}M$. Then for $x \in C_r$, we have $dx \in F_{r-1}M$, $g(x) - g'(x) - Ddx$ is a cycle belonging to M'_r, and we define Dx to be any element of M'_{r+1} such that $dDx = g(x) - g'(x) - Ddx$.

Notice that $g(1 \otimes N_r) \subset F_r M'$ if M' has a regular filtration (i.e. $M'_r \subset F_r M'$), and then $g(A \otimes N_r) \subset F_r M'$, since for $x \in A'$, $m \in F_r M'$ we have $x \cdot m \in F_r M'$. The same reasoning shows that $DF_r M \subset F_{r+1} M'$, or that D is of degree 1.

Definitons: If (A,N,M) and (A',N',M') are constructions, a map of the first into the second consists of a DGA homomorphism $f: A \longrightarrow A'$ together with a filtration preserving DGA homomorphism $g: M \longrightarrow M'$ which is compatible with f. Under the preceding conditions the map of constructions will be said to be compatible with f. Further, since g is filtration preserving, g induces $g^r: E^r(M) \longrightarrow E^r(M')$. Now consider Λ as an $H(A)$ module by defining $x.a = x \cdot \varepsilon(a)$ for $x \in \Lambda$, $a \in H(A)$. Similarly consider Λ as an $H(A')$ module. Then $N = \Lambda \otimes_{H(A)} E^1(M)$, and $N' = \Lambda \otimes_{H(A')} E^1(M')$, and there is a DGA homomorphism $\overline{g}: N \longrightarrow N'$ induced by g^1, or by g.

Theorem 1: Let (A,N,M) be a free construction, (A',N',M') an acyclic construction with a regular filtration, and $f: A \longrightarrow A'$ a DGA homomorphism. Then there is a map of (A,N,M) into (A',N',M') compatible with f. Further the induced homomorphism $\overline{g}_*: H(N) \longrightarrow H(N')$ is independent of the choice of such a map.

Proof: The first part of this theorem is just a restatement of Theorem 1. To prove the last part suppose $g, g': M \longrightarrow M'$ are compatible with f. Let D be a homotopy between g and g' satisfying the conditions of Theorem 1, and define $\bar{D}: N \longrightarrow N'$ by $\bar{D}x = pDx$ for $x \in C_r$ where C_r is a basis for N_r as in Theorem 1, and $p: M' \longrightarrow N'$ is the projection map of the construction (A', N', M'). One verifies easily that $d\bar{D} + \bar{D}d = \bar{g} - \bar{g}'$.

Theorem 2: Suppose that (A,N,M) and (A',N',M') are constructions, $f: A \longrightarrow A'$ and $g: M \longrightarrow M'$ are DGA homomorphisms which determine a map of constructions, and N, N' are free Λ-modules. Under these conditions if $f_x: H(A) \longrightarrow H(A')$ is an isomorphism and $g_x: H(M) \longrightarrow H(M')$ is an isomorphism, then $\bar{g}_x: H(N) \longrightarrow H(N')$ is also an isomorphism.

The preceding theorem is almost a special case of Theorem A of chapter 4. The difference is that we have not assumed that the isomorphism $H(A) \otimes N \longrightarrow E^1(M)$ is compatible with differential operators. This, however, is the case if $H_0(A) = \Lambda$. With $H_0(A) = \Lambda$, the map $p: E^1_{q,0} \longrightarrow N_q$ is an isomorphism, and therefore the differential operator d' is of the correct form on $\sum_q E^1_{q,0}$. Now as a left $H(A)$ module, $E^1(M) = H(A) \otimes N$, and $d'(x \otimes y) = (-1)^{\dim x} x \cdot d'(1 \otimes y) = (-1)^{\dim x} x(1 \otimes dy) = (-1)^{\dim x} x \otimes dy$, and we see that in this case the differential operator is just the usual one in $H(A) \otimes N$.

We will now indicate the changes necessary in the proof of Theorem A to prove the above theorem without the assumption that $H_0(A) = \Lambda$. Let M'' be the relative mapping cylinder of $g:M \longrightarrow M'$, and N'' the relative mapping cylinder of $\bar{g}:N \longrightarrow N'$. It is easily seen that $E^1(M'') = H(A) \otimes N''$, and to use the same proof as before we need to know that $E^1_{q,0}(M'') = 0$ for $q \leq p$ implies that $E^1_{p,q}(M'') = 0$ for all q.

Let $N^\# = H_0(A) \otimes N''$. We have a differential operator in $N^\#$ induced by d^1. Further $E^1(M'') = H(A) \otimes_{H_0(A)} N^\#$. Let G be any right $H_0(A)$ module, and define $H(N^\#; G)$ to be $H(G \otimes_{H_0(A)} N^\#)$. Now $E^2_{p,q}(M'') = H_p(N^\#; H_q(A))$, and $N'' = \Lambda \otimes_{H_0(A)} N^\#$. Therefore to prove the theorem it suffices to show that $H_q(N^\#; H_0(A)) = 0$ for $q \leq p$ implies that $H_q(N^\#; G) = 0$ for $q \leq p$ for any right $H_0(A)$ module G. However, the fact that $H_q(N^\#; H_0(A)) = 0$ for $q \leq p$ implies that $H_q(N^\#; F) = 0$ for $q \leq p$ where F is any free $H_0(A)$ module. Suppose now that $0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$ is an exact sequence of right $H_0(A)$ modules. Then since $N^\#$ is a free $H_0(A)$ module (this follows since N'' is a free Λ-module), the sequence

$$0 \longrightarrow R \otimes_{H_0(A)} N^\# \longrightarrow F \otimes_{H_0(A)} N^\# \longrightarrow G \otimes_{H_0(A)} N^\# \longrightarrow 0$$

is exact, and there is a resulting exact sequence

$$\cdots \longrightarrow H_q(N;R) \longrightarrow H_q(N;F) \longrightarrow H_q(N;G) \longrightarrow H_{q-1}(N;R) \longrightarrow \cdots \quad ;$$

Consequently for F free we have $H_q(N;G) \simeq H_{q-1}(N;R)$ for $q \leq p$, and by induction this implies the desired result.

Having given some properties of constructions, we shall now show how they arise. We shall first prove that any twisted Cartesian product (Γ, B, E) (Definition 2.13) gives rise to a construction, provided is a monoid complex. To do this some preliminary definitions are needed.

<u>Definitions</u>: If (Γ, B, E) is a twisted Cartesian product, let $\nabla' : C(\Gamma)_N \otimes C(B)_N \longrightarrow C(E)_N$ be the composition of the natural map $\nabla : C(\Gamma)_N \otimes C(B)_N \longrightarrow C(\Gamma \times B)_N$ of the Eilenberg-Zilber Theorem (Chapter 3, p. 17) and the identification of $C(E)_N$ and $C(\Gamma \times B)_N$ as groups. We shall say that a simplex $\sigma \in E$ is of <u>filtration</u> p if its projection lies in the p-skeleton of B, i.e. may be written as $s_{i_1} \ldots s_{i_r} \tau$ where $\tau \in B$ is a simplex of dimension less than or equal to p. Define $F_p C(E)_N$ to be the subgroup generated by simplexes of filtration p. Further when Γ is a monoid complex consider $C(E)_N$ as a left $C(\Gamma)_N$ module by using the diagram

$$C(\Gamma)_N \otimes C(E)_N \xrightarrow{\nabla} C(\Gamma \times E)_N \longrightarrow C(E)_N$$

all maps being the natural ones.

<u>Proposition</u>: If Γ is a monoid complex, and (Γ, B, E) is a twisted Cartesian product, then $(C(\Gamma)_N, C(B)_N, C(E)_N)$ is a construction with a regular filtration.

Proof: All statements which need to be verified follow at once except the assertion that $\nabla : C(\Gamma)_N \otimes C(B)_N \longrightarrow C(E)_N$ commutes with d^0 and induces an isomorphism $\nabla' : H(C(\Gamma)_N) \otimes C(B)_N \longrightarrow E^1(C(E)_N)$. We shall prove this by showing that $E^0(C(E)_N) = E^0(C(\Gamma \times B)_N)$, that this identification is compatible with d^0, and that the proposition is true for a Cartesian product.

First identify E and $\Gamma \times B$ as sets. Then we have to consider $\partial_i(\sigma \times \tau)$ where $\sigma \times \tau$ is of filtration p. If $i > 0$ it does not matter whether we mean the i-th face operator in E or $\Gamma \times B$ by ∂_i. If $i = 0$ we still have the relation $\partial_0(\sigma \times \tau) = \partial_0 \sigma \cdot \partial_0(1 \times \tau)$. The fact that $\sigma \times \tau$ is of filtration p means that $\tau = s_{i_0} \cdots s_{i_r} \tau'$ where $\tau' \in B$ has dimension less than or equal to p. If τ' has dimension less than p, then $\sigma \times \tau$ represents the zero element in $C(E)_N$. Therefore assume that dimension $(\tau') = p$. Now $\partial_0(1 \times \tau) = \partial_0(1 \times s_{i_0} \cdots s_{i_r} \tau') = \partial_0 s_{i_0} \cdots s_{i_r}(1 \times \tau')$. Assuming, as we may, that $i_0 > \cdots > i_r$, it follows that the element $\partial_0(1 \times \tau) = s_{i_0 - 1} \cdots s_{i_r - 1} \partial_0(1 \times \tau')$ is of filtration $(p-1)$ unless $i_r = 0$. In this case $\partial_0(1 \times \tau) = s_{i_0 - 1} \cdots s_{i_{r-1} - 1}(1 \times \tau')$, and this formula is independent of whether we mean the 0'th face operator of E or $\Gamma \times B$ by ∂_0. Thus, we have shown that $E^0(C(E)_N) = E^0(C(\Gamma \times B)_N)$. It therefore remains to show that

$\nabla' : H(C(\Gamma)_N) \otimes C(B)_N \longrightarrow E^1(C(\Gamma \times B)_N)$ is an isomorphism. To show this, recall that we have defined a map

$f: C(\Gamma \times B)_N \longrightarrow C(\Gamma)_N \otimes C(B)_N$ (Chapter 3) such that $f\nabla$ is the identity[1] and ∇f is homotopic to the identity. Since f is filtration preserving, $f'\nabla'$ is the identity, and to prove the proposition we need only show that $\nabla' f'$ is the identity. For this it suffices to know that the homotopy of f with the identity is of degree 0. However, this is indeed the case, for the homotopy is natural.

The following comments may help to clarify the last assertion. The fact that the homotopy is natural means that if $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$ are maps of semi-simplicial complexes, then the homotopy commutes with the induced map of $C(X \times Y)_N \longrightarrow C(X' \times Y')_N$. However, any simplex of a Cartesian product $X \times Y$ is the image of a simplex of $\Delta_p \times \Delta_q$ for some p and q, and every simplex of Δ_p or Δ_q can be obtained by applying face and degeneracy operations to the basic simplex. Therefore the fact that the homotopy is natural means that it may be expressed by using face and degeneracy operations. However, from the very definition of the filtration on the chains of a Cartesian product or a twisted Cartesian product product it is evident that the filtration can not be raised by applying face and degeneracy operations.

<u>Definition</u>: A construction (A,N,M) satisfies the <u>condition B'</u> if

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1 In Chapter 3 it only stated that $f\nabla$ is homotopic to the identity. However, one verifies easily that it is actually equal to the identity.

1) $\varepsilon : N_0 \xrightarrow{\approx} \Lambda$, and

2) $x \in Z_q(M)$ and $\varepsilon(x) = 0$ imply that there exists a unique $y \in \nabla(1 \otimes N_{q+1})$ such that $dy = x$.

The construction satisfies the condition B if it satisfies the condition B' and is free.

Theorem 3: If (A.N,M) is a free construction, (A',N',M') is a construction satisfying the condition B', and $f:A \longrightarrow A'$ is a DGA homomorphism, then there is a unique map of (A,N,M) into (A',N',M') such that $\nabla(1 \otimes N)$ maps into $\nabla(1 \otimes N')$.

One we note that the condition B' implies that the construction is acyclic, the proof of this theorem is entirely similar to the proof of Theorem 1, except that at each stage where a choice had to be made in the proof of the earlier theorem, there is now available a unique element of $\nabla(1 \otimes N')$ satisfying the required conditions.

Theorem 4: If A is a DGA algebra, and kernel $\varepsilon : A \longrightarrow \Lambda$ is a free Λ-module, there exists a construction (A,N,M) satisfying the condition B. Further if (A,N',M') is another such construction, then there is a unique isomorphism of (A,N,M) with (A,N',M') which maps $\nabla(1 \otimes N)$ into $\nabla(1 \otimes N')$.

The uniqueness is clear from the preceding theorem. It remains to prove existence. This will be done in two different ways. The first way is perhaps more intuitive, but is valid only if Λ is a principal ideal domain.

First proof of existence: We assume now that Λ is a principal ideal domain. Recall that over a principal ideal domain any submodule of a free module is free. Therefore $\hat{A}$ = kernel $\varepsilon : A \longrightarrow \Lambda$ is automatically free. Proceeding with the construction, let $N_0 = \Lambda$, $M_0 = A_0 \otimes N_0 - A_0$, let $N_1 = \hat{A}_0$, $M_1 = A_1 \otimes N_0 + A_0 \otimes N_1$, and define $d : 1 \otimes N_1 \longrightarrow A_0$ to be the natural map.

Suppose that N_q and M_q are defined for $q \leq r$ so as to satisfy the condition B. We have $M_q = \sum_{i+j=q} A_i \otimes N_j$. Define N_{r+1} = kernel $d : M_r \longrightarrow M_{r-1}$, and $M_{r+1} = \sum_{i+j=r+1} A_i \otimes N_j$. Further define $d : 1 \otimes N_{r+1} \longrightarrow M_r$ to be the natural map. It is now evident that (A,N,M) is a construction satisfying the condition B.

Second proof of existence: Again let $\hat{A}$ denote kernel $\varepsilon : A \longrightarrow \Lambda$. Define $\bar{B}^0(A) = \Lambda$, and for $n > 0$, $\bar{B}^n(A)$ to be the tensor product of $\hat{A}$ with itself n-times, and denote an element of $\bar{B}^n(A)$ by $[a_1, \ldots, a_n]$. Define a new gradation in $\bar{B}^n(A)$ by setting dimension $[a_1, \ldots, a_n] = n + \sum \alpha_i$ where α_i = dimension a_i. Define $\bar{B}(A)$ to be $\sum \bar{B}^n(A)$, and $B(A)$ to be $A \otimes \bar{B}(A)$. The object now is to place a differential operator in $B(A)$ so that $(A, \bar{B}(A), B(A))$ is a construction satisfying the condition B.

Denote $A \otimes \bar{B}^n(A)$ by $B^n(A)$ and denote an element of this module by $a[a_1,\ldots,a_n]$. Define $s: B(A) \longrightarrow B(A)$ by setting $s(a[a_1,\ldots,a_n]) = [a - \varepsilon(a), a_1,\ldots,a_n]$. We want s to be a contracting homotopy for $B(A)$, i.e. we want the relation $ds + sd = 1 - \varepsilon$ to hold, where 1 is the identity map. Since $B(A)$ is to be a left A-module we shall have the relation $d(a \cdot x) = (da) \cdot x + (-1)^{\alpha} a \cdot dx$, where α = dimension a. Therefore it suffices to define d on $\bar{B}(A)$. On $\bar{B}_0(A)$, d is zero. On $\bar{B}_1(A)$ define $d[a_1] = a_1 \in A \otimes \bar{B}_0(A)$. Assume that d is defined on $\bar{B}^r(A)$ for $r \leq n$, such that $d: \bar{B}^r(A) \longrightarrow B^r(A)$. A typical element of $\bar{B}^{n+1}(A)$ may be written as $[a_1,\ldots,a_{n+1}] = s\, a_1 [a_2,\ldots,a_{n+1}]$. Define $d[a_1,\ldots,a_{n+1}] = a_1[a_2,\ldots,a_{n+1}] - sda_1[a_2,\ldots,a_{n+1}]$. Then $dd[a_1,\ldots,a_{n+1}] = d\,a_1[a_2,\ldots,a_{n+1}] - dsda_1[a_2,\ldots,a_{n+1}]$, and assuming by induction that $ds + sd = 1 - \varepsilon$ this last expression is zero. Consequently d is defined, and $d^2 = 0$.

To show that this construction satisfies the condition B, suppose that $x \in B(A)_0$ and $\varepsilon(x) = 0$; then $x = d[x]$. Suppose that we also have $x = dy$, where $y \in \bar{B}(A)$; then $y = s(z)$ where $\varepsilon(z) = 0$, and $d([x] - s(z)) = ds(x-z) = 0$. However, $(x-z) = ds(x-z) + sd(x-z) = 0$, and $x = z$, so that $y = [x]$. Now suppose that $x \in B(A)_q$, $q > 0$, and that $dx = 0$. We have $x = dsx$, where $sx \in \bar{B}(A)$, and if $x = dy$

where $y \in \bar{B}(A)$ then $y = sz$, and $ds(x-z) = 0$. This means that $x-z = sd(x-z)$, $s(x-z) = ssd(x-z) = 0$, and consequently $y = sz = sx$. The proof of the theorem is now complete.

In neither of the preceding proofs have we shown how to obtain the differential operator in N in the construction (A,N,M). The construction, however, is free, so that $N = \Lambda \otimes_A M$, and the differential operator in N is the natural induced one.

Proposition: Let Γ be a monoid complex, and let (A,N,M) be the construction arising from the twisted Cartesian product $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$. Then (A,N,M) satisfies the condition B'.

Proof: $\bar{W}_0$ has one element (cf. definition 2.17), and consequently $\nabla : A_0 \otimes N_0 \longrightarrow M_0$ is an isomorphism. Further if S is the contracting homotopy for $W(\Gamma)$ used in the proof of 2.15, then S satisfies the identity $S^2 = s_0 S$, and $S : W_q \longrightarrow 1_{q+1} \times \bar{W}_{q+1}$ is onto. Consequently, denoting by S the induced contracting homotopy on M, we have $S : M \longrightarrow \nabla(1 \otimes N)$, $S^2 = 0$, and $S : \hat{M} \longrightarrow \nabla(1 \otimes \hat{N})$ is an epimorphism (recall that if C is a DGA module, then $\hat{C} = \text{kernel } \varepsilon : C \longrightarrow \Lambda$). Suppose, therefore that if $x \in M_q$ is such that $\varepsilon(x) = 0$ for $q = 0$, or $dx = 0$ for $q < 0$, then $x = dSx$. If $x = dy$, where $y \in \nabla(1 \otimes N)$, then $y = Sz$ for some $d \in \hat{M}$, and $d\,S(x-z) = 0$. Consequently $x - z = Sd(x-z)$, $S(x-z) = 0$, and $y = Sx$. This proves the desired result.

Definition: Let (A,N,M) and (A',N',M') be constructions. Consider $(A \otimes A', N \otimes N', M \otimes M')$. Define $\nabla : A \otimes A' \otimes N \otimes N' \longrightarrow M \otimes M'$ by $\nabla(a \otimes a' \otimes n \otimes n') = (-1)^{\alpha\beta} \nabla(a \times n) \otimes \nabla(a' \times n')$ where α = dimension a' and β = dimension n. Suppose that $M \otimes M'$ is provided with the usual filtration, i.e. $F_p(M \otimes M') = \sum_{r+s=p} F_r M \otimes F_s M'$, and the usual differential operator. Consider $M \otimes M'$ as a left $A \otimes A'$ module by defining $(a \otimes a').(m \otimes m') = (-1)^{\alpha\gamma} a.m \otimes a'm'$ where α = dimension a, and γ = dimension m.

Proposition: If (A,N,M) and (A',N',M') are constructions whose underlying modules are free over Λ, then $(A \otimes A', N \otimes N', M \otimes M')$ is a construction whose underlying modules are free over Λ. If in addition

1) (A,N,M) and (A',N',M') are free, then $(A \otimes A', N \otimes N', M \otimes M')$ is free, and

2) if (A,N,M) and (A',N',M) are acyclic, then $(A \otimes A', N \otimes N', M \otimes M')$ is acyclic.

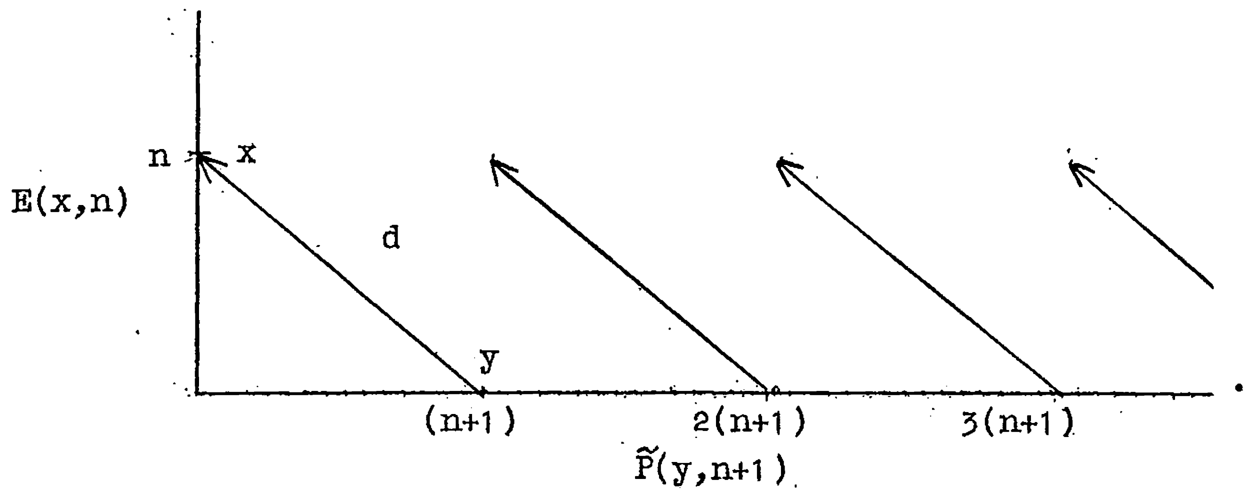
The proof of this proposition follows immediately from the definitons.

Corollary: If A, A' are DGA algebras such that $\hat{A}, \hat{A}'$ are free as Λ-modules, and $(A \otimes A', N, M)$ is an acyclic construction such that the underlying modules are free over Λ, then $H(\bar{B}(A) \otimes \bar{B}(A')) \simeq H(N)$. If $H(\bar{B}(A'))$ is a free Λ-module, then $H(\bar{B}(A)) \otimes H(\bar{B}(A')) \simeq H(N)$.

Notation: Let $E(x,n)$ denote the exterior algebra over Λ with one generator x of dimension n. In other words $E(x,n)_q = 0$ for $q \neq 0,n$; $E(x,n)_0 \simeq \Lambda$ with basis element 1, the unit of $E(x,n)$; and $E(x,n)_n \simeq \Lambda$, with basis element x. In the algebra $x^2 = 0$.

Let $\tilde{P}(y,n)$ denote the divided polynomial ring with basic element y in dimension n. In other words $\tilde{P}(y,n)_q = 0$ unless q is of the form kn for some non-negative integer k, $\tilde{P}(y,n)_{kn} \simeq \Lambda$ with basis element y_k, $y_0 = 1$ is the unit of the algebra $\tilde{P}(y,n)$, $y_1 = y$, and the product in the algebra is defined by $y_i y_j = \binom{i+j}{i} y_{i+j}$.

Notice that for n odd, both $E(x,n)$ and $\tilde{P}(y,n+1)$ are anti-commutative. For each n we define a free acyclic construction $(E(x,n),\tilde{P}(y,n+1),M)$ as follows: since the construction is free $\nabla : E(x,n) \otimes \tilde{P}(y,n+1) \xrightarrow{\approx} M$, and we will assume that ∇ is the identity map as far as modules are concerned. Define $d(1 \otimes y_{k+1}) = x \otimes y_k$, $d(x \otimes y_k) = 0$. Now M is an algebra with an additive base $\{x \otimes y_k, 1 \otimes y_k\}$. Further $d((1 \otimes y_i)(1 \otimes y_j)) = d(1 \otimes y_i y_j) = d(\binom{i+j}{i}(1 \otimes y_{i+j}) = \binom{i+j}{i} x \otimes y_{i+j-1}$, and $d(1 \otimes y_i)(1 \otimes y_j) + (1 \otimes y_i)d(1 \otimes y_j) = (x \otimes y_{i-1})(1 \otimes y_j) + (1 \otimes y_i)(x \otimes y_{j-1}) = (\binom{i+j-1}{i-1} + \binom{i+j-1}{i}) x \otimes y_{i+j-1} = \binom{i+j}{i} x \otimes y_{i+j-1}$. These calculations show that d is an anti-derivation on the algebra M. Moreover, it is clear that the algebra M is acyclic. Its structure is described by the diagram



Combining the results of the calculation just made, the comparison theorem for constructions, and the previous proposition concerning constructions over tensor products, we obtain the following result. Suppose that $(E(x_1,n_1) \otimes \ldots \otimes E(x_k,n_k), N, M)$ is an acyclic construction with N and M free Λ-modules. Suppose further that n_i is odd for $i = 1,\ldots,k$. In this case $H(N) \simeq \tilde{P}(y_1,n_1+1) \otimes \ldots \otimes \tilde{P}(y_k,n_k+1)$. This result is quite weak, but we have a much stronger result due to A. Borel [2].

<u>Theorem</u>: Suppose that (A,N,M) is an acyclic construction such that the underlying Λ-modules are free, and that $H(A) \simeq E(x_1,n_1) \otimes \ldots \otimes E(x_k,n_k)$, where n_i is odd, for $i = 1,\ldots,k$. In this case $H(N) \simeq \tilde{P}(y_1,n_1+1) \otimes \ldots \otimes \tilde{P}(y_k,n_k+1)$.

<u>Proof</u>: It is sufficient to prove this theorem for the construction $(A,\bar{B}(A),B(A))$. In other words it is sufficient to prove that $H(\bar{B}(A)) \simeq \tilde{P}(y_1,n_1+1) \otimes \ldots \otimes \tilde{P}(y_k,n_k+1)$. To do this we shall look at a spectral sequence for $\bar{B}(A)$.

As usual let $\hat{A}$ = kernel $\varepsilon : A \longrightarrow \Lambda$, and recall that if we define $\bar{B}^k(A) = \hat{A} \otimes \ldots \otimes \hat{A}$, the tensor product being taken k-times, then $\bar{B}(A) = \Lambda + \bar{B}'(A) + \ldots + \bar{B}^k(A) + \ldots$. In $\bar{B}^k(A)$ the dimension of a typical element $[a_1, \ldots, a_k]$ is $\Sigma \alpha_i + k$, where α_i is the dimension of a_i .

Define $F_p(\bar{B}(A)) = \Sigma_{k \leq p} \bar{B}^k(A)$. Then $E'(\bar{B}(A)) = \Lambda + H(\hat{A}) + \ldots H(\hat{A} \otimes \ldots \otimes \hat{A}) + \ldots$ with the appropriate conventions concerning dimensions. Now if $\hat{H}(A)$ is a free Λ-module, then $H(\hat{A} \otimes \ldots \otimes \hat{A}) \simeq \hat{H}(A) \otimes \ldots \otimes \hat{H}(A)$, and $E'(\bar{B}(A)) = \bar{B}(H(A))$. Further it is not difficult to verify that in this case $E^2(\bar{B}(A)) = H(\bar{B}(H(A)))$. However, we have more data available. We have assumed that $H(A) \simeq E(x_1, n_1) \otimes \ldots \otimes E(x_k, n_k)$. Consequently by our earlier remark $H(\bar{B}(H(A))) \simeq \tilde{P}(y_1, n_1 + 1) \otimes \ldots \otimes \tilde{P}(y_k, n_k + 1)$. This means that the total degree or dimension of every element of $E^2(\bar{B}(A))$ is even, and therefore that $E^2(\bar{B}(A)) = E^\infty(\bar{B}(A))$. We then have $\tilde{P}(y_1, n_1 + 1) \otimes \ldots \otimes P(y_k, n_k + 1) = E^\infty(\bar{B}(A)) = E^0(H\bar{B}(A))$. Since $E^0(H(\bar{B}(A)))$ is a free Λ-module we now see that $H(\bar{B}(A)) \simeq \tilde{P}(y_1, n_1 + 1) \otimes \ldots \otimes \tilde{P}(y_k, n_k + 1)$, which is the desired result. Note that this last isomorphism is not natural.

[1] Séminaire Henri Cartan, 7e anne: 1954/1955. Algèbres d'Eilenberg Mac Lane et Homotopie.

[2] A, Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. vol. 57 (1953), pp.115-207.

Errata to Chapter IV

p.4-2 line 9: read "functor $H:A \longrightarrow \mathcal{G}_\Lambda$" instead of "functor $H:A \longrightarrow \Lambda$."

p.4-10 line 5: read "theorem of Serre [5]" instead of "theorem of Serre [ ]."

References for Chapter IV

[1] J. Leray, L' anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue, Journ. de Math., tome XXIX (1950), pp. 1-139.

[2] W. S. Massey, Exact couples in algebraic topology (Parts I and II), Ann. of Math., 56 (1952), pp. 364-396.

[3] H. Cartan and S. Eilenberg, Homological Algebra .

[4] S. Eilenberg and N. E. Steenrod, Foundations of Algebraic Topology, Princeton, 1952.

[5] J. P. Serre, Homologie singulière des espaces fibrés, Ann. of Math., vol. 50 (1951) pp. 425-505.