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FRÉCHET MANIFOLDS AS DIFFEOLOGIC SPACES

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In recent years numerous generalizations of the concept of a manifold have evolved. The concept of Souriau diffeologic space (see [9] and [10]) seems to be the most general among these. In [3], there was given a scheme which enables to define various concepts of differential geometry for a wide generalization of the notion of a structure on manifold. For diffeologic spaces, which are called *sets with smooth structure*, such a scheme gives a possibility to introduce the concepts of a tangent bundle, of vector and tensor fields, of a differential form, etc.

In the present article we consider a natural diffeologic structure of a C^∞-manifold modeled on Fréchet space (a manifold of that kind is called a Fréchet manifold [7]). We demonstrate that the diffeologic morphism between Fréchet manifolds is a morphism between manifolds, and the diffeology of Fréchet manifold M determines uniquely its manifold structure. In addition, we find the tangent bundle of diffeologic space M, tensor and exterior degrees of this bundle. This enables us to define tensor fields and differential forms on Fréchet manifold in a manner similar to the case of finite-dimensional manifolds. Finally, this gives the possibility to include the theory (see the review of the theory in [8])of regular Fréchet-Lie groups (this theory contains, in particular, the theory of all the classical primitive infinite-dimensional groups of diffeomorphisms of closed manifolds) into the theory of Souriau diffeologic groups. Henceforth we suppose all the manifolds, their smooth mappings, vector and tensor fields, etc. to be differentiable of class C^∞. In the article we use the concept of an integral of a function with values in Fréchet space along a segment of the real line; we apply also the properties of these integrals (see [2] and [7]).

§ 1. Diffeologic spaces

Let us recall basic notions of the theory of diffeologic spaces (see [3], [9] and [10]), restricting our consideration to the class C^∞.

Let us denote by S the category, whose all objects are open sets in the spaces $\mathbb{R}^n$ (n=0,1,...), and whose morphisms are arbitrary smooth mappings. For an arbitrary set Z we denote by $S(Z)$ the set of all the mappings of the sets $U\in\mathrm{Ob}S$ into Z, and by $\alpha(f)$ we mean the domain of definition of the mapping $f\in S(Z)$.

Definition 1. A subset Φ of the set $S(Z)$ is called a *diffeology* if it satisfies the following properties:

1) any constant mapping from $S(Z)$ is in Φ;

2) if $f\in\Phi$, $V\in\mathrm{Ob}S$, $g\in\mathrm{Hom}_S(V,\alpha(f))$, then $f\circ g\in\Phi$;

3) if $f\in S(Z)$ and $\{U_i\}$ is an open covering of $\alpha(f)$ such that $f|_{U_i}$ is in Φ for all the i, then $f\in\Phi$.

A set Z, being endowed with a diffeology Φ, is called a *diffeologic space* or, briefly, *d-space.*

Definition 2. Let (Z_i,Φ_i) (i=1,2) be d-spaces. A mapping $h:Z_1\to Z_2$ is called a *diffeologic morphism*, or a *d-morphism*, if for any $f\in\Phi_1$ we have $h\circ f\in\Phi_2$.

Thus, d-spaces form a category D. Let us give several examples of d-spaces.

Example 1. If X is a finite-dimensional manifold, the set of all the smooth mappings contained by $S(X)$ is a diffeology on X. Any smooth mapping between finite dimensional manifolds is a d-morphism.

Example 2. Let E be a Fréchet space, i.e., a metrizable complete linear space whose topology is determined by a countable system of seminorms. If W is an open subset of E (or, in general, Fréchet manifold M, see [7]), then the set of all the smooth mappings which are in $S(W)$ (in $S(M)$, respectively) is the diffeology $\Phi(W)$ ($\Phi(M)$, respectively) on W (M).

Let Φ_1 and Φ_2 be diffeologies on a set Z. We say that Φ_1 *is weaker than* Φ_2 if $\Phi_1 \subset \Phi_2$. It is clear that any set of mappings of a family of finite-dimensional manifolds (or even of d-spaces) into Z generates a diffeology on Z; namely, the weakest diffeology among all the diffeologies, for which all the mappings of this set are being d-morphisms.

For d-space (Z,Φ) let us define the category $C(\Phi)$, whose set of objects is Φ, and, for $f_1,f_2 \in \Phi$, by a morphism between f_1 and f_2 let us understand the mapping $g \in \mathrm{Hom}_S$ such that $f_1 = f_2 \circ g$. We obtain the covariant functor $I_\Phi : C(\Phi) \longrightarrow S$, setting $I_\Phi(f) = \alpha(f)$ and $I_\Phi(g) = g$, where $g : f_1 \longrightarrow f_2$ ($f, f_1, f_2 \in \Phi$).

Now let F be a covariant (contravariant) functor between S and the small category of sets Set. We extend F on the category D, by setting $F(Z,\Phi) = \varinjlim F \circ I_\Phi$ for d-space (Z,Φ) ($F(Z,\Phi) = \varprojlim F \circ I_\Phi$, respectively) and by defining naturally values of F on d-morphisms, in accordance with the definitions of the direct limit and of the inverse one. It is clear that the morphisms of functors between S and Set can be easily extended to morphisms of corresponding functors between D and Set.

In particular, when F is a covariant functor between S and the category of finite dimensional manifolds, we shall use the construction pointed above by clearing off the structure of manifold. Then for a d-space (Z,Φ) the set of natural mappings $F \circ I_\Phi(f) \longrightarrow$ $\longrightarrow F(Z,\Phi) = \varinjlim F \circ I_\Phi$ ($f \in \Phi$) generates a diffeology on $F(Z,\Phi)$. For example, in the capacity of F we can take functors which send $U \in \mathrm{Ob}S$ to the total space of tangent bundle TU, or to its p-th tensor degree T^pU, or to its p-th exterior degree Λ^pTU, etc. The natural projections $TU \longrightarrow U$, $T^pU \longrightarrow U$ and $\Lambda^pTU \longrightarrow U$ induce d-morphisms $T(Z,\Phi) \longrightarrow Z$, $T^p(Z,\Phi) \longrightarrow Z$ and $\Lambda^pT(Z,\Phi) \longrightarrow Z$, call them projections, which determine the corresponding diffeologic bundles. Notice that, in contrast to the case of manifolds, the fibres of these bundles, in general, do not obtain a structure of a linear space. A point ξ of a fibre $T(Z,\Phi)$ at a point $z \in Z$ is called a *tangent vector* at z.

If we take as F the covariant functor τ_p (Ω^p), which sends $U \in \mathrm{Ob}S$ to the linear space of tensor fields of type $(0,p)$ (differential p-forms) on U, then for the d-space (Z,Φ) we obtain the set $\tau_p(Z,\Phi)$ ($\Omega^p(Z,\Phi)$). Due to definition of the inverse limit, these sets are linear spaces, and their points are called *tensor fields of type* $(0,p)$ or, respectively, *differential p-forms on* (Z,Φ).

It is clear that exterior differential and its main properties can be extended to differential p-forms on (Z,Φ).

§ 2. The tangent bundle of an Fréchet space, its tensor and exterior degree

Let E be a Fréchet space whose topology is determined by a countable system of seminorms $\| \; \|_n$ ($n=1,2,\ldots$). We can always set that $\|x\|_n \leq \|x\|_{n+1}$ ($n=1,2,\ldots$), this is assumed in the sequel. Being W an open subset in E, it have been demonstrated in §1, that there is a natural diffeology $\Phi(W)$ on W such that for $W=E$ it is compatible with the linear structure on E.

Let us prove the following simple generalization of Hain's lemma (see [6]).

LEMMA 1. *If $\{x_n\}$ is a sequence of points of the space E with $\|x_n\|_n=O(\exp(-2^n))$ for $n\to\infty$, then there is a smooth mapping $f:\mathbb{R}\to E$ such that $f(1-2^{-n})=x_n$ and $f(1)=0$.*

Proof. We select a C^∞-function $\theta:\mathbb{R}\to\mathbb{R}$ which satisfies the following conditions: $\theta(t)=0$ when $t\le 0$, and $\theta(t)=1$ when $t\ge 1$. We set $t_n=1-2^{-n}$ and define the function $s_n:[t_{n-1},t_n]\to[0,1]$ by the equality $s_n(t)=2^n(t-t_{n-1})$ $(n=1,2,\ldots)$. The desired mapping f is defined in the following manner:

$$f(t)=\begin{cases} x_0, & \text{if } t\le 0;\\ \theta\circ s_n(t)x_n+(1-\theta\circ s_n(t))x_{n-1}, & \text{if } t_{n-1}\le t\le t_n;\\ 0, & \text{if } t\ge 1.\end{cases}$$

It is clear that $f(t_n)=x_n$ and f is a smooth mapping at the intervals $t<0$ and $t>0$. One can see easily that for $t\in[t_{n-1},t_n]$ and for each p there holds $\|f^{(p)}(t)\|_n=O(2^{pn}\exp(-2^n))$ for $n\to\infty$. By induction, it follows that $f^{(p)}(0)$ does exist, being equal to zero.

From Lemma 1 we obtain easily the following consequences.

Corollary 1. Let W be an open set of Fréchet space E_1, M be a Fréchet manifold modeled on a Fréchet space E_2 and $g:W\to M$ be a d-morphism. Then g is a continuous mapping.

Corollary 2. If on a linear space E there are given two structures of Fréchet space, which determine the same diffeology, then these structures are the same; i.e., the topology of Fréchet space is determined uniquely by the diffeology of this space.

Let us denote by L_p the algebraic p-th tensor degree of a linear space L. Assume $f\in\Phi(W)$, and let $U=\alpha(f)$ be an open set of $\mathbb{R}^n$. Let also $f'_p(x):\mathbb{R}^n_p\to L_p$ be the p-th tensor degree of the derivative $f'(x)$ of the mapping f at the point $x\in U$. We consider the mapping $f'_p:T^pU=U\times\mathbb{R}^n_p\to W\times E_p$, given by the condition $f'_p(x,\xi_p)=(f(x),f'_p(x)(\xi_p))$, where $x\in U$ and $\xi_p\in\mathbb{R}^n_p$. It is easy to see that all the f'_p $(f\in\Phi(W))$ are morphisms between the vector bundles of class C^∞, which are compatible with morphisms of the category $C(\Phi(W))$.

THEOREM 1. *The set $W\times E_p$ and the set of mappings f'_p $(f\in\Phi(W))$ determine $T^p(W,\Phi(W))$.*

Proof. It is sufficient to show that the canonical mapping $i_p:T^p(W,\Phi(W))\to W\times E_p$, being such that for any $f\in\Phi(W)$ we have that $f'_p=i_p\circ f_{*,p}$, where $f_{*,p}:T^pU\to T^p(W,\Phi(W))$, is a natural mapping determined by the direct $\varinjlim T^p\circ I_\Phi$, is bijective.

Let $\xi_p=\sum_{1\le i\le m}\xi^i_1\otimes\ldots\otimes\xi^i_p\in E_p$, where $\xi^i_1,\ldots,\xi^i_p\in E$. It is clear that $\xi_p\in L_p$, where L is the subspace of E generated by the vectors $\xi^i_1,\ldots,\xi^i_p$ $(i=1,\ldots,m)$. Selecting a basis of L, we identify L with the space $\mathbb{R}^N$, where $N=\dim L$. The inclusion $L\subset E$ determines a linear mapping $f:\mathbb{R}^N\to E$. If $x\in W$ and τ_x is the translation of E onto a vector x, then $g=\tau_x\circ f$ is an affine mapping between $\mathbb{R}^N$ and E. It is clear, that we may identify ξ_p with a point ξ'_p in such a fibre of a bundle $T^p\mathbb{R}^N$ at a point 0, and $g'_p(0,\xi_p)=(x,\xi_p)$. It follows that the mapping i_p is surjective. There occurs the following lemma.

LEMMA 2. *Let $f\in\Phi(W)$, let $U=\alpha(f)$ be a neighborhood of zero in $\mathbb{R}^n$, $f(0)=0$ and $f'(0)=0$. Then there exists a mapping b between U and the space of symmetric bilinear forms on $\mathbb{R}^n$*

with values in E, such that for any vector $a_1,a_2\in\mathbb{R}^n$ the function $b(x)(a_1,a_2)$ $(x\in U)$ is a smooth function on U with values in E and $f(x)=b(x)(x,x)$ $(x\in U)$.

Indeed, we can take

$$\int_0^1 ds\int_0^1 sf''(stx)dt$$

as $b(x)$.

Now let $\zeta\in T^p(W,\Phi(W))$. It is clear that there exists a mapping $f\in\Phi(W)$ such that $U=\alpha(f)$ is a neighborhood of zero in $\mathbb{R}^n$, and there exists a tensor $\zeta'\in\mathbb{R}^n_p$ such that $f_{*,p}(0,\zeta')=\zeta$. Consider the linear mapping $f'(0):\mathbb{R}^n\longrightarrow E$ and the affine mapping $a=\tau_{f(x)}\circ f'(0):\mathbb{R}^n\longrightarrow E$, where $\tau_{f(x)}$ is the shift of E on a vector $f(x)$. We obviously have $f'_p(0,\zeta')=a'_p(0,\zeta')$. By applying Lemma 2 to the mapping $f-a$, we obtain that $f(x)=a(x)+b(x)(x,x)$ $(x\in U)$. Let e_i $(i=1,\dots,n)$ be the standard basis of $\mathbb{R}^n$. Then $b(x)(x,x)=\sum_{1\le i,j\le n} b(x)(e_i,e_j)x_ix_j$, where x_i are the coordinates of x. Take the mapping $g:U\times\mathbb{R}^N\longrightarrow E$, where $N=\frac{n(n+1)}{2}$, which is given by the equality $g(x,x_{ij})=a(x)+\sum_{1\le i,j\le n} b(x)(e_i,e_j)x_{ij}$, where $x_{ij}=x_{ji}$ are the coordinates in $\mathbb{R}^N$, and take the mappings $h_1,h_2:U\longrightarrow U\times\mathbb{R}^N$, which are set by the conditions $h_1(x)=(x,0)$ and $h_2(x)=(x,x_ix_j)$, where x_i are the coordinates of $x\in U$. Evidently, $g\circ h_1=a$ and $g\circ h_2=f$. One can see easily that $a_{*,p}(0,\zeta')=\zeta$, because the images of the tensor ζ' at the point $0\in U$ with respect to h_1 and h_2 coincide. This implies the injectivity of τ_p.

Now let again $f\in\Phi(W)$, $\alpha(f)=U$, and $\tilde f'_p(x):\Lambda^p\mathbb{R}^n\longrightarrow\Lambda^pE$ be the p-th exterior degree of the mapping $f'(x)$ at a point $x\in U$ and $\tilde f':\Lambda^pTU=U\times\Lambda^p\mathbb{R}^n\longrightarrow W\times\Lambda^pE$ be given by the condition $\tilde f'(x,\eta_p)=(f(x),\tilde f'(x)(\eta_p))$ $(x\in U,\ \eta_p\in\Lambda^p\mathbb{R}^n)$.

In a manner similar to the Theorem 1 we prove

THEOREM 2. *The set $W\times\Lambda^pE$ and the set of mappings f' $(f\in\Phi(W))$ determine $\Lambda^pT(W,\Phi(W))$.*

Thus, in contrast to arbitrary d-space, the tangent bundle of d-space $(W,\Phi(W))$, the tensor and exterior degrees of this tangent bundle are vector bundles. It is also clear that for fibres of these bundles there are determined the tensor and exterior products, which have the routine properties.

THEOREM 3. *The diffeology of the tangent bundle $T(W,\Phi(W))=W\times E$ coincides with the diffeology $\Phi(W\times E)$ on the open set $W\times E$ of the Fréchet space $E\times E$.*

Proof. It is sufficient to show that the diffeology $\Phi(W\times E)$ is weaker than the diffeology of d-space $T(W,\Phi(W))$.

Let $f\in\Phi(W\times E)$ and $U=\alpha(f)$. Consider a mapping g of the class C^∞ between a neighborhood of subset $\{0\}\times U$ in $\mathbb{R}\times U$ and W, which are given by the equality $g(t,x)=f_1(x)+tf_2(x)$ $(t\in\mathbb{R}, x\in U)$, where $f_1=p_1\circ f$, $f_2=p_2\circ f$ and p_1,p_2 are the projections of $W\times E$ onto the factors W and E, respectively. If $\xi\in T_xU$, then $p_1\circ g_*((\frac{d}{dt})_0,\xi)=f_1(x)$, $p_2\circ g_*((\frac{d}{dt})_0,\xi)=p_2\circ f'_1(x)(\xi)+f_2(x)$. Let us determine a smooth mapping $s:U\longrightarrow T(\mathbb{R}\times U)$ by the condition $s(x)=((\frac{d}{dt})_0,0_x)$, where $x\in U$ and 0_x is the zero of T_xU. Then $p_1\circ g_*\circ s=f_1$ and $p_2\circ g_*\circ s=f$, i.e., $g_*\circ s=f$, this means that f belongs to the diffeology of $T(W,\Phi(W))$.

From Theorems 1 and 2 and from Corollary 2 (see §1) it follows that fibres of the bundle TW have a structure of Fréchet space, which is isomorphic to the structure of Fréchet space E.

§ 3. Morphisms between open sets of Fréchet space

Let E_i $(i=1,2)$ be a Fréchet space, W be an open set of E_1 and $f:W\to E_2$ be a d-morphism between $(W,\Phi(W))$ and $(E_2,\Phi(E_2))$. For any $p=1,2,\ldots$ we consider the mapping $f_p:W\times E_1^p\to E_2$, which is defined as follows:

$$f_p(x,h_1,\ldots,h_p) = \left\{\frac{d}{ds_1}\cdots\frac{d}{ds_p}f(x+s_1h_1+\ldots+s_ph_p)\right\}_{s_1=\ldots=s_p=0}$$

where $x\in W$, $h_1,\ldots,h_p\in E_1$, $s_1,\ldots,s_p\in\mathbb{R}$; we set $f_0=f$.

LEMMA 3. *The mapping f_p is a d-morphism and for any fixed x the function $f_p(x,h_1,\ldots,h_p)$ is a symmetric continuous p-linear form on E_1 with values in E_2.*

Proof. It is clear that f_p is a d-morphism, which is symmetric with respect to the variables $h_1,\ldots,h_p$. The continuity of $f_p(x,h_1,\ldots,h_p)$ follows from the theorem 1, and p-linearity follows from the definition of f_p.

From Lemma 3 it follows that the statement "f is a d-morphism" is equivalent to the statement "f satisfies the weakest definition among the definitions of C^∞ differentiability of mapping between open sets in Fréchet space" (see, for example, [4]), and, by this definition, the p-th derivative D^pf is determined by the equality $D^pf(x)(h_1,\ldots,h_p) = f_p(x,h_1,\ldots,h_p)$ $(x\in W,\ h_1,\ldots,h_p\in E_1)$. This definition of C^∞-differentiability is claimed in literature to be equivalent to all other definitions of differentiability in Fréchet spaces; however, the explicit proof is not presented anywhere. Therefore, we shall prove that the definition of differentiability implies so-called "b-differentiability" of class C^∞, which has implied all the others (cf. [8]-[10]).

Let us denote by $L_p(E_1,E_2)$ the space of all symmetric continuous p-linear forms on E_1 with values in E_2 and by $f^{(p)}$ – the mapping between W and $L_p(E_1,E_2)$, given by the condition $f^{(p)}(x)(h_1,\ldots,h_p)=f_p(x,h_1,\ldots,h_p)$ $(x\in W,\ h_1,\ldots,h_p\in E_1)$.

LEMMA 4. *If $f^{(p)}(x_0)=0$ $(x_0\in W)$, then there exists a d-morphism $g:(-\varepsilon,\varepsilon)\times E_1^{p+1}\to E_2$, where ε is a positive number such that $f_p(x_0+th,h_1,\ldots,h_p) = tg(t,h,h_1,\ldots,h_p)$ $(|t|<\varepsilon,\ h,h_1,\ldots,h_p\in E_1)$.*

Proof. By integrating the equality

$$\frac{d}{ds}f_p(x_0+sth,h_1,\ldots,h_p) = \left\{\frac{d}{ds_0}\cdots\frac{d}{ds_p}f(x_0+(s+s_0)th+s_1h_1+\ldots+s_ph_p)\right\}_{s_0=\ldots=s_p=0} = tf_{p+1}(x_0+sth,h,h_1,\ldots,h_p),$$

we obtain that $f_p(x_0+th,h_1,\ldots,h_p)=t\int_0^1 f_{p+1}(x_0+sth,h,h_1,\ldots,h_p)ds$. Now we have only to set $g(t,h,h_1,\ldots,h_p)=\int_0^1 f_{p+1}(x_0+sth,h,h_1,\ldots,h_p)ds$ and to notice that g is a d-morphism.

On $L_p(E_1,E_2)$ let us consider the topology of uniform convergence on bounded sets. Then there occurs

LEMMA 5. *The mapping $f^{(p)}:W\to L_p(E_1,E_2)$ is continuous for any $f\in\Phi(W)$.*

Proof. Let $x_0\in W$ and a mapping $k:W\to E_2$ be determined by the equality $k(x) = f(x) - f^{(p)}(x_0)(h,\ldots,h)$, where $h=x-x_0$. It is clear that $k^{(p)}(x_0)=0$ and, according to Lemma 4, there exists g such that $k_p(x_0+th,h_1,\ldots,h_p)=tg(t,h,h_1,\ldots,h_p)$.

Applying Theorem 1 to g, we obtain that for any positive integer n there exist positive numbers M_n, $\delta=\delta(n)<1$ and a positive integer $\alpha(n)\geq n$, all such that for $|t|, \|h\|_{\alpha(n)}$, $\|h_1\|_{\alpha(n)},\dots,\|h_p\|_{\alpha(n)} < \delta$ the inequality holds $\|g(t,h,h_1,\dots,h_p)\|_n < M_n$. Assume $\|h\|_{\alpha(n)} = \|h_1\|_{\alpha(n)}=\dots=\|h_p\|_{\alpha(n)}=1$ and $|t|<\delta^2$. Then for $\delta'>0$, which satisfies the inequality $|t|<\delta'\delta<\delta^2$, we have

$$\|k_p(x_0+th,h_1,\dots,h_p)\|_n = (\delta')^{-p}\|k_p(x_0+th,\delta' h_1,\dots,\delta' h_p)\|_n =$$
$$= |t|(\delta')^{-p-1}\|g(t/\delta',\delta' h,\delta' h_1,\dots,\delta' h_p)\|_n < M_n|t|(\delta')^{-p-1}.$$

Tending δ' to δ, we obtain

$$\|k_p(x_0+h')(h_1,\dots,h_p)\|_n\leq M_n\|h'\|_{\alpha(n)}\delta^{-p-1},$$

where $h'=th$.

Let B be a bounded set in E_1^p. Then there exists a sequence of positive numbers C_n $(n=1,2,\dots)$ such that for any $(h_1,\dots,h_p)\in B$ the inequalities $\|h_i\|_n\leq C_n$ hold for all i and n. Consequently,

$$\|k^{(p)}(x_0+h')(h_1,\dots,h_p)\|_n \leq M_nC^p_{\alpha(n)}\delta^{-p-1}\|h'\|_{\alpha(n)}$$

for all $h_1,\dots,h_p$ which satisfy the inequalities $\|h_i\|_{\alpha(n)}\leq C_{\alpha(n)}$; hence, the continuity of $k^{(p)}$ follows, therefore $f^{(p)}$ is continuous at the point x_0.

THEOREM 4. *If E_i is a Fréchet space $(i=1,2)$, W is an open set of E_1 and $f:W\longrightarrow E_2$ is a d-morphism, then f is a b-differentiable mapping of class C^∞.*

The proof follows easily from Lemma 5 by induction on p, because the canonical mapping $L_p(E_1,E_2)\longrightarrow L_1(E_1,L_{p-1}(E_1,E_2))$ is continuous and the continuous weak b-differentiability implies the b-differentiability (see [1]).

Notice that for Banach spaces this Theorem had been proved in [6].

§ 4. Fréchet manifolds

By Fréchet manifolds we mean a C^∞ manifold modeled on Fréchet space, which is defined in the standard manner by means of any definition of differentiability, because all of them are equivalent, as it has been demonstrated in §3.

THEOREM 5. *If M_i $(i=1,2)$ is a Fréchet manifold modeled on Fréchet space E_i and $f:M_1\longrightarrow M_2$ is a d-morphism, then f is a smooth mapping between manifolds.*

Proof. Assume $x\in M_1$ and let U_1 be an open neighborhood of the point x and let $k_1:U_1\longrightarrow E_1$ be the coordinate mapping of the manifold M_1. Following Theorem 1, the mapping $f\circ k_1^{-1}$: $k_1(U_1)\longrightarrow M_2$ is continuous and, therefore, for the coordinate mapping $k_2:U_2\longrightarrow E_2$ of the manifold M_2, which is set in a neighborhood U_2 of $f(x)$, there exists an open neighborhood U of x, which is lying in U_1, such that $f(U)\subset U_2$. Now it suffices to prove that the mapping $k_2\circ f\circ k_1^{-1}$, being restricted on U, is smooth; this fact follows from Theorem 5, being this mapping a d-morphism between the open set $k_1(U)\subset E_1$ and E_2.

Corollary 1. A structure of Fréchet manifold on M is uniquely determined by its diffeology.

Indeed, if there are two structures of Fréchet manifolds on the set M, which determine the same diffeology, then, according to Theorem 5, the identical mapping of M is a diffeomorphism of these structures.

Let us demonstrate how the diffeology $\Phi(W)$ determines coordinate mappings on M. Let $k_\alpha:U_\alpha\longrightarrow W_\alpha$ $(\alpha\in A)$, where W_α is an open set of Fréchet space, A is a family of coordinate mappings of the atlas on M. The topology on M is uniquely reconstructed as the strongest topology among all the topologies with all the d-morphisms between open sets of E and M being continuous, as soon as $h_\alpha = k_\alpha^{-1}$ $(\alpha \in A)$ are d-morphisms as mappings between W_α and M.

Let U be an open set of M. Let us define a diffeology $\Phi(U)$ on U as a set of mappings $f \in \Phi(M)$ such that $f(\alpha(f)) \subset U$. Then a homeomorphism k of U with an open set W of E is a coordinate mapping if and only if it is a d-isomorphism of $(U,\Phi(U))$ with $(W,\Phi(W))$.

Using Theorems 1-3 we obtain the following consequence:

Corollary 2. If M is a Fréchet manifold modeled on E, the tangent bundle TM is the tangent bundle of M as d-space, whose diffeology coincides with the diffeology of TM as Fréchet manifold. The algebraic p-th tensor degree and the algebraic p-th exterior degree of the vector bundle TM are the p-th tensor degree $T^p(M,\Phi(M))$ and the p-th exterior degree $\Lambda^p T(M,\Phi(M))$ of TM as a d-space.

Notice that, in accordance with the general definition of tensor field of type $(0,p)$ (of differential p-form) on d-space, on Fréchet manifold this object is defined by d-morphism between $T^p(M,\Phi(M))$ $(\Lambda^p T(M,\Phi(M)))$ and $\mathbb{R}$, which is linear on fibres of this bundle. One can define an arbitrary tensor field of type (q,p) on M as a d-morphism between $T^p(M,\Phi(M))$ and $T^q(M,\Phi(M))$, which maps linearly a fibre of the bundle $T^p(M,\Phi(M))$ at any point $x \in M$ into a fibre of the bundle $T^q(M,\Phi(M))$ at the same point. Even for Banach manifolds the definition of an arbitrary tensor field of type (q,p) is apparently new, and our definition of a tensor field of type $(0,p)$ or of a differential p-form is formally weaker, than the customary definition.

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