On G-CW complexes and a theorem of J. H. C. Whitehead

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0. Introduction

In this note we shall define a G-CW complex for a general topological group G and establish the elementary properties of it following those of a CW complex [4]. In these the G-homotopy extension property and the G-cellular approximation theorem are important and the generalized theorem of J. H. C. Whitehead seems to be most interesting.

When G is a finite group, a G-CW complex is the same concept as a G-complex of G. Bredon [1]. When G is a compact Lie group any differentiable G-manifold has a G-CW complex structure [2, Prop. (4.4)], which is used to make a representation of the equivariant K-theory by using the space of Fredholm operators.

In this note G denotes a fixed topological group. By a G-space we mean a topological space X which has a fixed topological left G-action $\mu: G\times X \to X$. We usually write gx for $\mu(g, x)$ and X/G for the orbit space which is topologized by the natural projection $\pi: X \to X/G$.

The alphabet H always denotes a closed subgroup of G, and $Hx=\{g\in G;\ Gx=x\}$ is an isotropy subgroup at x of X. On the contrary $Gx=\{gx;\ g\in G\}$ is the G-orbit of $x\in X$. This convention is due to Professor M. Sato.

1. Definition of a G-CW complex

A G-CW complex shall be defined to be a type of cell complex with a fixed topological G-action. We first recall the definition of a cell complex.

DEFINITION (1.1) A pair (X, K) of a Hausdorff space X and a family of (open) cells in X, $K=\{e_\lambda\subset X;\ \lambda\in\Lambda\}$, is called to be a *cell complex* if:

(a) $X=\cup e_\lambda$ ($\lambda\in\Lambda$, a disjoint union),

(b) each n-cell e^n has its characteristic continuous map of a closed n-simplex onto its closure, $\sigma: \Delta^n \to \bar{e}^n\subset X$, satisfying

(b1) $\sigma|(\Delta^n-\partial\Delta^n)$ is a homeomorphism onto e,

(b2) $\partial e^n\subset X^{n-1}$, where $\partial e=\bar{e}-e=\sigma(\partial\Delta)$ and X^{n-1} is the $(n-1)$ skeleton of X, the union of all cells whose dimensionalities do not exceed $(n-1)$.

This definition of a cell complex is due to J. H. C. Whitehead [4]. When a cell complex structure K is fixed, we write simply X for (X, K). The *cell complex closure* of a subset S of X, denoted by $X(S)$, is defined to be the smallest subcomplex which contains S. Then the topological closure of a cell, $\bar{e}$ does not necessarily coincide with $X(e)$.

DEFINITION (1.2) Let G be a topological group and X be a G-space. A cell complex (X, K) is called to be a *G-cell complex* if:

(c) the orbit space X/G is a Hausdorff space,

(d) G acts cellularly, that is, $e \in K$ implies $ge \in K$ for every $g \in G$,

(e) every point x of a (open) cell e has the same isotropy subgroup, which is denoted by He, and in particular each boundary point is fixed by He,

(f) if g is not contained in He, then ge is disjoint from e,

(g) the topology of the subspace $G\bar{e}$ is the identification topology determined by the induced G-characteristic map,

$$G\sigma(=\mu\circ(id_G \times \sigma)) : G \times \Delta^n \rightarrow G\bar{e} \subset X .$$

REMARK. (e) is the key condition and is equivalent to each of the following:

(e′) if $gx=x$ for some $x \in e$, then $gy=y$ for each $y \in \bar{e}$,

(e″) for each $g \in G$ the g-stationary subspace $X^g=\{x \in X ;\ gx=x\}$ forms a subcomplex.

Take a cell e, then we get the distinguished subfamily $\{ge ;\ g \in G\}$ of K whose spanning domain is just the orbit of e. Moreover by the Hausdorff property of the orbit space, the orbit of the closure of e is closed in X and we get:

(1.3) *$\overline{Ge}=G\bar{e}$ as subsets of X.*

We account this subfamily Ge one G-cell and call it the *G-cell* represented by e. Each G-cell has its G-characteristic map, $G\sigma : G \times \Delta^n \rightarrow \overline{Ge} \subset X$. This is continuous G-equivariant map onto the closure of Ge. Then we get an induced continuous G-map $G\sigma : G/He \times \Delta \rightarrow \overline{Ge}$ whose restriction to the interior, $G\sigma | G/He \times (\Delta - \partial\Delta)$ is a homeomorphism onto Ge by (e), (f) and (g). Thus a G-cell complex is the G-equivariant version of a cell complex. That is, a G-cell complex is a pair of a G-space X and a family of G-cells GK which satisfies that X and X/G are Hausdorff spaces and

(a) $X=\cup Ge$ ($Ge \in GK$, disjoint union),

(b) each G-n-cell has its G-characteristic identification map, $G\sigma : G \times \Delta^n \rightarrow \overline{Ge} \subset X$ with

(b1) $G\sigma | G/He \times (\Delta - \partial\Delta)$ is a homeomorphism onto Ge and (b2) $\partial\overline{Ge}^n \subset X^{n-1}$.

The image of every G-cell of X by the natural projection $\pi: X \to X/G$ induces one cell of the same dimension in the orbit space. The characteristic map is just $\sigma/G(=\pi\circ\sigma): \Delta^n \to \bar{e}/G \subset X/G$. Collecting these cells we may induce a cell complex structure on X/G canonically. When we take this structure, we get $\pi^{-1}((X/G)^n) = X^n$ for any n.

DEFINITION (1.4) When a G-complex (X, K) has only finite G-cells, X is called to be a *G-finite complex*. The words *G-locally finite* and *G-countable* are used in the similar ways.

The G-cell complex closure of a subset S of X, denoted by $GX(S)$ is defined to be the G-orbit of $X(S)$ which turns out to be the smallest G-invariant subcomplex which contains S. (Afterwards we call a G-invariant subcomplex to be a G-subcomplex.) Now we can define a G-CW complex.

DEFINITION (1.5) A G-cell complex (X, K) is called to be a G-CW complex if it satisfies the following two conditions (G-C) and (G-W):

(G-C) *G-closure finiteness*, that is, the G-cell complex closure of each cell, $GX(e)$ is a G-finite complex. This condition is equivalent to the condition that the induced cell complex structure on the orbit space is closure finite. To say in another way the cell complex closure $X(e)$ of each cell has intersection with only finite G-cells.

(G-W) *G-weak topology*, that is, X has the identification topology with respect to the onto G-characteristic map of X, $G\sigma_X: \mathrm{E}_X = \amalg G \times \Delta_\lambda\ (\lambda \in \Lambda) \to X$, where E_X is topologized as a topological disjoint union. This topology coincides with the weak topology with respect to the closed covering $\{\overline{Ge}, Ge \in GK\}$ of X by (g). When X has this topology, the orbit space X/G has the weak topology with respect to the characteristic maps σ/G's.

Therefore we get

PROPOSITION (1.6) *If X is a G-CW complex, then the orbit space X/G has a canonical CW complex structure.*

The following lemma is easily deduced by the general argument about weak topologies.

LEMMA (1.7) *If we assume (G-C), (G-W) is equivalent to say that X has the weak topology with respect to its G-finite subcomplexes X_0's.*

When G is a compact, compact Lie or finite group, we may simplify the definition of a G-CW complex as follows.

The case when G is a compact group.

By the lemma (1.8) we may omit (c). When X is a 1st countable G-space, by the proposition (1.10) we only need to require that the induced cell complex structure on the orbit space is a CW complex structure instead of (G-C) and (G-W).

LEMMA (1.8) *Let G be a compact group. If a G-space X is a-Hausdorff space, then so is the orbit space X/G.*

PROOF: Take a $G\times G$-space $X\times X$, then the orbit space is $X/G\times X/G$ and the image of the diagonal subspace of $X\times X$ by the natural projection is the diagonal subspace of $X/G\times X/G$. Since the natural projection is a closed map because of the compactness of $G\times G$, the closedness of the latter is deduced from that of the former, which shows that the lemma is valid. q.e.d.

When G is a compact group, by the Hausdorff property of X we get the condition (g). Hence from the following lemma we deduce the proposition (1.10).

LEMMA (1.9) *Let G be a compact group and X be a G-cell complex which satisfies the 1st axiom of countability as a space. Then X has the weak topology with respect to the closed covering $\{\overline{Ge}; Ge\in GK\}$ of X if and only if the orbit space X/G has the weak topology with respect to the induced cell complex structure.*

PROPOSITION (1.10) *Let G be a compact group and X be a G-cell complex which satisfies the 1st axiom of countability. Then X is a G-*CW *complex if and only if the induced cell complex structure on the orbit space is a* CW *complex structure.*

PROOF OF LEMMA (1.9): We have already proved only-if-part without any assumption on G and X. We shall prove if-part by a contradiction. Let S be a subset of X and $S\cap\overline{Ge}$ be closed in $\overline{Ge}$ for every G-cell. We assume that S is not closed in X. Then there exists a convergent sequence $s_n\rightarrow s$ in X, where $s_n \in S$ and $s\notin S$. The subset $\{s_n, s\}=\{s_n : n \text{ (natural number)}\}\cup\{s\}$ is closed in X. Fix any cell e of X. Then $\{s_n, s\}\cap\overline{Ge}$ is closed in $\overline{Ge}$. Because $S\cap\overline{Ge}$ is closed in $\overline{Ge}$, $S\cap\{s_n, s\}\cap\overline{Ge}=\{s_n\}\cap\overline{Ge}$ is closed in $\overline{Ge}$. Put $t_n=\pi(s_n)$ and $t=\pi(s)$, where π is the natural projection: $X\rightarrow X/G$. Since π is a closed map because of the compactness of G, $\{t_n\}\cap\pi(\overline{Ge})$ is closed in $\pi(\overline{Ge})$. Because the orbit space has the weak topology with respect to these closed cells, $\pi(\overline{Ge})$'s, $\{t_n\}$ is closed in the orbit space. Therefore $t\in\{t_n\}$. Let n_0 be one of the indexes. For the subsequence $\{s_n ; n\geqq n_0\}$ the above argument is also valid. Therefore the subsequence $\{s_n\}\cap\pi^{-1}(t)$ of $\{s_n\}$ is cofinal to $\{s_n\}$, and converges to s. Let e be a unique cell which contains s, then $\pi^{-1}(t)\subset Ge\subset\overline{Ge}$ and $\{s_n\}\cap\pi^{-1}(t)$ converges to s in $\overline{Ge}$. But because $S\cap\pi^{-1}(t)\cap\overline{Ge}$ is closed in $\overline{Ge}$,

$$S\cap \overline{Ge}\cap \{s_n, s\}\cap \pi^{-1}(t)=\{s_n\}\cap \pi^{-1}(t)$$

is closed in $\overline{Ge}$. Since s is not contained in $\{s_n\}\cap\pi^{-1}(t)$, this is a contradiction.

q.e.d.

The case when G is a compact Lie group.

By the lemma (1.11) we may omit (f).

LEMMA (1.11) *When G is a compact Lie group, we may deduce* (f) *from* (d) *and* (e) *for a cell complex which is also a G-space.*

PROOF: Fix a cell e and put $H'e=\{g\in G;\ ge=e\}$. Then $H'e$ is a closed subgroup of G and contains He as a normal subgroup. The group $H'e/He$ operates on e freely because of (e). Identifying the boundary of e into one point, we may think the compact Lie group $H'e/He$ operates on a sphere of dimension greater than one with one fixed point and freely elsewhere. If the connected component of $H'e/He$ is not an identity group, it contains an 1-dimensional torus and hence contains a periodic transformation of a prime order. This contradicts with the theorem of P. A. Smith on a periodic transformation [3], which shows that $H'e/He$ is a finite group. Thus if $H'e/He$ is not an identity group, it contains an element which is not an identity and of finite order and hence contains a periodic transformation of a prime order. Then we may conclude that $H'e/He$ is an identity group and hence $H'e=He$, using the theorem of P. A. Smith in the above way. If $ge\cap e$ is not empty, then $ge=e$ because of (d) and hence $g\in H'e=He$. This shows (f) is deduced from (d) and (e). q.e.d.

The case when G is a finite group

REMARK (1.12) When G is a finite group, a G-CW complex itself must be a CW complex. Thus the definition of a G-CW complex is reduced to a G-space which has a CW complex structure satisfying (d) and (e″). The latter is the definition of a G-complex of G. Bredon in [1] for a finite group G. Hence a G-CW complex is a generalized concept of a G-complex of Bredon.

Last we shall show some examples of G-CW complexes.

EXAMPLES (1.13)

(1) *The natural* $SO(n)$*-action on* $SO(n+1)$. The orbit space is an n-dimensional sphere S^n. Giving the natural riemannian metric on S^n, we may lift every point except the south pole into the rotation along the unique minimum geodesic spanning from the north pole to it. Then this lifting has a unique continuous extension over the closed n-disk D^n into $SO(n+1)$, which represents a G-n-cell. A point over the south pole represents another G-0-cell. This G-CW complex consists of only two G-cells and is not a CW complex.

(2) *When G is a compact Lie group, any differentiable G-manifold has a G-CW complex structure.* This is proved in [2].

2. Elementary properties of a G-CW complex

In this section (X, K) or X denotes a fixed G-CW complex. We shall prove some properties of (X, K) following the corresponding properties of a CW complex established in [4] by J. H. C. Whitehead.

(A) *A G-map, $f: X \to Y$ into any G-space Y, is continuous provided $f|\bar{e}$ is continuous for each cell $e \in K$.*

PROOF: Let $f_e = f|\bar{e}$ be continuous, then the composition with the characteristic map $f_e \circ \sigma : \Delta \to Y$ is continuous. Moreover, the G-map $f \circ G\sigma : G \times \Delta \to Y$ is continuous because of the continuity of the G-action on X. Thus $f|\overline{Ge}$ is continuous with respect to the identification topology of $G\bar{e}$. Therefore, $f: X \to Y$ is continuous with respect to the G-weak topology of X. q.e.d.

It is easy to generalize (A) to the case when X is a closed or open G-subspace of a G-CW complex.

(B) *A G-subcomplex, (Y, L) of (X, K), is a closed G-subspace of X and the topology induced from X is the G-weak topology in (Y, L).*

PROOF: Let $S \subset Y$ be a subspace such that $S \cap Y_0$ is closed in Y_0 for each G-finite subcomplex Y_0 of Y. Since Y_0 is closed in X, $S \cap Y_0$ is a closed subspace of X. Let X_0 be any G-finite subcomplex of (X, K). Then $Y_0 = Y \cap X_0$ is a G-finite subcomplex of (Y, L) and $S \cap X_0 = S \cap Y \cap X_0 = S \cap Y_0$. Therefore, $S \cap X_0$ is closed in X_0, whence S is closed in X because of the lemma (1.7), and a fortiori in Y. Therefore, (Y, L) has the G-weak topology. Also, taking $S = Y$, it follows that Y is closed. q.e.d.

By definition a G-space X is called to be *G-connected*, *G-compact* and *G-paracompact* if and only if the orbit space X/G is connected, compact and paracompact respectively. We get the following properties (C), (D), (E) and (G) from the corresponding statements about a CW complex. Also (F) is deduced from the lemma (1.7).

(C) *If X is G-connected, then so is the skeleton X^n for each $n > 0$.*

(D) *If $S \subset X$ is a compact subspace, then $GX(S)$ is a G-finite subcomplex.*

(E) *If a G-cell complex (Y, L) and also Y^n for each $n \geqq 0$, all have the G-weak topology, then Y is a G-CW complex.*

Let $f: X \to Y$ be a G-map of X onto a G-closure finite complex Y, which has the identification topology determined by f. Further let the G-subcomplex $GY(f\bar{e})$ be G-finite for each cell $e \in K$.

(F) *Subject to these conditions Y is a G-CW complex.*

(G) *X is G-paracompact.*

Let (X, K) be a G'-cell complex and (Y, L) be a G''-cell complex. The product $G=G'\times G''$-cell complex of (X, K) and (Y, L) is defined to be the pair of the product G-space $X\times Y$ and a family of the product cell $K\times L=\{e=e'\times e''\,;\ e'\in K, e''\in L\}$. In fact because $\partial e=\partial e'\times e''\cup e'\times\partial e''$, $(g'\times g'')(e'\times e'')=g'e'\times g''e''$ and $He=He'\times He''$, this forms a G-cell complex. But even if (X, K) is a G'-CW complex and (Y, L) is also a G''-CW complex, the product complex $(X\times Y, K\times L)$ is not necessarily a G-CW complex. Some sufficient conditions are given in the following (H).

(H) *If one of the following conditions* (i), (ii) *with* (iia) *or* (ii) *with* (iib) *is satisfied, then the product complex $X\times Y$ is a G-CW complex*:

(i) *Y is a G''-finite G''-CW complex which is also a locally compact space.*

(ii) *G is a compact group and $X\times Y$ satisfies the 1st axiom of countability and also*

(iia) *Y is a G''-locally finite complex, or*

(iib) *both X and Y are G'-(G''-resp.) locally countable G'-(G''-resp.) CW complexes.*

PROOF: According to the proposition (1.10) the cases (ii) with (iia) and (ii) with (iib) are reduced to the cases of ordinary CW complexes. So we shall prove only the case (i) here. The weak topology of $X\times Y$ is the identification topology determined by the G-characteristic map $G\sigma_{X\times Y}: \mathrm{E}_{X\times Y}=\amalg G\times\Delta'\times\Delta''\to X\times Y$. But since Y is a G''-finite complex which satisfies (G''-W), the G-weak topology of $X\times Y$ is the identification topology determined by the product map $G\sigma_X\times\mathrm{id}: \mathrm{E}_X\times Y\to X\times Y$ by the lemma (1.7). Since $G'\sigma_X: \mathrm{E}_X\to X$ is an identification map, the product topology of $X\times Y$ coincides with the identification topology determined by the above map because of the locally compactness of Y. q.e.d.

(I) *A G-homotopy, $f_t: X\to Z$ into an arbitrary G-space Z is continuous provided $f_t|\bar{e}$ is continuous for each cell $e\in K$.*

This follows from (H) (i) for $Y=I$ with a trivial G-action and (A).

(J) (G-homotopy extension property) *Let $f_0: X\to Z$ be a given G-map of X into an arbitrary G-space Z. Let $g_t: Y\to Z$ be a G-homotopy of $g_0=f_0|Y$, where Y is a G-subcomplex of X. Then there is a G-homotopy $f_t: X\to Z$, such that $f_t|Y=g_t$.*

PROOF: Put $X_r=Y\cup X^r$ $(r\geqq -1, X_{-1}=Y)$ and assume that g_t has been extended to a G-homotopy, $f_t^{n-1}: X_{n-1}\to Z$ such that $f_0^{n-1}=f_0|X_{n-1}$ and $f_t^{n-1}|Y=g_t$ $(n\geqq 0)$. Take an n-cell e^n. The images of $f_t^{n-1}|\partial e$ and $f_0|\bar{e}$ are contained in Z^{H_e}

because of (e), where Z^{He} stands for the pointwise fixed subspace of Z by He. We may extend to the homotopy $f_{t,e}$ on $\bar{e}$ such that the image of $f_{t,e}|\bar{e}$ is contained in Z^{He}. For the points of Ge^n we define $f_{t,Ge}$ by $f_{t,Ge}(gx)=gf_{t,e}(x)\in gZ^{He}=Z^{Hge}$ $(g\in G, x\in e)$. By the properties (d), (e) and (f), g is determined unique up to He. That is, if $g'x'=gx$ for x', $x\in e$ then $x'=x$ and $g'=gh$ for some $h\in He$ so that $g'f_{t,e}(x')=gf_{t,e}(x)$ (since $f_{t,e}(x)\in Z^{He}$), which shows that this definition is valid. In this way the G-homotopy $f_t^{n-1}: X_{n-1}\to Z$ may be extended throughout $X_{n-1}\cup Ge^n$ for each G-n-cell $Ge\subset X_{n-1}-Y$. The extension on X_n is completed by taking one representing n-cell from each G-n-cell and following the procedure above. Starting with $f_t^{-1}=g_t$ it follows by an induction on n that there is a sequence of G-homotopies, $f_t^n: X_n\to Z$ $(n=0,1,\cdots)$, such that $f_0^n=f_0|X_n$ and $f_t^n|X_{n-1}=f_t^{n-1}$. It follows from (I) that G-homotopy $f_t: X\to Z$ which satisfies the requirements of (J) is given by $f_t|X_n=f_t^n$. Even if X is not G-finite, each inducting step has the only finite preceding steps because of the G-closure-finiteness.

q.e.d.

3. A sufficient condition for extending a G-map

By a similar argument as in the proof of (J) we get

PROPOSITION (3.1) *Let Z be a G-space and $Y\subset X$ be a G-CW complex pair the dimensions of whose cells do not exceed $N\leqq\infty$. If for each closed subgroup H of G the pointwise fixed subspace of Z by H, Z^H is not empty, arcwise connected and $\pi_n(Z^H)$ vanishes for $n<N+1\leqq\infty$, then any G-map of Y into Z can be extended equivariantly on X.*

The relative version of this proposition is stated as follows.

Let $\phi=Z_{-1}\subset Z_0\subset Z_1\cdots$ be a sequence of G-subspaces of a given G-space, Z, such that any G-map: $(G/H\times\Delta^n, G/H\times\partial\Delta^n)\to(Z, Z_{n-1})$, is G-homotopic rel. $G/H\times\partial\Delta^n$ to a G-map, $G/H\times\Delta^n\to Z_n$ $(n=0,1,2,\cdots)$, where H is any closed subgroup of G. Let $Y\subset X$ be a given G-subcomplex, which may be empty, and let $f_0: X\to Z$ be a G-map such that $f_0Y^n\subset Z_n$ for each $n=0,1,\cdots$.

(K) *There is a G-homotopy $f_t: X\to Z$ rel. Y, such that $f_1X^n\subset Z_n$ for each $n=0,1,\cdots$.*

PROOF: When $n=-1$, the statement is trivial. Therefore, we may assume there is a G-homotopy $f_t^{n-1}: X^{n-1}\to Z$ rel. $Y\cap X^{n-1}$ such that $f_0^{n-1}=f_0|X^{n-1}$ and $f_1^{n-1}X^{n-1}\subset Z_{n-1}$ as an induction hypothesis. Let e^n be an n-cell of X which is not contained in Y and has the G-characteristic map $G\sigma: G/He\times\Delta\to\overline{Ge}\subset X$. Define a G-map $F'_e: (G/He)\times\Delta^n\times 0\cup(G/He)\times\partial\Delta^n\times I\to Z$ by $F'_e(g,s,0)=f_0(G\sigma(g,s))$

($s\in\Delta^n$) and $F'_e(g, s, t)=f_t^{n-1}(G\sigma(g, s))$ ($s\in\partial\Delta^n$). By the induction hypothesis we get $F'_e(G/He\times\partial\Delta^n\times 1)=f_1^{n-1}(G\sigma(G/He\times\partial\Delta^n))\subset Z_{n-1}$. Then there is a G-extension of F'_e, $F_e: G/He\times\Delta^n\times I\to Z$, such that $F_e(G/He\times\Delta^n\times 1)\subset Z_n$. F_e induces a continuous G-map of $Ge\times I$ into Z which is an extension of f_t^{n-1}, therefore we get a G-homotopy $f_t^n: X^n\to Z$ rel. $X^n\cap Y$ such that $f_t^n|X^{n-1}=f_t^{n-1}$, $f_0^n=f|X^{n-1}$ and $f_1^n X^n \subset Z_n$. By the induction on n we get f_t^n for any n. Define $f_t: X\to Z$ by $f_t|X^n =f_t^n$, which satisfies the requirements of (K). q.e.d.

LEMMA (3.2) *Let $Z\supset C$ be a G-space pair and H be a closed subgroup of G. If C^H is not empty, and *) $\pi_n(Z^H, C^H)$ vanishes, then any G-map $Gf:(G/H\times\Delta^n, G/H\times\partial\Delta^n)\to(Z, C)$ is G-homotopic rel. $G/H\times\partial\Delta^n$ to a G-map $G/H\times\Delta^n\to C$.*

PROOF: If we restrict Gf in $H/H\times\Delta^n$, we get a (non G-equivariant) map $f: (\Delta^n, \partial\Delta^n)\to(Z^H, C^H)$. This map is homotopic rel. $\partial\Delta^n$ to a map $f_1:\Delta^n\to C^H$. Let $f_t:\Delta^n\to Z^H$ be this homotopy. Define $Gf_t: G/H\times\Delta^n\to Z$ by $Gf_t(g, s)=gf_t(s)$ ($g\in G$, $s\in\Delta^n$). This is well defined since $f_t(s)\in Z^H$. It is obvious $Gf_0=Gf$ and Gf_t gives a G-homotopy rel. $G/H\times\partial\Delta^n$ of Gf_0 to $Gf_1: G/H\times\Delta^n\to C$. q.e.d.

As a corollary of this lemma and (K) we get the following generalization of Proposition (3.1).

PROPOSITION (3.3) *Let $Z\supset C$ be a G-space pair and $Y\subset X$ be a G-CW complex pair the dimensions of whose cells do not exceed $N\leqq\infty$. If for each closed subgroup H of G (which appears as an isotropy subgroup of a G-CW complex X) C^H is not empty and *) $\pi_n(Z^H, C^H)$ vanishes for each $n<N+1\leqq\infty$, then any G-map: $(X, Y)\to(Z, C)$ is homotopic rel. Y to a G-map: $X\to C$.*

4. *G*-cellular approximation theorem

We shall prove the lemmas (4.1) and (4.2) by the simplicial approximation and then get the G-cellular approximation theorem (4.4) by the compactness of disks and (K).

LEMMA (4.1) *Let $f:(\Delta^k, \partial\Delta^k)\to(\Delta^n, \partial\Delta^n)$ be a continuous map between disks. Then f is homotopic rel. $f^{-1}(\partial\Delta^n)$ to a map of Δ^k into $\partial\Delta^n$, provided $k<n$.*

PROOF: We may assume that the given map f is surjective. Fix a standard piecewise linear structure on each of Δ^k and Δ^n. We can find a compact neigh-

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*) If X or Y is not arcwise connected, we mean by $\pi_k(X, Y)$ the homotopy classes of maps of $(\Delta^k, \partial\Delta^k)$ into (X, Y). Therefore $\pi_n(X)$ corresponds to a union of the n-th homotopy groups of all arcwise connected components of X, and in particular $\pi_0(X)$ stands for the number of arcwise connected components of X. Moreover $\pi_k(X, Y)=0$ implies that any continuous map $f:(\Delta^k, \partial\Delta^k)\to(X, Y)$ is homotopic rel. $\partial\Delta^k$ to a map $f':\Delta^k\to Y$.

borhood N of $f^{-1}(\partial\Delta^n)$ which at the same time is a piecewise linear submanifold with boundary and whose image is in the open subspace $H\Delta^n=\Delta^n-(1/2)\Delta^n$ of Δ^n. Approximating f on the boundary ∂N of the neighborhood to a simplicial map, we get a continuous map

$$F:\partial N\times[-1,1]\to H\Delta^n$$

with

$$F|\partial N\times(-1,1)=f|N,\ F(x,-t)=F(x,t)\ \text{and}\ F|\partial N\times 0=\text{a simplicial map.}$$

This gives a continuous map

$$f':(\Delta^k)'=(\Delta^k-\mathring{N})\cup\partial N\times[-1,1]\cup N\to\Delta^n$$

with

$$f'|\partial N\times[-1,1]=F,\ f'|(\Delta^k-\mathring{N})=f|(\Delta^k-\mathring{N})\ \text{and}\ f'|N=f|N.$$

Approximate f' on $(\Delta^k-\mathring{N})\cup\partial N\times[-1,0]$ rel. $\partial N\times 0$ to a simplicial map, and we obtain a third continuous map $f'':(\Delta^k)'\to\Delta^n$ which is not a surjection and then is homotopic rel. $f^{-1}(\partial\Delta^n)$ to a map of $(\Delta^k)'$ into Δ^n. Since there is a homotopy rel. $f^{-1}(\partial\Delta^n)$ equivalence map: $(\Delta^k,\partial\Delta^k)\to((\Delta^k)',\partial(\Delta^k)')$ which transforms f' to a map homotopic to f, this completes the proof. q.e.d.

LEMMA (4.2) *Let $Z=G/H'\times\Delta^n$ and $C=G/H'\times\partial\Delta^n$. Then any continuous map $f:(\Delta^k,\partial\Delta^k)\to(Z^H,C^H)$ is homotopic rel. $f^{-1}(C^H)$ to a map of Δ^k into C^H for $k<n$ and any closed subgroup H of G.*

PROOF: Composite f with the projection: $Z^H=(G/H')^H\times\Delta^n\to\Delta^n$. Then we obtain a continuous map $f':(\Delta^k,\partial\Delta^k)\to(\Delta^n,\partial\Delta^n)$ which is homotopic rel. $(f')^{-1}(\partial\Delta^n)$ to a map of Δ^k into $\partial\Delta^n$ by the previous lemma. This gives a homotopy rel. $f^{-1}(C^H)$ of f to a map of Δ^k into C^H. q.e.d.

PROPOSITION (4.3) *Let X be a G-CW complex. Then ${}^*\pi_k(X^H,(X^n)^H)=0$, provided $k\leq n$.*

PROOF: Let $f:(\Delta^k,\partial\Delta^k)\to(X^H,(X^n)^H)$ be a continuous map. Since $f(\Delta^k)$ is compact, $f(\Delta^k)$ intersects with only finite G-cells in X. Let $Ge_1^m, Ge_2^m,\cdots,Ge_k^m$ be G-m-cells of the highest dimension which intersect with $f(\Delta^k)$. Then we can consider f to be a map of $(\Delta^k,\partial\Delta^k)$ into $(Z^H,(X^n)^H)$ where $Z=Ge_1^m\cup Ge_2^m\cup\cdots\cup Ge_k^m\cup X^{m-1}$. Put $C=Ge_2^m\cup\cdots\cup Ge_k^m\cup X^{m-1}$. Since the difference between Z and C is only one G-cell Ge_1^m, by the technique of the previous lemma we obtain a homotopy rel. $f^{-1}(C^H)$ of f to a map $f':\Delta^k\to C^H$, provided $k<m$. Repeating this procedure we finally get a homotopy rel. $\partial\Delta^k$ of f to a map $f'':\Delta^k\to(X^n)^H$.

q.e.d.

THEOREM (4.4) (G-cellular approximation theorem) *Let $f: X \to Y$ be a G-map between G-CW complexes. Then f is G-homotopic to a G-map $f': X \to Y$ such that $f'(X^n) \subset Y^n$ for any n.*

PROOF: We only need to combine the proposition (4.3) with the proposition (3.3) and (K). q.e.d.

5. Theorem of J.H.C. Whitehead

We shall prove the theorem (5.3) which is a generalization to the equivariant case of the theorem of J.H.C. Whitehead.

LEMMA (5.1) *Let $\varphi: C \to Z$ be a G-map between two G-spaces, and $X \supset Y$ be a G-CW complex pair the dimensions of whose cells do not exceed $\aleph \leqq \infty$. If for each closed subgroup H of G (which appears as an isotropy subgroup of a G-CW complex X) C^H and Z^H are non-empty and the induced map $\varphi_*: {}^{*)}\pi_n(C^H) \to {}^{*)}\pi_n(Z^H)$ is bijective for $n < \aleph$ and surjective for $n = \aleph$, then for any G-map pair $g: X \to Z$, $f': Y \to C$ with $g | Y = \varphi \circ f'$, there exists a G-map $f: X \to C$ such that $f | Y = f'$ and $\varphi \circ f$ is G-homotopic rel. C to g.*

PROOF: Let M be a mapping cylinder of $\varphi: C \to Z$. Then M^H coincides with the mapping cylinder of $\varphi^H: C^H \to Z^H$ for each closed subgroup H of G. Thus ${}^{*)}\pi_n(M^H, C^H)$ vanishes for $n < \aleph + 1$ (For $n = 0$, the assumption means that every arcwise connected component of M^H contains only one arcwise connected component of C^H. Hence for $n \geqq 1$, we can use the exact sequence in the Hurwicz homotopy theory.). Therefore, we may deduce this lemma from the proposition (3.3). q.e.d.

THEOREM (5.2) *Let $\varphi: X \to Y$ be a G-map between two G-spaces. If each of X^H and Y^H is not empty for each closed subgroup H of G, then the followings are equivalent:*

(1) *for each closed subgroup H of G, the induced map*

$$\varphi_*: {}^{*)}\pi_n(X^H) \to {}^{*)}\pi_n(Y^H)$$

is bijective for $1 \leqq n < \aleph$ and surjective for $n = \aleph$.

(2) *the induced map*

$$\varphi_*: [K;\ X]_G \to [K;\ Y]_G$$

is bijective for $\dim K < \aleph$ *and surjective for* $\dim K = \aleph$ *for any G-CW complex K, where $[\cdot\ ;\ \cdot]_G$ stands for the set of G-homotopy classes of G-maps.*

PROOF: (1) implies (2) because of the lemma (5.1). If we take $K = G/H \times (\Delta^n - \partial\Delta^n)$, it is obvious that (2) implies (1). q.e.d.

As a corollary of this theorem we get

THEOREM (5.3) *Let $\varphi:(X, A)\to(Y, B)$ be a G-map between two G-CW complex pairs. If X^H, A^H, Y^H and B^H are non-empty and the induced maps*

$$\varphi_*: {}^{*)}\pi_n(X^H)\to{}^{*)}\pi_n(Y^H)$$

and

$$\varphi_*: {}^{*)}\pi_n(A^H)\to{}^{*)}\pi_n(B^H)$$

are bijective for $1\leqq n\leqq\max(\dim X, \dim Y)$ and each closed subgroup H which appears as an isotropy subgroup in X or Y, then

$\varphi:(X, A)\to(Y, B)$ *is a G-homotopy equivalence map.*

PROOF: Put $K=B$. Then $\varphi|A$ has a G-homotopy left inverse ψ because the induced map $\varphi_*:[B; A]_G\to[B, B]_G$ is an isomorphism. By the G-homotopy extension property we get a G-map $\psi':Y\to Y$ which is G-homotopic to identity and satisfies $\psi'|B=\psi$. Then by the lemma (5.1) we get a G-map $\psi'':Y\to X$ such that $\psi''B=\psi$ and $\varphi\circ\psi''=\psi'$ is G-homotopic to identity of Y. That is, ψ'' is a G-homotopy left inverse of φ. Also we get a G-homotopy left inverse of ψ'' and hence by an algebraic argument we can show that (ψ'', ψ) is a G-homotopy inverse of $(\varphi, \varphi|B)$. q.e.d.

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