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Canonical L^p-spaces Associated with an Arbitrary Abstract von Neumann Algebra

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

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To my wife

Sumiko

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ABSTRACT OF THE DISSERTATION

Canonical L^p-spaces Associated with
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Professor Masamichi Takesaki, Chair

The paper is devoted to a construction of canonical L^p-spaces, $1 \leq p \leq \infty$, from a given arbitrary abstract von Neumann algebra.

We start from an abstract von Neumann algebra $\mathfrak{M}$. Namely, $\mathfrak{M}$ is a C^*-algebra, which is the dual space of a (unique) Banach space $\mathfrak{M}_*$, the predual. Without fixing a distinguished functional on $\mathfrak{M}$, we construct the canonical standard form $(\mathfrak{M}, \mathfrak{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathfrak{H}_{\mathfrak{M}})_+)$ and Banach spaces $L^p(\mathfrak{M})$, $1 \leq p \leq \infty$, the canonical L^p-spaces associated with the algebra $\mathfrak{M}$ in question. Our canonical L^p-spaces have the expected properties such as duality.

In Chapter I, for later use, we develop the relative modular theory for positive linear functionals on a von Neumann algebra, which are not necessarily faithful. Although the theory was originally studied for faithful ones, we show that, with natural modification, almost all properties remain valid for non-faithful ones. Especially, we obtain the relative K.M.S.-condition, a necessary and sufficient

condition for a Radon-Nikodym cocycle $(D\varphi : D\varphi_0)_t$, $t \in \mathbb{R}$, to admit certain analytic continuation, and the fact that functionals are "close" if and only if their Radon-Nikodym cocycles are "close."

Chapter II is devoted to a canonical construction of the standard form. The algebra $\mathfrak{M}$ in question is not a priori represented as operators so that at first we give a canonical construction of a Hilbert space. We begin with introducing a notion of new addition and scalor multiplication (by positive numbers) on $\mathfrak{M}_*^+$, the positive part of $\mathfrak{M}_*$, by making use of Radon-Nikodym cocycles studied in the previous chapter. By extending these new operations linearly, we obtain the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$. Thus, $\mathfrak{M}_*^+$ $(= (\mathcal{H}_{\mathfrak{M}})_+)$ is sitting inside the space as a positive cone. We then let $\mathfrak{M}$ act on the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$ canonically. This action is constructed by modular automorphism groups associated with individual functionals in $\mathfrak{M}_*^+$. We thus obtain the canonical standard form $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$.

The final Chapter III is devoted to a construction of the canonical L^p-spaces, $1 \leq p < \infty$ $(L^\infty(\mathfrak{M}) = \mathfrak{M})$. We introduce, for each p, a new linear structure on $\mathfrak{M}_*$ by using relative modular operators on $\mathcal{H}_{\mathfrak{M}}$, the canonical Hilbert space, and polar decompositions of functionals in $\mathfrak{M}_*$. The predual $\mathfrak{M}_*$ with the new linear structure is our $L^p(\mathfrak{M})$, the canonical L^p-space associated with $\mathfrak{M}$. Since we deal with only relative modular operators on the single and canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$, our L^p-spaces are functorially attached to the algebra itself. We then show that the canonical L^p-spaces have the expected properties. As classical L^p-spaces constructed from a faithful normal semi-finite trace, the duality between $L^p(\mathfrak{M})$ and $L^q(\mathfrak{M})$, $1/p + 1/q = 1$, is obtained as a consequence of a certain inequality concerning the norm.

Introduction

The thesis is devoted to a construction of the canonical standard form in the sense of Haagerup, [17], and canonical L^p-spaces, $1 \leq p \leq \infty$, associated with a given arbitrary abstract von Neumann algebra, [38]. The construction is based on the relative modular theory as well as the Tomita-Takesaki theory, [43], and carried out without fixing a distinguished functional on the algebra in question. Before going into the details of the organization of the thesis, we examine the history and motivation of the subject.

The study of von Neumann algebras was initiated by Murray and von Neumann, [30]. Their tool for the classification theory was a dimension function, or equivalently a trace. Thus a pair $(\mathfrak{M}, \tau)$ consisting of a von Neumann algebra and a faithful trace on it is naturally an important object. We may regard this pair as a non-commutative integration theory. In fact, when $\mathfrak{M}$ is commutative, there exists a measure space $(X; d\mu)$ such that $\mathfrak{M}$ is exactly $L^\infty(X; d\mu)$, and the integral $\tau = \int_X \cdot \, d\mu$ gives rise to a trace on $\mathfrak{M}$. Thus, many authors developed theories of non-commutative integrations based on the theory of traces. The non-commutative L^p-spaces of Kunze, [29], Ogasawara-Yoshinaga, [32], are based on Segal's work [39] (the theory of a gage and measurable operators). Later, Nelson, [31], somewhat simplified the above mentioned works. At the same time as Segal, Dixmier, [12], also constructed his theory of non-commutative L^p-spaces using a slightly different (but equivalent) method.

Following the development of the theory of left Hilbert algebras, that is, the Tomita-Takesaki theory, one can effectively study von Neumann algebras, which do not admit a trace, namely von Neumann algebras of type III. Also, we can construct successful theories of non-commutative L^p-spaces from a truly non-commutative pair $(\mathfrak{M}, \varphi_0)$, namely φ_0 is just a faithful state, or weight, [4], on the von Neumann algebra $\mathfrak{M}$ in question. There are several ways to construct such L^p-spaces: Haagerup's L^p-spaces, whose L^1-version was also obtained by Woronowicz, [45], rely on the crossed product technique Connes-Hilsum's L^p-spaces, [7], [23], are constructed without using a crossed product. We saw that such non-commutative L^p-spaces can be also obtained by making use of the complex interpolation method (due to Calderón), [27], and that the study of (the positive parts of) L^p-spaces is exactly the study of a one parameter family of positive cones introduced by Araki, [2], [25], [26].

We now examine the reason why the theory of non-commutative L^p-spaces is important. Firstly, non-commutative L^p-spaces, especially L^2-space, provide a powerful tool for the study of a von Neumann algebra itself. Indeed, through the study of (quasi-) Hilbert algebra, many important results on a semi-finite von Neumann algebra were obtained, [11], [15], [35]. Furthermore, the above mentioned theory of left Hilbert algebras, [41], which has been playing a central role in the recent development of the theory of von Neumann algebras, may be considered as a non-commutative L^2-space. Secondly, as mentioned explicitly by von Neumann in the introduction of [30], the main purpose of the study of von Neumann algebras is its application to other fields of mathematics and physics. And some important applications were actually obtained through the theory of non-commutative L^p-spaces. For example, applications to unitary representation theory, [29], and to quantum physics, [16].

So far, we have some constructions of non-commutative L^p-spaces. The next question is whether or not one can construct such L^p-spaces in a canonical fashion. Recently, there have been many applications of the theory of von Neumann algebras to geometry, [8], and the categorical point of view is being used for the study of von Neumann algebras, [37]. Thus, it is desirable to construct the canonical L^p-spaces associated with a given von Neumann algebra $\mathfrak{M}$ itself, without fixing a functional on it. In the commutative case, the von Neumann algebra $\mathfrak{M} = L^\infty(X; d\mu)$ depends only on the measure class of $d\mu$, not the measure itself, and lattice theorists constructed canonical L^p-spaces in this set up. In a similar motivation, Blattner, [3], showed that one can obtain induced representations canonically, which was strongly influenced by Mackey's notion of the intrinsic Hilbert space (consisting of half densities).

In the thesis, as mentioned at the beginning, we start from a given arbitrary abstract von Neumann algebra $\mathfrak{M}$, that is, a C^*-algebra, [13], which is the dual space of a (unique) Banach space $\mathfrak{M}_*$, the predual. (The predual $\mathfrak{M}_*$ is also considered as the space of $\sigma(\mathfrak{M},\mathfrak{M}_*)$-continuous functionals on $\mathfrak{M}$.) One should note that $\mathfrak{M}$ is not a priori represented as a space of operators on a Hilbert space. Our goal is to construct the canonical L^p-spaces, $1 \leq p \leq \infty$, as well as the canonical Hilbert space on which $\mathfrak{M}$ acts canonically. The thesis consists of three chapters. The material of each chapter is briefly described in what follows.

In Chapter I, we develop the relative modular theory of positive functionals in $\mathfrak{M}_*^+$. Although the theory was originally developed for only faithful functionals in $\mathfrak{M}_*^+$, [5], [10], it is more convenient to allow also non-faithful ones for our purpose. We show that, with natural modification, almost all known properties remain valid for non-faithful ones. Among them, we show the relative K.M.S. condition, the possibility of analytic continuation of a Radon-Nikodym cocycle $(D\varphi;D\varphi_0)_t$, $t \in \mathbb{R}$, under certain conditions, and the fact that functionals are "close" if and only if their Radon-Nikodym cocycles are "close."

Chapter II is devoted to a construction of the canonical standard form. At first, we define new addition, scalor multiplication (by positive numbers), and an inner product on $\mathfrak{M}_*^+$ by using Radon-Nikodym cocycles. Extending these operations linearly, we obtain the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$. Here, $\mathfrak{M}_*^+$ $(= (\mathcal{H}_{\mathfrak{M}})_+)$ can be imbedded into $\mathcal{H}_{\mathfrak{M}}$ as a positive cone. We then let $\mathfrak{M}$ act on $\mathcal{H}_{\mathfrak{M}}$ canonically, by using modular automorphism groups associated with individual functionals in $\mathfrak{M}_*^+$. We thus obtain the canonical standard form. $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$.

Chapter III is devoted to a construction of the canonical L^p-spaces, $1 \leq p < \infty$. $(L^\infty(\mathfrak{M}) = \mathfrak{M})$ At first, we collect some known properties on measurable operators and crossed products. Then, we prove certain properties of homogeneous operators, which are indispensable in our construction of L^p-spaces. After this preparation, we introduce, for each p, a new linear structure and norm on $\mathfrak{M}_*$

by making use of only "canonical" relative modular operators on the single and canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$, constructed in the previous chapter. The predual $\mathfrak{M}_*$ together with the above mentioned new structure is our $L^p(\mathfrak{M})$, the canonical L^p-space associated with the algebra $\mathfrak{M}$ in question. We show that our L^p-spaces have the expected properties such as duality.

Finally, our standard reference on the general theory of von Neumann algebras are [14], [43], while the one on the Tomita-Takesaki theory is [41]. We also use freely the results and notations of standard forms, which are found in [2], [6], [17].

Chapter I Relative Modular Theory

In this chapter, we develop the relative modular theory for positive linear functionals on a von Neumann algebra, which are not necessarily faithful.

Throughout the chapter, we fix a von Neumann algebra $\mathfrak{M}$ and a standard form $(\mathfrak{M},\mathfrak{H},J,\mathcal{P}^\natural)$ in the sense of Haagerup, [17]. Namely, $\mathcal{P}^\natural$ is a self-dual cone in a Hilbert space $\mathfrak{H}$, on which $\mathfrak{M}$ acts. A unitary involution J is determined by $\mathcal{P}^\natural$, and $J\mathfrak{M}J = \mathfrak{M}'$, the commutant of $\mathfrak{M}$. It is known that the map: $\xi \in \mathcal{P}^\natural \mapsto \omega_\xi \in \mathfrak{M}_*^+$ is homeomorphic with respect to the norm topologies in $\mathcal{P}^\natural$ and $\mathfrak{M}_*^+$. Here, $\omega_\xi \in \mathfrak{M}_*^+$ is given by $\omega_\xi(x) = (x\xi|\xi)$, $x \in \mathfrak{M}$.

Let φ_0 (resp. φ) be a faithful functional (resp. functional) in $\mathfrak{M}_*^+$ with a unique implementing vector ξ_0 (resp. ξ) in $\mathcal{P}^\natural$, that is, $\varphi_0 = \omega_{\xi_0}$ (resp. $\varphi = \omega_\xi$). We shall fix these two functionals in this chapter, except in Lemma 1.4.1 and Theorem 1.4.5, 1.4.6.

§1.1 The Gelfand-Naimark-Segal Construction for a "Mixed" Functional.

In this section, we study the Gelfand-Naimark-Segal construction (abbreviated as G.N.S.-construction), determined by a "mixed" functional which will be made precise shortly.

At first we remark that the support projection p of φ (as a functional) is exactly the projection onto the smallest closed subspace $[\mathfrak{M}'\xi]$ containing $\mathfrak{M}'\xi$, and that $\mathfrak{M}\xi_0$ and $\mathfrak{M}'\xi_0$ are both dense in $\mathfrak{H}$ as φ_0 being faithful.

Let $\mathfrak{N}$ be the tensor product of $\mathfrak{M}$ and the 2×2-matrix algebra

$M_2(\mathbb{C})$, realized as

$$\mathfrak{n} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \; a,b,c,d \in \mathfrak{m} \right\},$$

and let $\bar{p}$ be a projection in $\mathfrak{n}$ given by

$$\bar{p} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} .$$

We denote the reduced algebra $\mathfrak{n}$ by the projection $\bar{p}$ by $\mathfrak{n}_{\bar{p}}$ as usual, that is,

$$(1.1.1) \quad \mathfrak{n}_{\bar{p}} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \; a \in \mathfrak{m}, \; b \in \mathfrak{m}p, \; c \in p\mathfrak{m}, \; d \in p\mathfrak{m}p \right\}.$$

We also consider the "mixed" functional χ on $\mathfrak{n}_{\bar{p}}$ determined by

$$\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \varphi_0(a) + \varphi(d) = \omega_{\xi_0}(a) + \omega_{\xi}(d) .$$

We compute

$$\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \chi\left(\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$= \chi\left(\begin{bmatrix} a^*a + c^*c, & a^*b + c^*d \\ b^*a + d^*c, & b^*b + d^*d \end{bmatrix}\right)$$

$$= \varphi_0(a^*a + c^*c) + \varphi(b^*b + d^*d) .$$

<u>Lemma 1.1.1.</u> The functional χ on $\mathfrak{n}_{\bar{p}}$ is faithful.

Proof. For a, b, c, d as in (1.1.1), we assume that

$$\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 .$$

The above calculation implies: $\varphi_0(a^*a) + \varphi_0(c^*c) = 0$ and $\varphi(b^*b) + \varphi(d^*d) = 0$. The first (resp. second) equality means $a = c = 0$ (resp. $b = d = 0$) since φ_0 is faithful (resp. b^*b, d^*d belong to $p\mathfrak{M}p$ and p is the support projection of φ). (Q.E.D.)

The calculation before the lemma implies

$$\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \omega_{\xi_0}(a^*a + c^*c) + \omega_\xi(b^*b + d^*d)$$

$$= \|a\xi_0\|^2 + \|c\xi_0\|^2 + \|b\xi\|^2 + \|d\xi\|^2 .$$

We set $p' = JpJ \in \mathfrak{M}'$ so that p' is the projection onto $[\mathfrak{M}\xi]$. We notice that $c\xi_0 \in p\mathfrak{M}\xi_0 \subseteq p\mathfrak{H}$, $b\xi \in \mathfrak{M}p\xi = \mathfrak{M}\xi \subseteq p'\mathfrak{H}$, and $d\xi \in p\mathfrak{M}p\xi = p\mathfrak{M}\xi \subseteq pp'\mathfrak{H}$. Thus, the above equality shows that a pre-Hilbert space $\mathfrak{n}_{\bar{p}}$ equipped with the inner product induced by the "mixed" functional χ is isometrically mapped into $\mathfrak{H} \oplus p\mathfrak{H} \oplus p'\mathfrak{H} \oplus pp'\mathfrak{H}$, via

$$\eta_\chi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a\xi_0 \\ c\xi_0 \\ b\xi \\ d\xi \end{bmatrix}$$

Furthermore, the image is clearly dense in $\mathfrak{H} \oplus p\mathfrak{H} \oplus p'\mathfrak{H} \oplus pp'\mathfrak{H}$ so that the Hilbert space $\mathfrak{H}_\chi$, the completion of $\mathfrak{n}_{\bar{p}}$, can be identified with $\mathcal{H} \oplus p\mathcal{H} \oplus p'\mathcal{H} \oplus pp'\mathcal{H}$, which we shall actually do throughout the chapter.

Next, we determine the induced representation π_χ of $h_{\bar{p}}$ on the Hilbert space $\mathcal{H} \oplus p\mathcal{H} \oplus p'\mathcal{H} \oplus pp'\mathcal{H}$. For a, b, c, d and e, f, g, h as (1.1.1), we simply compute

$$\pi_\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\pi_\chi\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \pi_\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$$

$$= \pi_\chi\left(\begin{bmatrix} ae + bg, & af + bh \\ ce + dg, & cf + dh \end{bmatrix}\right).$$

On the other hand, by the above identification, we get

$$\pi_\chi\left(\begin{bmatrix} ae + bg, & af + bh \\ ce + dg, & cf + dh \end{bmatrix}\right) = \begin{bmatrix} (ae + bg)\xi_0 \\ (ce + dg)\xi_0 \\ (af + bh)\xi \\ (cf + dh)\xi \end{bmatrix}$$

$$= \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix}\begin{bmatrix} e\xi_0 \\ g\xi_0 \\ f\xi \\ h\xi \end{bmatrix}.$$

Thus, we conclude that the induced representation is given by

$$\pi_\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in h_{\bar{p}}.$$

Finally, the cyclic and separating vector ξ_χ in $\mathcal{H}_\chi$ for $h_{\bar{p}}$, which gives rise to the original χ as a vector functional, is given by

$$\xi_\chi = \pi_\chi\left(\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}\right) = \begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix} .$$

§1.2 Relative Modular Operators

By identifying $\mathfrak{n}_{\underline{p}}$ with $\pi_\chi(\mathfrak{n}_{\underline{p}})$, we have the triple $(\mathfrak{n}_{\underline{p}}, \mathcal{H}_\chi, \xi_\chi)$. Namely, the Hilbert space $\mathcal{H}_\chi = \mathcal{H} \oplus p\mathcal{H} \oplus p'\mathcal{H} \oplus pp'\mathcal{H}$, the von Neumann algebra

$$\mathfrak{n}_{\underline{p}} = \left\{ \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} ; a \in \mathfrak{M}, b \in \mathfrak{M}p, c \in p\mathfrak{M}, d \in p\mathfrak{M}p \right\}$$

acting on $\mathcal{H}_\chi$ naturally, and the cyclic and separating vector $\begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix}$ in $\mathcal{H}_\chi$ for $\mathfrak{n}_{\underline{p}}$. We compute important objects associated with this triple in the Tomita-Takesaki theory, [41].

We set

$$S_\chi\left(\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix}\begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix}^*\begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix} .$$

This densely-defined (conjugate linear) operator on $\mathcal{H}_\chi$ is known to be closable (S-operator). By looking at each component, we also set

(i) $S_{\varphi_0\varphi_0} = S_{\varphi_0} : a\xi_0 \in \mathfrak{m}\xi_0 \mapsto a^*\xi_0 \in \mathfrak{m}\xi_0$ (the usual S-operator determined by $(\mathfrak{m},\mathcal{H},\xi_0)$),

(ii) $S_{\varphi\varphi_0} : c\xi_0 \in p\mathfrak{m}\xi_0 \mapsto c^*\xi \in \mathfrak{m}p\xi = \mathfrak{m}\xi$,

(iii) $S_{\varphi_0\varphi} : b\xi \in \mathfrak{m}p\xi = \mathfrak{m}\xi \mapsto b^*\xi_0 \in p\mathfrak{m}\xi_0$,

(iv) $S_{\varphi\varphi} : d\xi \in p\mathfrak{m}p\xi = p\mathfrak{m}\xi \mapsto d^*\xi \in p\mathfrak{m}p\xi = p\mathfrak{m}\xi$.

(Notice that $S_{\varphi_0\varphi}$ and $S_{\varphi\varphi}$ are well-defined.) Since we compute

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix}^* \begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix} = \begin{bmatrix} a^* & c^* & 0 & 0 \\ b^* & d^* & 0 & 0 \\ 0 & 0 & a^*p' & c^*p' \\ 0 & 0 & b^*p' & d^*p' \end{bmatrix} \begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix}$$

$$= \begin{bmatrix} a^*\xi_0 \\ b^*\xi_0 \\ c^*\xi \\ d^*\xi \end{bmatrix},$$

we get

$$S_\chi = \begin{bmatrix} S_{\varphi_0} & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi_0\varphi} & 0 \\ 0 & S_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & 0 & S_{\varphi\varphi} \end{bmatrix}.$$

As S_χ being densely-defined and closable, we certainly conclude that

(i) $S_{\varphi\varphi_0}$ is a densely-defined closable operator from $p\mathcal{H}$ to $p'\mathcal{H}$,

(ii) $S_{\varphi_0\varphi}$ is a densely-defined closable operator from $p'\mathcal{H}$ to $p\mathcal{H}$,

(iii) $S_{\varphi\varphi}$ is a densely-defined closable operator on $pp'\mathcal{H}$.

Furthermore, we get

$$S_\chi^* \bar{S}_\chi = \begin{bmatrix} S_{\varphi_0} & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi_0\varphi} & 0 \\ 0 & S_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & 0 & S_{\varphi\varphi} \end{bmatrix}^* \begin{bmatrix} \bar{S}_{\varphi_0} & 0 & 0 & 0 \\ 0 & 0 & \bar{S}_{\varphi_0\varphi} & 0 \\ 0 & \bar{S}_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{\varphi\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} S^*_{\varphi_0} & 0 & 0 & 0 \\ 0 & 0 & S^*_{\varphi\varphi_0} & 0 \\ 0 & S^*_{\varphi_0\varphi} & 0 & 0 \\ 0 & 0 & 0 & S^*_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \bar{S}_{\varphi_0} & 0 & 0 & 0 \\ 0 & 0 & \bar{S}_{\varphi_0\varphi} & 0 \\ 0 & \bar{S}_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{\varphi\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} S^*_{\varphi_0}\bar{S}_{\varphi_0} & 0 & 0 & 0 \\ 0 & S^*_{\varphi\varphi_0}\bar{S}_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & S^*_{\varphi_0\varphi}\bar{S}_{\varphi_0\varphi} & 0 \\ 0 & 0 & 0 & S^*_{\varphi\varphi}\bar{S}_{\varphi\varphi} \end{bmatrix}.$$

Here, the bar on the top of the S-operators means their closures.

Definition 1.2.1. We set $\Delta_{\varphi_0} = S^*_{\varphi_0}\bar{S}_{\varphi_0}$, $\Delta_{\varphi\varphi_0} = S^*_{\varphi\varphi_0}\bar{S}_{\varphi\varphi_0}$, $\Delta_{\varphi_0\varphi} = S^*_{\varphi_0\varphi}\bar{S}_{\varphi_0\varphi}$, and $\Delta_{\varphi\varphi} = S^*_{\varphi\varphi}\bar{S}_{\varphi\varphi}$ respectively. Among them, we call $\Delta_{\varphi\varphi_0}$ and $\Delta_{\varphi_0\varphi}$ relative modular operators. (See Remark 1.2.4.)

Clearly, Δ_{φ_0} (resp. $\Delta_{\varphi\varphi_0}$, $\Delta_{\varphi_0\varphi}$, $\Delta_{\varphi\varphi}$) is a non-singular positive self-adjoint operator on $\mathfrak{H}$ (resp. $p\mathfrak{H}$, $p'\mathfrak{H}$, $pp'\mathfrak{H}$). We notice that Δ_{φ_0} is exactly the usual modular operator associated with $(\mathfrak{M},\mathfrak{H},\xi_0)$, while $\Delta_{\varphi\varphi} = \Delta_\varphi$ is the one associated with the cyclic and separating vector ξ in $pp'\mathfrak{H}$ for the von Neumann algebra $p'p\mathfrak{M}p$, which is isomorphic to the reduced algebra $\mathfrak{M}p$. The above calculation shows

$$\Delta_\chi = \begin{bmatrix} \Delta_{\varphi_0} & 0 & 0 & 0 \\ 0 & \Delta_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & \Delta_{\varphi_0\varphi} & 0 \\ 0 & 0 & 0 & \Delta_{\varphi\varphi} \end{bmatrix}.$$

Next, we have to determine the modular conjugation J_χ, which is the phase part of $\bar{S}_\chi$. To do so, we begin with computing the commutant $\mathfrak{n}'_{\bar{p}}$ of $\mathfrak{n}_{\bar{p}}$ acting on $\mathcal{H}_\chi = \mathcal{H} \oplus p\mathcal{H} \oplus p'\mathcal{H} \oplus pp'\mathcal{H}$.

<u>Lemma 1.2.2.</u> The commutant $\mathfrak{n}'_{\bar{p}}$ is given by

$$\mathfrak{n}'_{\bar{p}} = \left\{ \begin{bmatrix} a' & 0 & b'p' & 0 \\ 0 & pa' & 0 & pb'p' \\ p'c' & 0 & p'dp' & 0 \\ 0 & pp'c' & 0 & pp'dp' \end{bmatrix} ;\ a', b', c', d' \in \mathfrak{m}' \right\}$$

Proof. Let $\mathfrak{R}$ be a von Neumann algebra acting on $\mathcal{H} \oplus \mathcal{H} \oplus p'\mathcal{H} \oplus p'\mathcal{H}$ given by

$$\mathfrak{R} = \left\{ \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} ;\ a, b, c, d \in \mathfrak{m} \right\}.$$

We notice that our von Neumann algebra $\mathfrak{n}_{\bar{p}}$ is exactly the reduced algebra $\mathfrak{R}_{\bar{q}}$ with a projection

$$\bar{q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p' & 0 \\ 0 & 0 & 0 & pp' \end{bmatrix} \quad \text{in } \mathfrak{R},$$

so that Proposition II, 3.10, [43], yields $\mathfrak{n}'_{\bar{p}} = (\mathfrak{R}_{\bar{q}})' = (\mathfrak{R}')_{\bar{q}}$.

On the other hand, the commutant $(\mathfrak{m} \otimes M_2(C))' = \mathfrak{m}' \otimes C1$ is realized as

$$\left\{ \begin{bmatrix} a' & 0 & b' & 0 \\ 0 & a' & 0 & b' \\ c' & 0 & d' & 0 \\ 0 & c' & 0 & d' \end{bmatrix} ;\ a', b', c', d' \in \mathfrak{m}' \right\},$$

and $\mathfrak{R}$ is the induced algebra $(\mathfrak{m} \otimes M_2(C))_{\bar{r}}$ with a projection

$$\bar{r} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p' & 0 \\ 0 & 0 & 0 & p' \end{bmatrix} \quad \text{in the commutant } (\mathfrak{m} \otimes M_2(C))'.$$

Thus, by Proposition II, 3.10, [43],

$$\mathfrak{R}' = (\mathfrak{m} \otimes M_2(C))'_{\bar{r}}$$

$$= \left\{ \begin{bmatrix} a' & 0 & b'p' & 0 \\ 0 & a' & 0 & b'p' \\ p'c' & 0 & p'd'p' & 0 \\ 0 & p'c' & 0 & p'd'p' \end{bmatrix} ;\ a', b', c', d' \in \mathfrak{m}' \right\},$$

so that we conclude that $\mathfrak{n}'_{\bar{p}} = (\mathfrak{R}')_{\bar{q}}$ is exactly the set described in the lemma. (Q.E.D.)

In the proof of the following proposition, the fact that ξ and ξ_0 belong to $P^\natural$ is essential:

<u>Proposition 1.2.3</u>. The modular conjugation operator J_χ determined by $\bar{S}_\chi$ is

$$J_\chi = \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & 0 & J \end{bmatrix},$$

or more precisely,

$$J_\chi = \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & 0 & J_1 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & 0 & J_3 \end{bmatrix}, \text{ where } J_1 \text{ (resp. } J_2, J_3)$$

is the restriction of J to $p'\mathcal{H}$ (resp. $p\mathcal{H}$, $pp'\mathcal{H}$).

Proof. According to Theorem 1, [2], J_χ is characterized by the following four conditions:

(i) J_χ is a unitary involution,

(ii) $J_\chi \xi_\chi = \xi_\chi$,

(iii) $J_\chi \mathfrak{h}_{\bar{p}} J_\chi = \mathfrak{h}'_{\bar{p}}$,

(iv) $(xJ_\chi xJ_\chi \xi_\chi | \xi_\chi) \geq 0$, $x \in \mathfrak{h}_{\bar{p}}$.

The conditions (i), (ii) are easily checked. By the previous lemma, we have known the commutant $\mathfrak{h}'_{\bar{p}}$ so that (iii) can be also checked by straight-forward calculation. Thus, it suffices to show that the operator J_χ described in the proposition satisfies (iv).

For any

$$x = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} ; \; a,b,c,d \text{ as } (1.1.1) ,$$

direct computation shows

$$xJ_\chi xJ_\chi \xi_\chi = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} \begin{bmatrix} JaJ & 0 & JbJ & 0 \\ 0 & pJaJ & 0 & pJbJ \\ JcJ & 0 & JdJ & 0 \\ 0 & pJcJ & 0 & pJdJ \end{bmatrix} \begin{bmatrix} \xi_0 \\ 0 \\ 0 \\ \xi \end{bmatrix} ,$$

$$= \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} \begin{bmatrix} JaJ\xi_0 \\ pJbJ\xi \\ JcJ\xi_0 \\ pJdJ\xi \end{bmatrix}$$

$$= \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} \begin{bmatrix} JaJ\xi_0 \\ JbJ\xi \\ JcJ\xi_0 \\ JdJ\xi \end{bmatrix}$$

$$= \begin{bmatrix} aJaJ\xi_0 + bJbJ\xi \\ cJaJ\xi_0 + dJbJ\xi \\ ap'JcJ\xi_0 + bp'JdJ\xi \\ cp'JcJ\xi_0 + dp'JdJ\xi \end{bmatrix} = \begin{bmatrix} aJaJ\xi_0 + bJbJ\xi \\ cJaJ\xi_0 + dJbJ\xi \\ ap'JcJ\xi_0 + bJdJ\xi \\ cp'JcJ\xi_0 + dJdJ\xi \end{bmatrix} .$$

Here, the third equality follows from the facts $pJbJ = JbJp$,

$pJdJ = JdJp$, while the fifth follows from the fact

$$JdJ\xi \in Jp\mathfrak{M}pJ\xi = p'\mathfrak{M}'p'\xi = p'\mathfrak{M}'\xi \subseteq p'p\mathfrak{H} .$$

Thus, we get

$$(xJ_\chi xJ_\chi \xi_\chi | \xi_\chi) = (aJaJ\xi_0 + bJbJ\xi | \xi_0) + (cp'JcJ\xi_0 + dJdJ\xi | \xi)$$

$$= (aJaJ\xi_0 | \xi_0) + (bJbJ\xi | \xi_0) + (cJcJ\xi_0 | \xi) + (dJdJ\xi | \xi) .$$

We notice that ξ_0 and ξ belong to $\mathcal{P}^\natural$, which is globally invariant under each $yJyJ$, $y \in \mathfrak{M}$. It follows from the self-duality of $\mathcal{P}^\natural$ that $(xJ_\chi xJ_\chi \xi_\chi | \xi_\chi) \geq 0$ as desired. (Q.E.D.)

Thus, the polar decomposition $\bar{S}_\chi = J_\chi \Delta_\chi^{\frac{1}{2}}$ is expressed in terms of 4×4-matrices, as follows:

$$\begin{bmatrix} \bar{S}_{\varphi_0} & 0 & 0 & 0 \\ 0 & 0 & \bar{S}_{\varphi_0\varphi} & 0 \\ 0 & \bar{S}_{\varphi\varphi_0} & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & 0 & J \end{bmatrix} \begin{bmatrix} \Delta_{\varphi_0}^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \Delta_{\varphi\varphi_0}^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \Delta_{\varphi_0\varphi}^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \Delta_{\varphi\varphi}^{\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} J\Delta_{\varphi_0}^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & J\Delta_{\varphi_0\varphi}^{\frac{1}{2}} & 0 \\ 0 & J\Delta_{\varphi\varphi_0}^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & J\Delta_{\varphi\varphi}^{\frac{1}{2}} \end{bmatrix} .$$

By comparing each component, one gets the following polar decompositions:

$$\bar{S}_{\varphi_0} = J\Delta_{\varphi_0}^{\frac{1}{2}}, \quad \bar{S}_{\varphi_0\varphi} = J\Delta_{\varphi_0\varphi}^{\frac{1}{2}}, \quad \bar{S}_{\varphi\varphi_0} = J\Delta_{\varphi\varphi_0}^{\frac{1}{2}}, \quad \bar{S}_{\varphi\varphi} = J\Delta_{\varphi\varphi}^{\frac{1}{2}} .$$

In what follows, we have to deal with the above operators simultaneously as operators on a single Hilbert space, namely $\mathcal{H}$, hence we put the following:

Remark 1.2.4. Unless the contrary is stated, the operator $\Delta_{\varphi\varphi_0}$ (resp. $\Delta_{\varphi_0\varphi}$, $\Delta_{\varphi\varphi} = \Delta_\varphi$) is regarded as a positive self-adjoint operator on $\mathcal{H}$ with support $p\mathcal{H}$ (resp. $p'\mathcal{H}$, $pp'\mathcal{H}$).

Before proceeding further, we notice that $J\Delta_\varphi J = \Delta_\varphi^{-1}$, $J\Delta_{\varphi\varphi_0}J = \Delta_{\varphi_0\varphi}^{-1}$, and $J\Delta_{\varphi_0\varphi}J = \Delta_{\varphi\varphi_0}^{-1}$. In fact, they can be obtained by expressing the relation $J_\chi\Delta_\chi J_\chi = \Delta_\chi^{-1}$ in terms of 4×4-matrices. We also notice that $\Delta_{\varphi\varphi_0}^{\frac{1}{2}}\xi_0 = \xi$, $\Delta_{\varphi_0\varphi}^{\frac{1}{2}}\xi = \xi_0$, and $\Delta_{\varphi\varphi}^{\frac{1}{2}}\xi = \xi$. These are consequences of $\Delta_\chi^{\frac{1}{2}}\xi_\chi = J_\chi\bar{S}_\chi\xi_\chi = J_\chi\xi_\chi = \xi_\chi$. Here, the last equality follows from Proposition 1.2.3 and the fact that ξ_0 and ξ belong to $P^\natural$. Finally, we notice that, from our construction, $\mathfrak{m}\xi_0$ is a core of $\Delta_{\varphi\varphi_0}^{\frac{1}{2}}$ (as well as $\Delta_{\varphi_0}^{\frac{1}{2}}$, [41]).

§1.3 Radon-Nikodym Cocycles.

In this section, by considering the modular automorphism σ_t^χ, we define Radon-Nikodym cocycles.

For

$$t \in \mathbb{R},\ x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } \mathfrak{n}_p, \text{ we compute}$$

$$\Delta_\chi^{it}\pi_\chi(x)\Delta_\chi^{-it}$$

$$= \begin{bmatrix} \Delta_{\varphi_0}^{it} & 0 & 0 & 0 \\ 0 & \Delta_{\varphi\varphi_0}^{it} & 0 & 0 \\ 0 & 0 & \Delta_{\varphi_0\varphi}^{it} & 0 \\ 0 & 0 & 0 & \Delta_{\varphi\varphi}^{it} \end{bmatrix} \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & ap' & bp' \\ 0 & 0 & cp' & dp' \end{bmatrix} \begin{bmatrix} \Delta_{\varphi_0}^{-it} & 0 & 0 & 0 \\ 0 & \Delta_{\varphi\varphi_0}^{-it} & 0 & 0 \\ 0 & 0 & \Delta_{\varphi_0\varphi}^{-it} & 0 \\ 0 & 0 & 0 & \Delta_{\varphi\varphi}^{-it} \end{bmatrix}$$

$$= \begin{bmatrix} \Delta_{\varphi_0}^{it} a\Delta_{\varphi_0}^{-it}, & \Delta_{\varphi_0}^{it} b\Delta_{\varphi\varphi_0}^{-it}, & 0, & 0 \\ \Delta_{\varphi\varphi_0}^{it} c\Delta_{\varphi_0}^{-it}, & \Delta_{\varphi\varphi_0}^{it} d\Delta_{\varphi\varphi_0}^{-it}, & 0, & 0 \\ 0, & 0, & \Delta_{\varphi_0\varphi}^{it} ap'\Delta_{\varphi_0\varphi}^{-it}, & \Delta_{\varphi_0\varphi}^{it} bp'\Delta_{\varphi\varphi}^{-it} \\ 0, & 0, & \Delta_{\varphi\varphi}^{it} cp'\Delta_{\varphi_0\varphi}^{-it}, & \Delta_{\varphi\varphi}^{it} dp'\Delta_{\varphi\varphi}^{-it} \end{bmatrix}.$$

Since this must belong to $\mathfrak{n}_{\underline{p}}$ again, we know that $\Delta_{\varphi_0}^{it} b\Delta_{\varphi\varphi_0}^{-it} \in \mathfrak{M}p$, $\Delta_{\varphi\varphi_0}^{it} c\Delta_{\varphi_0}^{-it} \in p\mathfrak{M}$, and $\Delta_{\varphi\varphi_0}^{it} d\Delta_{\varphi\varphi_0}^{-it} \in p\mathfrak{M}p$, and that, in the above 4×4-matrix, the (i,j)-component, $i,j = 1,2$, multiplied by p' is exactly the $(i+2,j+2)$-component. In particular, $\Delta_{\varphi\varphi_0}^{it} d\Delta_{\varphi\varphi_0}^{-it}$ is exactly $\sigma_t^\varphi(d)$, where σ_t^φ is the modular automorphism group determined by the faithful functional φ on $p\mathfrak{M}p$, which is isomorphic to $p'p\mathfrak{M}p$.

<u>Definition 1.3.1</u>. For $b \in \mathfrak{M}p$ (resp. $c \in p\mathfrak{M}$), we set $\sigma_t^{\varphi_0\varphi}(b) = \Delta_{\varphi_0}^{it} b\Delta_{\varphi\varphi_0}^{-it}$ (resp. $\sigma_t^{\varphi\varphi_0}(c) = \Delta_{\varphi\varphi_0}^{it} c\Delta_{\varphi_0}^{-it}$), $t \in \mathbb{R}$.

By the above argument, $\{\sigma_t^{\varphi_0\varphi}\}_{t\in\mathbb{R}}$ (resp. $\{\sigma_t^{\varphi\varphi_0}\}_{t\in\mathbb{R}}$) gives rise to a one parameter family of isometries of the left ideal $\mathfrak{M}p$ (resp. the right ideal $p\mathfrak{M}$). We also notice that the modular automorphism group σ_t^χ associated with $(\mathfrak{n}_{\underline{p}}, \chi)$ is

$$(1.3.1)\qquad \sigma_t^{\chi}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \sigma_t^{\varphi_0}(a)\ , & \sigma^{\varphi_0\varphi}(b) \\ \sigma_t^{\varphi\varphi_0}(c), & \sigma_t^{\varphi}(d) \end{bmatrix}.$$

Definition 1.3.2 ([5], [9], [10]). For $t \in \mathbb{R}$, we set $(D\varphi : D\varphi_0)_t = \sigma_t^{\varphi\varphi_0}(p) = \Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it}$. (Radon-Nikodym cocycle (of φ with respect to φ_0).)

We collect some basic properties, which are easy consequences of the definition:

Lemma 1.3.3. The Radon-Nikodym cocycle $(D\varphi : D\varphi_0)_t$, $t \in \mathbb{R}$, enjoys the following properties:

(i) For each $t \in \mathbb{R}$, $(D\varphi : D\varphi_0)_t$ is a partial isometry in $p\mathfrak{M}$ with the initial (resp. final) projection $\sigma_t^{\varphi_0}(p)$ (resp. p).

(ii) The family $\{(D\varphi : D\varphi_0)_t\}_{t\in\mathbb{R}}$ is a strong continuous 1-cocycle for σ^{φ_0}, that is, $(D\varphi : D\varphi_0)_{t+s} = (D\varphi : D\varphi_0)_t\sigma_t^{\varphi_0}((D\varphi : D\varphi_0)_s)$; $t,s \in \mathbb{R}$, as its name indicates.

(iii) The Radon-Nikodym cocycle $(D\varphi : D\varphi_0)_t$, $t \in \mathbb{R}$, intertwines $\sigma_t^{\varphi_0}$ and σ_t^{φ}, that is, $(D\varphi : D\varphi_0)_t\sigma_t^{\varphi_0}(x)(D\varphi : D\varphi_0)_t^* = \sigma_t^{\varphi}(pxp)$, $x \in \mathfrak{M}$.

Proof. (i) From the construction, $(D\varphi : D\varphi_0)_t$ is a partial isometry in $p\mathfrak{M}$, and we simply compute

$$(D\varphi : D\varphi_0)_t^*(D\varphi : D\varphi_0)_t = (\Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it})^*(\Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it})$$

$$= \Delta_{\varphi_0}^{it}\Delta_{\varphi\varphi_0}^{-it}\Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it}$$

$$= \Delta_{\varphi_0}^{it} p \Delta_{\varphi_0}^{-it} \qquad \text{(See Remark 1.2.4.)}$$

$$= \sigma_t^{\varphi_0}(p) ,$$

$$(D\varphi : D\varphi_0)_t (D\varphi : D\varphi_0)_t^* = \Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it}\Delta_{\varphi_0}^{it}\Delta_{\varphi\varphi_0}^{-it}$$

$$= \Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi\varphi_0}^{-it} = p .$$

(ii) The statement concerning the continuity is trivial. We simply compute

$$(D\varphi : D\varphi_0)_{t+s} = \Delta_{\varphi\varphi_0}^{i(t+s)}\Delta_{\varphi_0}^{-i(t+s)}$$

$$= \Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it}\Delta_{\varphi_0}^{it}(\Delta_{\varphi\varphi_0}^{is}\Delta_{\varphi_0}^{-is})\Delta_{\varphi_0}^{-it}$$

$$= (D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}((D\varphi : D\varphi_0)_s) .$$

(iii) For each $x \in \mathfrak{M}$, we compute

$$(D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(x)(D\varphi : D\varphi_0)_t^* = \Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it}\Delta_{\varphi_0}^{it} x \Delta_{\varphi_0}^{-it}(\Delta_{\varphi\varphi_0}^{it}\Delta_{\varphi_0}^{-it})$$

$$= \Delta_{\varphi\varphi_0}^{it} x \Delta_{\varphi\varphi_0}^{-it} .$$

Then, we notice that $\Delta_{\varphi\varphi_0}^{it} x \Delta_{\varphi\varphi_0}^{-it} = \sigma_t^{\varphi}(pxp)$ as remarked before Definition 1.3.1. (Q.E.D.)

§1.4 Properties of Radon-Nikodym Cocycles.

In this section, we shall obtain less obvious properties of Radon-Nikodym cocycles for later use.

Lemma 1.4.1 (cf. Theorem 15.3, [41]). Let ψ be a faithful functional in $\mathfrak{M}_*^+$. For $x \in \mathfrak{M}_+$, the following two statements are equivalent:

(i) $\psi(x \cdot x^*) = x^*\psi x \leq \ell\psi$ with some $\ell \geq 0$, that is, $\psi(xyx^*) \leq \ell\psi(y)$ for any $y \in \mathfrak{M}^+$,

(ii) The function: $t \in R \mapsto \sigma_t^\psi(x) \in \mathfrak{M}$ extends to a function $\sigma_z^\psi(x)$ which is bounded and σ-weakly continuous (resp. analytic) on $-\frac{1}{2} \leq \operatorname{Im} z \leq 0$ (resp. $-\frac{1}{2} < \operatorname{Im} z < 0$), and $\|\sigma_{-i/2}^\psi(x)\| \leq \sqrt{\ell}$.

Proof. Let ζ be a unique implementing vector in $P^\natural$ for ψ, that is $\psi = \omega_\zeta$. Firstly, we assume that $\psi(x \cdot x^*) \leq \ell\psi$. Then the map: $y\zeta \in \mathfrak{M}\zeta \mapsto yx^*\zeta \in \mathfrak{M}\zeta$ extends to a bounded operator a' in $\mathfrak{M}'$ with $\|a'\| \leq \sqrt{\ell}$. In fact, we estimate

$$\|yx^*\zeta\|^2 = (yx^*\zeta|yx^*\zeta) = (xy^*yx^*\zeta|\zeta)$$

$$= \psi(xy^*yx^*) \leq \ell\psi(y^*y) = \ell\|y\zeta\|^2 .$$

Let Δ_ψ be the modular operator associated with $(\mathfrak{M}, \mathfrak{H}, \zeta)$ so that $J\Delta_\psi^{\frac{1}{2}}y\zeta = y^*\zeta$, $y \in \mathfrak{M}$. The vector $a'\zeta = x^*\zeta$ belongs to $\mathfrak{D}(\Delta_\psi^{-\frac{1}{2}})$ and one gets

$$\Delta_\psi^{-\frac{1}{2}}x^*\zeta = \Delta_\psi^{-\frac{1}{2}}a'\zeta = Ja'^*\zeta = b\zeta ,$$

with $b = Ja'^*J \in \mathfrak{M}$, $\|b\| \leq \sqrt{\ell}$.

For $c', d' \in \mathfrak{M}'$, we introduce a function

$$f(t) = (\sigma_t^\psi(x)c'\zeta|d'\zeta) = (d'^*c'\zeta|\Delta_\psi^{it}x^*\zeta), \quad t \in R .$$

It follows from $\Delta_\psi^{-\frac{1}{2}}x^*\zeta = b\zeta$ that $f(t)$ extends to a function $f(z)$, which is bounded and continuous (resp. analytic) on $-\frac{1}{2} \leq \mathrm{Im}\, z \leq 0$ (resp. $-\frac{1}{2} < \mathrm{Im}\, z < 0$). In fact, the bound for $|f(z)|$ on $-\frac{1}{2} \leq \mathrm{Im}\, z \leq 0$ is obtained by

$$\begin{aligned} |(d'^*c'\zeta|\Delta_\psi^{i\bar{z}}x^*\zeta)| &\leq \|d'^*c'\zeta\|\|\Delta_\psi^{i\bar{z}}x^*\zeta\| \\ &\leq \|d'^*c'\zeta\|\{\|\Delta_\psi^{-\frac{1}{2}}x^*\zeta\|^2 + \|x^*\zeta\|^2\}^{\frac{1}{2}} \\ &= \|d'^*c'\zeta\|\{\|b\zeta\|^2 + \|x^*\zeta\|^2\}^{\frac{1}{2}} . \end{aligned}$$

Here, the second inequality is a consequence of the spectral decomposition theorem for the positive self-adjoint operator Δ_ψ.

To get a better estimate for $|f(z)|$, we consider $|f(z)|$ on two boundaries. For $t \in \mathbb{R}$, we estimate

$$\begin{aligned} |f(t)| = |(\sigma_t^\psi(x)c'\zeta|d'\zeta)| &\leq \|\sigma_t^\psi(x)\|\|c'\zeta\|\|d'\zeta\| \\ &= \|x\|\|c'\zeta\|\|d'\zeta\| . \end{aligned}$$

On the other hand, for $-\frac{i}{2} + t$, $t \in \mathbb{R}$, we estimate

$$\begin{aligned} |f(-\tfrac{i}{2} + t)| &= |(d'^*c'\zeta|\Delta_\psi^{-\frac{1}{2}+it}x^*\zeta)| \\ &= |(d'^*c'\zeta|\Delta_\psi^{it}\Delta_\psi^{-\frac{1}{2}}x^*\zeta)| \\ &= |(d'^*c'\zeta|\Delta_\psi^{it}b\zeta)| \\ &= |(d'^*c'\zeta|\sigma_t^\psi(b)\zeta)| \\ &= |(\sigma_t^\psi(b)c'\zeta|d'\zeta)| \end{aligned}$$

$$\leq \|\sigma_t^{\psi}(b)\| \|c' \zeta\| \|d' \zeta\|$$

$$\leq \sqrt{\ell}\ \|c' \zeta\| \|d' \zeta\| .$$

Thus, it follows from the Phrägmen-Linderöf theorem that, for $-\frac{1}{2} \leq \text{Im } z \leq 0$,

$$|f(z)| \leq \text{Max}(\sqrt{\ell}, \|x\|) \|c' \zeta\| \|d' \zeta\| .$$

Thus, the density of $\mathfrak{M}\zeta$ in $\mathfrak{H}$ yields the possibility of the desired analytic continuation $\sigma_z^{\psi}(x)$, and clearly we get $\|\sigma_{-i/2}^{\psi}(x)\| \leq \sqrt{\ell}$ from the above estimate on one of two boundaries.

Conversely, we assume the possibility of the analytic continuation obtained above. For each $a \in \mathfrak{M}_+$, we compute

$$\begin{aligned}
\psi(xax^*) &= \|a^{\frac{1}{2}}x^*\zeta\|^2 = \|Ja^{\frac{1}{2}}JJx^*\zeta\|^2 \\
&= \|Ja^{\frac{1}{2}}JJJ\Delta_{\psi}^{\frac{1}{2}}x\zeta\|^2 \\
&= \|Ja^{\frac{1}{2}}J\sigma_{-i/2}^{\psi}(x)\zeta\|^2 \\
&= \|\sigma_{-i/2}^{\psi}(x)Ja^{\frac{1}{2}}J\zeta\|^2 \\
&\leq \|\sigma_{-i/2}^{\psi}(x)\|^2 \|Ja^{\frac{1}{2}}J\zeta\|^2 \\
&\leq \ell\|a^{\frac{1}{2}}\zeta\|^2 = \ell\psi(a) .
\end{aligned}$$

(Q.E.D.)

<u>Theorem 1.4.2</u>. For $c \in \mathfrak{p}\mathfrak{M}$, the following two conditions are equivalent:

(i) $\varphi(c \cdot c^*) \leq \ell\varphi_0$ with some $\ell \geq 0$,

(ii) The function: $t \in \mathbb{R} \mapsto \sigma_t^{\varphi\varphi_0}(c) \in p\mathfrak{M}$ extends to a function $\sigma_z^{\varphi\varphi_0}(c)$ which is bounded and σ-weakly continuous (resp. analytic) on $-\frac{1}{2} \leq \operatorname{Im} z \leq 0$ (resp. $-\frac{1}{2} < \operatorname{Im} z < 0$), and $\|\sigma_{-i/2}^{\varphi\varphi_0}(c)\| \leq \sqrt{\ell}$.

Proof. At first, we assume (ii). Then the function

$$
(1.4.1) \qquad t \in \mathbb{R} \mapsto \sigma_t^{\chi}\left(\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \sigma_t^{\varphi\varphi_0}(c) & 0 \end{bmatrix}
$$

admits an analytic continuation so that the above lemma implies that, for each $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in (\mathfrak{n}_{\underline{p}})_+$, $a \in \mathfrak{M}_+$, one gets

$$
\chi\left(\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}^*\right) \leq \ell\chi\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right),
$$

that is, $\varphi(cac^*) \leq \ell\varphi_0(a)$.

Conversely, we now assume (i). By the previous lemma and (1.4.1), it suffices to show

$$
\chi\left(\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}^*\right) \leq \ell\chi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right),
$$

for any $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in (\mathfrak{n}_{\underline{p}})_+$. However, we simpy calculate

$$
\chi\left(\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}^*\right) = \varphi(cxc^*)
$$

$$
\leq \ell\varphi_0(x) \qquad (x \in \mathfrak{M}_+)
$$

$$\leq \ell(\varphi_0(x) + \varphi(w)) \qquad (w \in p\mathfrak{M}_+ p)$$

$$= \ell\chi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right). \qquad \text{(Q.E.D.)}$$

In particular, with $c = p$, we have:

Corollary 1.4.3. The following two statements are equivalent:

(i) $\varphi \leq \ell\varphi_0$ with some $\ell \geq 0$

(ii) The Radon-Nikodym cocycle: $t \in R \mapsto (D\varphi : D\varphi_0)_t \in p\mathfrak{M}$ extends to a function $(D\varphi : D\varphi_0)_z$ which is bounded and σ-weakly continuous (resp. analytic) on $-\frac{1}{2} \leq \operatorname{Im} z \leq 0$ (resp. $-\frac{1}{2} < \operatorname{Im} z < 0$), and $\|(D\varphi : D\varphi_0)_{-i/2}\| \leq \sqrt{\ell}$.

Before stating the next result, we notice

$$\sigma_t^{\varphi\varphi_0}(c) = (D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c), \quad \sigma_t^{\varphi\varphi_0}(b^*) = \sigma_t^{\varphi_0\varphi}(b)^* .$$

In fact, the first equality is trivial from the definition, while the second follows from the fact that σ_t^χ is $*$-preserving (see (1.3.1)). These two equalities and Lemma 1.3.3(iii) imply

$$\sigma_t^\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \sigma_t^{\varphi_0}(a) & , & \sigma_t^{\varphi_0}(b)(D\varphi : D\varphi_0)_t \\ (D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c), & & \sigma_t^{\varphi}(d) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_t^{\varphi_0}(a) & , & (D\varphi : D\varphi_0)_t^* \sigma_t^{\varphi}(pb) \\ (D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c), & & \sigma_t^{\varphi}(d) \end{bmatrix}.$$

Theorem 1.4.4. (Relative Kubo-Martin-Schwinger condition). For $b \in \mathfrak{M}p$, $c \in p\mathfrak{M}$, there exists a function $f(z)$ $(= f_{bc}(z))$ which is bounded and continuous (resp. analytic) on $-1 \leq \operatorname{Im} z \leq 0$ (resp. $-1 < \operatorname{Im} z < 0$) with boundary values:

$$f(t) = \varphi_0(b(D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c)), \quad t \in \mathbb{R},$$
$$f(t - i) = \varphi((D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c)b), \quad t \in \mathbb{R}.$$

Conversely, let $\{u_t\}_{t\in\mathbb{R}}$ be a one parameter family of partial isometries in $\mathfrak{M}$ satisfying the properties stated in Lemma 1.3.3. If, for each $b \in \mathfrak{M}p$, $c \in p\mathfrak{M}$, there exists a function $f(z)$ ($= f_{bc}(z)$) which is bounded (resp. analytic) on $-1 \leq \operatorname{Im} z \leq 0$ (resp. $-1 < \operatorname{Im} z < 0$) with the boundary values described above, then u_t is exactly $(D\varphi : D\varphi_0)_t$, $t \in \mathbb{R}$.

Before proving the result, we remark two facts. Firstly, in the second half of the theorem, the continuity on $-1 \leq \operatorname{Im} z \leq 0$ is not assumed, which is a consequence of the other conditions. Secondly the above theorem (a characterization of $(D\varphi : D\varphi_0)_t$ in terms of the relative K.M.S.-condition) yields that the Radon-Nikodym cocycle $(D\varphi : D\varphi_0)_t$, $t \in \mathbb{R}$, does not depend on the standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ which we fixed at the beginning of the chapter. Actually, $(D\varphi : D\varphi_0)_t$ is a canonical object attached to the pair (φ, φ_0).

Proof. For $x = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ in $\mathfrak{n}_{\underline{p}}$, straight-forward calculation shows

$$\chi(x\sigma_t^{\chi}(y)) = \varphi_0(b(D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c)),$$
$$\chi(\sigma_t^{\chi}(y)x) = \varphi((D\varphi : D\varphi_0)_t \sigma_t^{\varphi_0}(c)b) .$$

Thus, the first half of the theorem is a special case of the usual K.M.S.-condition for a single functional χ on $\mathfrak{n}_{\bar{p}}$, [41].

To show the second half, we assume that a family $\{u_t\}_{t\in R}$ satisfies the properties stated in the second half of the theorem. We then notice that

$$\beta_t\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \sigma_t^{\varphi_0}(a) \ , & \sigma_t^{\varphi_0}(b)u_t^* \\ u_t\sigma_t^{\varphi_0}(c), & \sigma_t^{\varphi}(d) \end{bmatrix}, \quad t \in R,$$

gives rise to a one parameter family $\{\beta_t\}_{t\in R}$ of mappings from $\mathfrak{n}_{\bar{p}}$ into itself. The assumption and the usual K.M.S.-condition for both φ_0 and φ yield that, for each $x,y \in \mathfrak{n}_{\bar{p}}$, there exists a function $g(z)$ which is bounded (resp. analytic) on $-1 \leq \mathrm{Im}\, z \leq 0$ (resp. $-1 < \mathrm{Im}\, z < 0$) with boundary values:

$$g(t) = \chi(x\beta_t(y)), \quad t \in R$$

$$g(t - i) = \chi(\beta_t(y)x), \quad t \in R .$$

Thus, $\{\beta_t\}_{t\in R}$, satisfies the usual K.M.S.-condition for a single functional χ so that β_t must be σ_t^{χ} by [14] (see also Theorem 1, [21]), that is, $(D\varphi : D\varphi_0)_t = u_t$. (Q.E.D.)

Finally, we prove two theorems which assure that two functionals are "close" if and only if their Radon-Nikodym cocycles (with respect to a fixed faithful functional) are "close."

<u>Theorem 1.4.5</u>. Let $\{\varphi_n\}$ be a sequence in $\mathfrak{m}_*^+$. If $\{\varphi_n\}$ converges to a functional φ in $\mathfrak{m}_*^+$ in norm, then, for each $t \in R$,

$\{(D\varphi_n : D\varphi_0)_t\}_n$ converges to $(D\varphi : D\varphi_0)_t$ in the strong*-topology. Furthermore, the convergence is uniform on t in each finite interval.

Proof. We notice that $\mathfrak{M}\xi_0$ is a common core for positive self-adjoint operators $\Delta^{\frac{1}{2}}_{\varphi\varphi_0}$ and $\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0}$, $n = 1,2,3,\ldots$. In particular, $(\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)\mathfrak{M}\xi_0$ is dense. For $\zeta = (\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)x\xi_0$, $x \in \mathfrak{M}$, in this dense subspace, we compute

$$\begin{aligned}
&\{(\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1} - (\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)^{-1}\}\zeta \\
&\qquad = (\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1}(\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)x\xi_0 - x\xi_0 \\
&\qquad = (\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1}\{(\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)x\xi_0 - (\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)x\xi_0\} \\
&\qquad = (\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1}\{\Delta^{\frac{1}{2}}_{\varphi\varphi_0}x\xi_0 - \Delta^{\frac{1}{2}}_{\varphi_n\varphi_0}x\xi_0\} \\
&\qquad = (\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1}Jx^*(\xi_\varphi - \xi_{\varphi_n}) ,
\end{aligned}$$

where ξ_{φ_n}, ξ_φ are unique implementing vectors in $P^\natural$ for φ_n, φ respectively. The positivity of each $\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0}$ guarantees $\|(\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1}\| \leq 1$ and $\{\xi_{\varphi_n}\}_n$ converges to ξ_φ by the assumption (see [2], [17]) so that the above quantity converges to 0 for each ζ, that is, $\{(\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1}\}_n$ converges to $(\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)^{-1}$ in the strong topology. We set a bounded continuous function f on [0,1] by

$$f(t) = \begin{cases} 0, & \text{if } t = 0 \\ (\log(t^{-1} - 1)^2 + i)^{-1}, & \text{if } 0 < t \leq 1 \end{cases}$$

Kaplansky's argument, [24] (see also Theorem II, 4.7, [43]), shows that $\{f((\Delta^{\frac{1}{2}}_{\varphi_n\varphi_0} + 1)^{-1})\}_n$ converges to $f((\Delta^{\frac{1}{2}}_{\varphi\varphi_0} + 1)^{-1})$ in the strong topology, that is, a sequence $\{\log \Delta_{\varphi_n\varphi_0}\}_n$ of (unbounded) self-adjoint operators converges to $\log \Delta_{\varphi\varphi_0}$ in the strong resolvent sense. Thus, the result follows from Trotter's theorem on the resolvent convergence, [36].

The following result, which will not be used later, is of independent interest, and is inspired by [21].

Theorem 1.4.6. Let $\{\varphi_n\}_{n=1,2,3,\dots}$ be a sequence of faithful functionals in $\mathfrak{M}_*^+$ satisfying $\varphi_n \leq \ell\varphi_0$ with some $\ell \geq 0$. If, for each $t \in \mathbb{R}$, a sequence $\{(D\varphi_n : D\varphi_0)_t\}_n$ is convergent in the strong*-topology, then there exists a faithful φ in $\mathfrak{M}_*^+$ such that $\{\varphi_n\}$ converges to φ in $\sigma(\mathfrak{M}_*,\mathfrak{M})$-topology, and, for each $t \in \mathbb{R}$, $(D\varphi : D\varphi_0)_t$ is the strong*-limit of $\{(D\varphi_n : D\varphi_0)_t\}_n$.

Proof. Let u_t, $t \in \mathbb{R}$, denote the strong*-limit of the sequence $\{(D\varphi_n : D\varphi_0)_t\}_n$. Since each $(D\varphi_n : D\varphi_0)_t$ is $\sigma^{\varphi_0} - 1$ cocycle, so is u_t. Also, $\{u_t\Delta^{it}_{\varphi_0}\}_{t\in\mathbb{R}}$ is a <u>weakly measurable</u> one parameter group of unitaries, that is, $(u_t\Delta^{it}_{\varphi_0}\zeta\,|\,\eta)$ is a measurable function on t for any $\zeta, \eta \in \mathcal{H}$. It follows from a result of von Neumann, [36], that $\{u_t\}_{t\in\mathbb{R}}$ is a continuous cocycle. We thus conclude that there exists a unique faithful semi-finite normal weight ω on $\mathfrak{M}$ satisfying $(D\omega : D\varphi_0)_t = u_t$, $t \in \mathbb{R}$.

Let ψ be an arbitrary accumulation point of the weakly relatively $\sigma(\mathfrak{M}_*,\mathfrak{M})$-compact set $\{\varphi_1, \varphi_2, \dots\}$ in $\mathfrak{M}_*^+$. It suffices to show $\psi = \omega$. (Then, $\{\varphi_1, \varphi_2, \dots\}$ admits a unique accumulation point.) By passing to a subsequence, we may assume that $\{\varphi_n\}$ converges to ψ in

$\sigma(\mathfrak{M}_*,\mathfrak{M})$-topology.

For each $x,y \in \mathfrak{M}$, let $f_n(z)$ $(= f_n^{x,y}(z))$, $n = 1,2,\ldots,$ denote the relative K.M.S. function determined by $(\varphi_n, \varphi_0, x, y)$ with boundary values:

$$f_n(t) = \varphi_0(x(D\varphi_n : D\varphi_0)_t \sigma_t^{\varphi_0}(y)), \quad t \in \mathbb{R},$$

$$f_n(t - i) = \varphi_n((D\varphi_n : D\varphi_0)_t \sigma_t^{\varphi_0}(y)x), \quad t \in \mathbb{R}.$$

Firstly, we examine the behavior of the sequence $\{f_n(z)\}$ on two boundaries. On the boundary $z = t \in \mathbb{R}$, the sequence $\{f_n(t)\}_n$ of the functions converges to $\varphi_0(x(D\omega : D\varphi_0)_t \sigma_t^{\varphi_0}(y))$ since $\{(D\varphi_n : D\varphi_0)_t\}_n$ tends to $(D\omega : D\varphi_0)_t$ in the strong*-topology and φ_0 is σ-weakly continuous. On the boundary $z = t - i$, $t \in \mathbb{R}$, the sequence $\{f_n(t - i)\}_n$ of the functions converges to $\psi((D\omega : D\varphi_0)_t \sigma_t^{\varphi_0}(y)x)$ due to [1]. Secondly, we examine the behavior of the sequence $\{f_n(z)\}$ on $-1 \leq \mathrm{Im}\, z \leq 0$. Since $\{\varphi_n\}$ is uniformly bounded, the sequence $\{f_n(z)\}_n$ is uniformly bounded on $-1 \leq \mathrm{Im}\, z \leq 0$. In particular, the sequence is a normal family on $-1 < \mathrm{Im}\, z < 0$. Thus, by passing to a subsequence, we may assume that $F_n(z)$ converges to $f(z) = \lim_n f_n(z)$ uniformly on each compact set in $-1 < \mathrm{Im}\, z < 0$. Also $f(z)$ is uniformly bounded on $-1 \leq \mathrm{Im}\, z \leq 0$ and has the boundary values:

$$f(t) = \varphi_0(x(D\omega : D\varphi_0)_t \sigma_t^{\varphi_0}(y)), \quad t \in \mathbb{R}$$

$$f(t - i) = \psi((D\omega : D\varphi_0)_t \sigma_t^{\varphi_0}(y)x), \quad t \in \mathbb{R}.$$

Since $f(z)$ is analytic on $-1 < \mathrm{Im}\, z < 0$, the second half of

Theorem 1.4.4 yields that $(D\omega : D\varphi_0)_t = (D\psi : D\varphi_0)_t$, $t \in \mathbb{R}$, that is, $\omega = \psi$. (Q.E.D.)

We remark that in this theorem the additional assumption (which is stronger than the one of Theorem 1.4.5) is indispensable. In fact, let $\{\varphi_n\}$ be an increasing sequence of faithful functionals in $\mathfrak{M}_*^+$ such that $\omega = \lim_n \varphi_n$ gives rise to a faithful semi-finite normal weight. Then, $\{(D\varphi_n : D\varphi_0)_t\}_n$ converges to $\{(D\omega : D\varphi_0)_t\}_n$ in the strong*-topology, [7].

Chapter II Canonical Standard Form

This chapter is devoted to a constuuction of the canonical standard form from a given von Neumann algebra.

Throughout the chapter, we fix an abstract von Neumann algebra $\mathfrak{M}$. We begin with constructing a Hilbert space by making use of Radon-Nikodym cocycles, studied in Chapter I. The construction is carried out without fixing a distinguished functional on $\mathfrak{M}$ so that this Hilbert space is canonically attached to $\mathfrak{M}$. We then let the algebra $\mathfrak{M}$ act on this canonical Hilbert space in a canonical fashion to obtain the canonical standard form.

Although our construction is canonical, it is convenient to consider a fixed faithful $\varphi_0 \in \mathfrak{M}_*^+$ for proofs (see Remark 2.2.15). We shall fix it throughout the chapter and denote the standard form constructed from φ_0, via the G.N.S.-construction, by $(\mathfrak{M} = \pi_0(\mathfrak{M}), \mathfrak{H}_0, J_0, \mathcal{P}_0^\natural)$, [17]. Namely, $(\pi_0, \mathfrak{H}_0)$ is the cyclic representation of $\mathfrak{M}$ induced by φ_0, and ξ_0 is the cyclic and separating vector in $\mathfrak{H}_0$ for $\mathfrak{M} = \pi_0(\mathfrak{M})$ satisfying $\varphi_0 = \omega_{\xi_0}$. The natural cone $\mathcal{P}_0^\natural$ is $(\Delta_{\xi_0}^{1/4}\mathfrak{M}_+\xi_0)^-$ as usual, where Δ_{ξ_0} is the modular operator satisfying $J_0\Delta_{\xi_0}^{\frac{1}{2}}x\xi_0 = x^*\xi_0$, $x \in \mathfrak{M}$. For arbitrary $\varphi \in \mathfrak{M}_*^+$, we shall denote a unique implementing vector in $\mathcal{P}_0^\natural$ for φ by ξ_φ, that is, $\varphi = \omega_{\xi_\varphi}$ $(\xi_{\varphi_0} = \xi_0)$.

§2.1 Canonical Hilbert Space.

In this section, we shall construct a Hilbert space from $\mathfrak{M}$ in a canonical fashion.

We begin with defining new addition and scalar multiplication (by positive numbers) on the positive part $\mathfrak{M}_*^+$ of the predual. When we deal with these new operations, we write $\sqrt{\varphi}$ instead of φ, $\varphi \in \mathfrak{M}_*^+$, to avoid confusion.

Definition 2.1.1. For $\varphi, \psi \in \mathfrak{M}_*^+$ and $\lambda \geq 0$, we write

(i) $\sqrt{\varphi} + \sqrt{\psi} = \sqrt{\chi}$, where $\chi \in \mathfrak{M}_*^+$ is given by

$$\chi(x) = (\varphi + \psi)(a^* x a), \quad x \in \mathfrak{M},$$

$$a = (D\varphi : D(\varphi + \psi))_{-i/2} + (D\psi : D(\varphi + \psi))_{-i/2} .$$

(ii) $\lambda \sqrt{\varphi} = \sqrt{\lambda^2 \varphi}$.

Remark 2.1.2. In the above definition, by cutting the algebra by the support of $\varphi + \psi$, we may assume that $\varphi + \psi$ is faithful. Since $\varphi, \psi \leq \varphi + \psi$, the above $a \in \mathfrak{M}$ makes sense by Corollary 1.4.3. We also remark that $\sqrt{\varphi} + \sqrt{\psi} = \sqrt{\varphi + \psi}$ if φ and ψ have mutually orthogonal supports.

The following can be considered as a non-commutative Hellinger integral, which will be explained shortly.

Definition 2.1.3. For $\varphi, \psi \in \mathfrak{M}_*^+$, we write

$$(\sqrt{\varphi} | \sqrt{\psi}) = (\varphi + \psi)((D\psi : D(\varphi + \psi))_{-i/2}^* (D\varphi : D(\varphi + \psi))_{-i/2}) .$$

Remark 2.1.4. We examine the above $(\cdot | \cdot)$ in the commutative case, $\mathfrak{M} = L^\infty(R; dx)$. In this case, $\mathfrak{M}_*^+$ is realized as $L^1(R; dx)_+$, the set of all positive L^1-functions. For $\varphi = f(x)$ $(\int \cdot f(x) dx)$

and $\psi = g(x)$ $(\int \cdot g(x)dx)$ in $L^1(R;dx)_+$, the Radon-Nikodym cocycles are given by

$$(D\varphi : D(\varphi + \psi))_t = f(x)^{it}/(f(x) + g(x))^{it} ,$$

$$(D\psi : D(\varphi + \psi))_t = g(x)^{it}/(f(x) + g(x))^{it} ,$$

which yields that

$$(D\varphi : D(\psi + \psi))_{-i/2} = \sqrt{f(x)}/\sqrt{f(x) + g(x)} ,$$

$$(D\psi : D(\psi + \psi))_{-i/2} = \sqrt{g(x)}/\sqrt{f(x) + g(x)} .$$

They are exactly the square roots of measure theoretic Radon-Nikodym derivatives, and one gets

$$(\sqrt{\varphi} \mid \sqrt{\psi}) = \int \frac{\sqrt{f(x)}}{\sqrt{f(x) + g(x)}} \ \frac{\sqrt{g(x)}}{\sqrt{f(x) + g(x)}} \ (f(x) + g(x))dx$$

$$= \int \sqrt{f(x)}\sqrt{g(x)}\ dx ,$$

which is known as the Hellinger integral between two finite measures $f(x)dx$ and $g(x)dx$ (see [23]).

The next lemma is important for technical reasons as well as motivation for the above definitions.

Lemma 2.1.5. Let φ, ψ be elements in $\mathfrak{m}_*^+$. We assume in addition that ψ is faithful and $\varphi \leq \ell\psi$ with some $\ell \geq 0$. Then $(D\varphi : D\psi)_{-i/2}\xi_\psi$ is exactly ξ_φ. We also have $p(D\varphi : D\psi)_{-i/2} = (D\varphi : D\psi)_{-i/2}$, where p is the support projection of φ.

Proof. By the uniqueness of a standard form (up to unitary equivalence), [17], we may assume $\psi = \varphi_0$. We consider two $\mathcal{H}_0$-valued functions f_1, f_2 given respectively by

$$f_1(z) = (D\varphi : D\varphi_0)_z \xi_0$$
$$f_2(z) = \Delta_{\varphi\varphi_0}^{iz} \xi_0 .$$

By Corollary 1.4.3, $f_1(z)$ is bounded and continuous (resp. analytic) on $-\frac{1}{2} \leq \operatorname{Im} z \leq 0$ (resp. $-\frac{1}{2} < \operatorname{Im} z < 0$). The same is true for $f_2(z)$ because ξ_0 belongs to the domain of $\Delta_{\varphi\varphi_0}^{\frac{1}{2}}$. However, for $z = t \in \mathbb{R}$ we have

$$f_1(t) = (D\varphi : D\varphi_0)_t \xi_0 = \Delta_{\varphi\varphi_0}^{it} \Delta_{\varphi_0}^{-it} \xi_0$$
$$= \Delta_{\varphi\varphi_0}^{it} \xi_0 = f_2(t) ,$$

so that two functions are identical by the uniqueness of analytic continuation. In particular, with $z = -\frac{i}{2}$, one gets

$$(D\varphi : D\varphi_0)_{-i/2} \xi_0 = \Delta_{\varphi\varphi_0}^{\frac{1}{2}} \xi_0 = \xi_\varphi .$$

The final statement is obvious because $p(D\varphi : D\varphi_0)_t = (D\varphi : D\varphi_0)_t$ due to Lemma 1.3.3,(i). (Q.E.D.)

By making use of the standard form $(\mathfrak{M}, \mathcal{H}_0, J_0, \mathcal{P}_0^\natural)$, which we fixed at the beginning, we consider the bijection

$$\Xi : \varphi \in \mathfrak{M}_*^+ \mapsto \xi_\varphi = \Xi(\varphi) \in \mathcal{P}_0^\natural .$$

<u>Lemma 2.1.6.</u> The bijection Ξ preserves addition and scalor

multiplication (by positive numbers), when we equip $\mathfrak{M}_*^+$ with the new operations introduced in Definition 2.1.1. In other words, the map $\sqrt{\varphi} \mapsto \Xi(\varphi)$ preserves addition and scalor multiplications.

Proof. When $\sqrt{\varphi} + \sqrt{\psi} = \sqrt{\chi}$, $\chi = (\varphi + \psi)(a^* x a)$, $x \in \mathfrak{M}$, as in Definition 2.1.1,(i), we have

$$\chi(x) = (xa\xi_{\varphi+\psi} | a\xi_{\varphi+\psi}) .$$

The previous lemma yields:

$$\begin{aligned} a\xi_{\varphi+\psi} &= (D\varphi : D(\varphi + \psi))_{-i/2}\xi_{\varphi+\psi} + (D\psi : D(\psi + \psi))_{-i/2}\xi_{\varphi+\psi} \\ &= \xi_\varphi + \xi_\psi , \end{aligned}$$

so that $\chi(x) = (x(\xi_\varphi + \xi_\psi)|\xi_\varphi + \xi_\psi)$, that is, $\Xi(\chi) = \xi_\chi = \xi_\varphi + \xi_\psi = \Xi(\varphi) + \Xi(\psi)$ since $\xi_\varphi + \xi_\psi$ belongs to $P_0^\natural$.

Also, for $\lambda \geq 0$, we compute

$$\lambda^2\varphi(x) = \lambda^2(x\xi_\varphi|\xi_\varphi) = (x(\lambda\xi_\varphi)|\lambda\xi_\varphi) ,$$

so that

$$\Xi(\lambda^2\varphi) = \lambda\Xi(\varphi) . \qquad \text{(Q.E.D.)}$$

The following result is immediate because we have the above lemma and the corresponding properties are all true in $P_0^\natural$:

<u>Corollary 2.1.7</u>. For $\varphi, \psi, \chi \in \mathfrak{M}_*^+$ and $\lambda, \mu \geq 0$, we have

(i) $\sqrt{\varphi} + \sqrt{\psi} = \sqrt{\psi} + \sqrt{\varphi}$,

(ii) $\sqrt{\varphi} + \sqrt{0} = \sqrt{\varphi}$,

(iii) $(\sqrt{\varphi} + \sqrt{\psi}) + \sqrt{\chi} = \sqrt{\varphi} + (\sqrt{\psi} + \sqrt{\chi})$,

(iv) $\sqrt{\varphi} + \sqrt{\chi} = \sqrt{\psi} + \sqrt{\chi}$ if and only if $\varphi = \psi$,

(v) $0\sqrt{\varphi} = \sqrt{0}$,

(vi) $(\lambda\mu)\sqrt{\varphi} = \lambda(\mu\sqrt{\varphi})$,

(vii) $\lambda(\sqrt{\varphi} + \sqrt{\psi}) = \lambda\sqrt{\varphi} + \lambda\sqrt{\psi}$.

In other words, $\mathfrak{m}_*^+$ equipped with the new operations is a commutative semi-group with the cancellation law.

Next we consider the non-commutative Hellinger integral given in Definition 2.1.3. Many properties, known for the classical Hellinger integral, [22], remain valid in our non-commutative situation.

Lemma 2.1.8. The non-commutative Hellinger integral enjoys the following properties:

(i) $(\cdot|\cdot)$ is a symmetric bilinear form on $\mathfrak{m}_*^+$, equipped with the new structure, which takes positive values.

(ii) For $\varphi, \psi \in \mathfrak{m}_*^+$, one gets $(\sqrt{\varphi}|\sqrt{\psi}) = 0$ if and only if φ and ψ have mutually orthogonal supports. In particular, $(\sqrt{\varphi}|\sqrt{\varphi}) = 0$ if and only if $\varphi = 0$.

Proof. Since we compute

$$\begin{aligned}(\sqrt{\varphi}|\sqrt{\psi}) &= (\varphi + \psi)((D\psi : D(\psi + \psi))^*_{-i/2}(D\varphi : D(\varphi + \psi))_{-i/2}) \\ &= ((D\varphi : D(\varphi + \psi))_{-i/2}\xi_{\varphi+\psi}|(D\psi : D(\varphi + \psi))_{-i/2}\xi_{\varphi+\psi}) \\ &= (\xi_\varphi|\xi_\psi) \qquad \text{(Lemma 2.1.5)},\end{aligned}$$

(i) follows from Lemma 2.1.6 and the self-duality of $\mathcal{P}_0^\natural$.

For (ii), it suffices to show that $(\xi_\varphi|\xi_\psi) = 0$ if and only if

φ and ψ have mutually orthogonal supports. First, we assume that φ and ψ have support projections p, q respectively and $p \perp q$. Then, by the last statement of Lemma 2.1.5, we conclude that

$$\begin{aligned}(\sqrt{\varphi}|\sqrt{\psi}) &= (\varphi + \psi)((D\psi : D(\varphi + \psi))^*_{-i/2} qp (D\varphi : D(\varphi + \psi))_{-i/2}) \\ &= 0 .\end{aligned}$$

Conversely, we assume that $(\xi_\varphi|\xi_\psi) = 0$. It follows from Theorem 4, [2], the projection onto $[\mathfrak{M}'\xi_\varphi]$ and the one onto $[\mathfrak{M}'\xi_\psi]$ are mutually orthogonal, that is, φ and ψ have mutually orthogonal supports.

(Q.E.D.)

The above proof shows that, through the bijection Ξ between $\mathcal{P}_0^\natural$ and $\mathfrak{M}_*^+$ (equipped with the new structure), the non-commutative Hellinger integral $(\cdot|\cdot)$ corresponds exactly to the inner product of the space $\mathcal{H}_0$.

By Corollary 2.1.7, $\mathfrak{M}_*^+$ equipped with the new structure is a commutative semi-group with the cancellation law. From this semi-group we obtain the real vector space in the usual way, which we shall denote by $(\mathcal{H}_\mathfrak{M})_{sa}$. Namely, $(\mathcal{H}_\mathfrak{M})_{sa} = \mathfrak{M}_*^+ \times \mathfrak{M}_*^+/\sim$ is the set of all equivalence classes $[\sqrt{\varphi},\sqrt{\psi}]$ of pairs $(\sqrt{\varphi},\sqrt{\psi})$ and the equivalence relation $\sim$ is determined by

$$(\sqrt{\varphi_1},\sqrt{\psi_1}) \sim (\sqrt{\varphi_2},\sqrt{\psi_2}) \text{ if and only if } \sqrt{\varphi_1} + \sqrt{\psi_2} = \sqrt{\varphi_2} + \sqrt{\psi_1} .$$

Furthermore, since this equivalence relation is compatible with the new structures on $\mathfrak{M}_*^+$, $(\mathcal{H}_\mathfrak{M})_{sa}$ is a real vector space by the following (well-defined) notion of addition and scalor multiplication (by real numbers):

$$\begin{cases} [\sqrt{\varphi_1},\sqrt{\varphi_2}] + [\sqrt{\psi_1}, \sqrt{\psi_2}] = [\sqrt{\varphi_1} + \sqrt{\psi_1}, \sqrt{\varphi_2} + \sqrt{\psi_2}] , \\ \lambda[\sqrt{\varphi},\sqrt{\psi}] = [\lambda\sqrt{\varphi}, \lambda\sqrt{\psi}] \quad \text{for} \quad \lambda \geq 0 , \\ \lambda[\sqrt{\varphi},\sqrt{\psi}] = [(-\lambda)\sqrt{\psi}, (-\lambda)\sqrt{\varphi}] \quad \text{for} \quad \lambda \leq 0 . \end{cases}$$

We now consider the map: $\sqrt{\varphi} \in \mathfrak{M}_*^+ \mapsto [\sqrt{\varphi}, 0] \in (\mathcal{H}_{\mathfrak{M}})_{sa}$. Clearly, this is injective and preserves the operations. We imbedd $\mathfrak{M}_*^+$ into $(\mathcal{H}_{\mathfrak{M}})_{sa}$ as a positive cone. We shall denote the image of this imbedding by $(\mathcal{H}_{\mathfrak{M}})_+$ and denote $[\sqrt{\varphi}, 0]$ simply by $\sqrt{\varphi}$. Since, for $\varphi, \psi \in \mathfrak{M}_*^+$, we calculate

$$\begin{aligned} [\sqrt{\varphi},\sqrt{\psi}] &= [\sqrt{\varphi},0] + [0,\sqrt{\psi}] \\ &= [\sqrt{\varphi},0] + (-1)[\sqrt{\psi},0] \\ &= \sqrt{\varphi} + (-1)\sqrt{\psi} , \end{aligned}$$

we shall write $\sqrt{\varphi} - \sqrt{\psi}$ instead of $[\sqrt{\varphi},\sqrt{\psi}]$ for convenience.

Since any element in $(\mathcal{H}_{\mathfrak{M}})_{sa}$ can be written as a difference of two elements in the positive cone $(\mathcal{H}_{\mathfrak{M}})_+$, we introduce a function $(\cdot|\cdot)$ on $(\mathcal{H}_{\mathfrak{M}})_{sa} \times (\mathcal{H}_{\mathfrak{M}})_{sa}$ by

$$\begin{aligned} (\sqrt{\varphi_1} - \sqrt{\varphi_2}|\sqrt{\psi_1} - \sqrt{\psi_2}) = (\sqrt{\varphi_1}|\sqrt{\psi_1}) - (\sqrt{\varphi_1}|\sqrt{\psi_2}) - (\sqrt{\varphi_2}|\sqrt{\psi_1}) \\ + (\sqrt{\varphi_2}|\sqrt{\psi_2}) . \end{aligned}$$

One can easily check that this is well-defined by using Lemma 2.1.8, (i).

<u>Theorem 2.1.9</u>. The above $(\cdot|\cdot)$ on $(\mathcal{H}_{\mathfrak{M}})_{sa} \times (\mathcal{H}_{\mathfrak{M}})_{sa}$ is

actually an inner product on the real vector space $(\mathcal{H}_{\mathfrak{M}})_{sa}$, and $(\mathcal{H}_{\mathfrak{M}})_{sa}$ is a real Hilbert space under this inner product. Also, $(\mathcal{H}_{\mathfrak{M}})_{+}$ is a self-dual positive cone in $(\mathcal{H}_{\mathfrak{M}})_{sa}$.

Proof. The bijection Ξ from $\mathfrak{M}_*^+$ (equipped with the new operations) onto $\mathcal{P}_0^\natural$ naturally extends to a bijective linear mapping from $(\mathcal{H}_{\mathfrak{M}})_{sa}$ onto $(\mathcal{H}_0)_{sa}$, which we shall denote by Ξ again. Here, $(\mathcal{H}_0)_{sa}$ is a real Hilbert space consisting of all fixed points of J_0. From the construction, $\Xi((\mathcal{H}_{\mathfrak{M}})_{+})$ is exactly $\mathcal{P}_0^\natural$.

As remarked after the proof of Lemma 2.1.8, the bilinear form $(\cdot|\cdot)$ on $(\mathcal{H}_{\mathfrak{M}})_{+}$ $(= \mathfrak{M}_*^+)$ corresponds exactly to the inner product restricted to $\mathcal{P}_0^\natural$ through Ξ. Thus, the function $(\cdot|\cdot)$ on $(\mathcal{H}_{\mathfrak{M}})_{sa} \times (\mathcal{H}_{\mathfrak{M}})_{sa}$ just introduced corresponds exactly to the inner product on $\mathcal{H}_0$ through the bijective linear mapping Ξ from $(\mathcal{H}_{\mathfrak{M}})_{sa}$ onto $(\mathcal{H}_0)_{sa}$. In other words, Ξ gives rise to a isomorphism from $((\mathcal{H}_0)_{sa}, \mathcal{P}_0^\natural, (\cdot|\cdot) =$ the inner product on $(\mathcal{H}_0)_{sa})$ onto $((\mathcal{H}_{\mathfrak{M}})_{sa}, (\mathcal{H}_{\mathfrak{M}})_{+}, (\cdot|\cdot))$. Thus, the theorem follows from the corresponding facts in the real Hilbert space $(\mathcal{H}_0)_{sa}$. (Q.E.D.)

Definition 2.1.10. Let $\mathcal{H}_{\mathfrak{M}}$ be the complexification of the above real Hilbert space $(\mathcal{H}_{\mathfrak{M}})_{sa}$. Since the complex Hilbert space $\mathcal{H}_{\mathfrak{M}}$ is obtained from $\mathfrak{M}$ in a canonical fashion, we call it the canonical Hilbert space (attached to $\mathfrak{M}$). Clearly, $(\mathcal{H}_{\mathfrak{M}})_{+}$ is again imbedded in $\mathcal{H}_{\mathfrak{M}}$ as a self-dual positive cone, and any element in $\mathcal{H}_{\mathfrak{M}}$ can be written as a linear combination of four elements in the cone $(\mathcal{H}_{\mathfrak{M}})_{+}$. We denote the unitary involution of $\mathcal{H}_{\mathfrak{M}}$ determined by the complex structure by $J_{\mathfrak{M}}$. Namely, $J_{\mathfrak{M}}$ is given by

$$J_{\mathfrak{m}}(\eta + i\zeta) = \eta - i\zeta, \quad \eta, \zeta \in (\mathcal{H}_{\mathfrak{m}})_{sa} .$$

We remark that Ξ from $(\mathcal{H}_{\mathfrak{m}})_{sa}$ onto $(\mathcal{H}_0)_{sa}$ naturally extends to a surjective isometry from $\mathcal{H}_{\mathfrak{m}}$ onto $\mathcal{H}_0$, under which the pair $((\mathcal{H}_{\mathfrak{m}})_+, J_{\mathfrak{m}})$ corresponds to $(\mathcal{P}_0^\natural, J_0)$.

Before proceeding further, we examine a functorial property of the correspondence: $\mathfrak{m} \mapsto \mathcal{H}_{\mathfrak{m}}$. Let $\mathfrak{n}$ be an abstract von Neumann algebra and θ be a normal *-homomorphism from $\mathfrak{m}$ onto $\mathfrak{n}$. It is known that there exists a central projection p such that θ vanishes on $(1 - p)\mathfrak{m}$ and θ gives rise to a normal *-isomorphism from $p\mathfrak{m}$ onto $\mathfrak{n}$. Thus, henceforth we shall deal with only normal *-isomorphisms whenever we consider functorial properties of our construction.

<u>Proposition 2.1.11</u>. Let $\mathfrak{n}$ be an abstract von Neumann algebra and θ be a normal *-isomorphism from $\mathfrak{m}$ onto $\mathfrak{n}$. Then θ naturally induces a surjective isometry u_θ from $\mathcal{H}_{\mathfrak{n}}$ onto $\mathcal{H}_{\mathfrak{m}}$, which maps $(\mathcal{H}_{\mathfrak{n}})_+$ onto $(\mathcal{H}_{\mathfrak{m}})_+$. In particular, u_θ intertwines $J_{\mathfrak{n}}$ and $J_{\mathfrak{m}}$.

Proof. The map $\theta_* : \varphi \in \mathfrak{n}_* \mapsto \varphi \circ \theta \in \mathfrak{m}_*$ gives rise to a surjective isometry from $\mathfrak{n}_*$ onto $\mathfrak{m}_*$. Furthermore, as θ being a *-isomorphism, $\theta_*(\mathfrak{n}_*^+) = \mathfrak{m}_*^+$.

For $\varphi, \psi \in \mathfrak{n}_*^+$ (ψ faithful), one obtains

$$(D(\theta_*\varphi) : D(\theta_*\psi))_t = \theta^{-1}((D\varphi : D\psi)_t) .$$

In fact, this follows from the second half of Theorem 1.4.4 because $\theta^{-1}((D\varphi : D\psi)_t)$ satisfies the relative K.M.S.-condition for $\theta_*\varphi$ and $\theta_*\psi$. Therefore, whenever $\varphi \leq \ell\psi$, one obtains

$$(2.1.1) \qquad (D(\theta_*\varphi) : D(\theta_*\psi))_{-i/2} = \theta^{-1}((D\varphi : D\psi)_{-i/2}) ,$$

which guarantees that the map θ_* from $\mathfrak{N}_*^+$ onto $\mathfrak{M}_*^+$ preserves the operations on both $\mathfrak{M}_*^+$ and $\mathfrak{N}_*^+$ introduced in Definition 2.1.1. Thus, through the construction of $\mathcal{H}_{\mathfrak{M}}$ and $\mathcal{H}_{\mathfrak{N}}$, θ_* extends to a bijective linear mapping u_θ from $\mathcal{H}_{\mathfrak{N}}$ onto $\mathcal{H}_{\mathfrak{M}}$, and clearly $u_\theta((\mathcal{H}_{\mathfrak{M}})_+) = (\mathcal{H}_{\mathfrak{N}})_+$.

The above (2.1.1) also implies that θ_* preserves the non-commutative Hellinger integral on $\mathfrak{N}_*^+$ and $\mathfrak{M}_*^+$. Thus, the above u_θ is an isometry. (Q.E.D.)

§2.2 Canonical Standard Form.

In the previous section, we gave a construction of the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$ (as well as $(\mathcal{H}_{\mathfrak{M}})_+$ and $J_{\mathfrak{M}}$). In this section we let $\mathfrak{M}$ act on $\mathcal{H}_{\mathfrak{M}}$ canonically to obtain the canonical standard form $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$.

For the moment, we fix $\varphi \in \mathfrak{M}_*^+$ and $x, y \in \mathfrak{M}$. By considering the reduced von Neumann algebra $\mathfrak{M}p$ by the support projection p of φ, we may assume that φ is faithful and gives rise to the modular automorphism σ_t^φ, $t \in \mathbf{R}$, on $\mathfrak{M}$. We define the function

$$f_y^{\varphi,x}(t) = \varphi(\sigma_{-t}^\varphi(x)x^*yx\sigma_t^\varphi(x^*)), \quad t \in \mathbf{R} ,$$

which enjoys the following property:

<u>Lemma 2.2.1.</u> The function $f_y^{\varphi,x}$ extends to a function $f_y^{\varphi,x}(z)$ which is bounded and continuous (resp. analytic) on $-\frac{1}{2} \leq \operatorname{Im} z \leq 0$ (resp. $-\frac{1}{2} < \operatorname{Im} z < 0$).

Proof. If we represent $\mathfrak{M}$ on $\mathfrak{H}_0$, we compute

$$f_y^{\varphi,x}(t) = (\sigma_{-t}^{\varphi}(x)x^* yx\sigma_t^{\varphi}(x^*)\xi_\varphi | \xi_\varphi)$$

$$= (yx\sigma_t^{\varphi}(x^*)\xi_\varphi | x\sigma_{-t}^{\varphi}(x^*)\xi_\varphi)$$

$$= (yx\Delta_\varphi^{it} x^*\xi_\varphi | x\Delta_\varphi^{-it} x^*\xi_\varphi) .$$

We consider two $\mathfrak{H}_0$-valued functions:

$$z \mapsto \Delta_\varphi^{iz} x^* \xi_\varphi ,$$

$$z \mapsto \Delta_\varphi^{-i\bar{z}} x^* \xi_\varphi .$$

Since $x^*\xi_\varphi$ belongs to the domain of $\Delta_\varphi^{\frac{1}{2}}$, the former is bounded and bountinuous (resp. analytic) on $-\frac{1}{2} \leq \text{Im } z \leq 0$ (resp. $-\frac{1}{2} < \text{Im } z < 0$), while the latter is bounded and continuous (resp. anti-analytic) on $-\frac{1}{2} \leq \text{Im } z \leq 0$ (resp. $-\frac{1}{2} < \text{Im } z < 0$). Thus, the function:

$$z \mapsto (yx\Delta_\varphi^{iz} x^*\xi_\varphi | x\Delta_\varphi^{-i\bar{z}} x^*\xi_\varphi)$$

is exactly the desired extension of $f_y^{\varphi,x}$ on $-\frac{1}{2} \leq \text{Im } z \leq 0$. (Q.E.D.)

As an easy consequence of the proof, we have:

<u>Corollary 2.2.2</u>. The map: $y \in \mathfrak{M} \rightarrow f_y^{\varphi,x}(-\frac{i}{2})$ is a positive normal linear functional on $\mathfrak{M}$.

Proof. Using the same notations as in the previous proof, we have

$$f_y^{\varphi,x}(-\frac{i}{2}) = (yx\Delta_\varphi^{\frac{1}{2}}x^*\xi_\varphi | x\Delta_\varphi^{\frac{1}{2}}x^*\xi_\varphi)$$

$$= (yxJ_0xJ_0\xi_\varphi | xJ_0xJ_0\xi_\varphi)$$

$$= \omega_{xJ_0xJ_0\xi_\varphi}(y) . \qquad \text{(Q.E.D.)}$$

Definition 2.2.3. Let $\rho(x)\varphi$ be a positive normal linear functional on $\mathfrak{M}$ obtained in the above corollary, that is,

$$(\rho(x)\varphi)(y) = f_y^{\varphi,x}(-\frac{i}{2}); \quad x,y \in \mathfrak{M}, \quad \varphi \in \mathfrak{M}_*^+ .$$

The map $\rho(x) : \varphi \in \mathfrak{M}_*^+ \mapsto \rho(x)\varphi \in \mathfrak{M}_*^+$ enjoys the following properties:

Proposition 2.2.4. The map $\rho(x)$ preserves the addition and scalor multiplication (by positive numbers) given in Definition 2.1.1, that is, $\rho(x) : \sqrt{\varphi} \in (\mathcal{H}_\mathfrak{M})_+ \mapsto \sqrt{\rho(x)\varphi} \in (\mathcal{H}_\mathfrak{M})_+$ is linear. Thus, it extends uniquely to a linear mapping from $\mathcal{H}_\mathfrak{M}$ into itself, which we denote by $\rho(x)$ again. The linear operator $\rho(x)$ on $\mathcal{H}_\mathfrak{M}$ is bounded and the map $\rho : x \in \mathfrak{M} \mapsto \rho(x) \in \mathcal{L}(\mathcal{H}_\mathfrak{M})$ is multiplicative.

Proof. The proof of Corollary 2.2.2 shows that, through the isometry from $\mathcal{H}_\mathfrak{M}$ onto $\mathcal{H}_0$, $\rho(x)$ corresponds to the linear operator xJ_0xJ_0 since $xJ_0xJ_0\xi_\varphi$ is a unique implementing vector for $\rho(x)\varphi \in \mathfrak{M}_*^+$ in $P_0^\natural$. Since xJ_0xJ_0 is certainly a bounded operator on $\mathcal{H}_0$, $\rho(x)$ belongs to $\mathcal{L}(\mathcal{H}_\mathfrak{M})$. Also ρ is multiplicative because we compute, for $x,y \in \mathfrak{M}$,

$$(xJ_0xJ_0)(yJ_0yJ_0) = xyJ_0xyJ_0 . \qquad \text{(Q.E.D.)}$$

For $x \in \mathcal{L}(\mathcal{H}_{\mathfrak{m}})$, the exponential $\exp x$ is given by

$$\exp x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n .$$

Clearly, the exponential function enjoys the following properties:

$$\|\exp x\| \leq \exp\|x\| ,$$

$$(\exp x)(\exp y) = \exp(x + y) \quad \text{if} \quad xy = yx .$$

It follows from the above lemma that the map:

$$t \in R \to \rho(\exp(tx)) \in \mathcal{L}(\mathcal{H}_{\mathfrak{m}})$$

gives rise to a one parameter group of operators in $\mathcal{L}(\mathcal{H}_{\mathfrak{m}})$.

Lemma 2.2.5. The above one parameter group is uniformly continuous so that it admits a bounded infinitesimal generator.

Proof. Through Ξ, $\rho(\exp(tx))$ corresponds to $\exp(tx)J_0\exp(tx)J_0$ as we have already seen. Since J_0 is a unitary involution, one computes

$$J_0\exp(tx)J_0 = J_0\left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n x^n\right) J_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} t^n (J_0 x J_0)^n$$

$$= \exp(tJ_0xJ_0) .$$

Since J_0xJ_0 is in $\mathfrak{m}'$, we conclude that

$$\exp(tx)J_0\exp(tx)J_0 = \exp(t(x + J_0xJ_0)) .$$

Therefore, the infinitesimal generator is $x + J_0xJ_0$. (Q.E.D.)

Definition 2.2.6. We denote the infinitesimal generator of the one parameter group: $t \in \mathbb{R} \mapsto \rho(\exp(tx)) \in \mathcal{L}(\mathcal{H}_{\mathfrak{M}})$ by $\delta(x)$.

Lemma 2.2.7. The map $\delta : x \in \mathfrak{M} \mapsto \delta(x) \in \mathcal{L}(\mathcal{H}_{\mathfrak{M}})$ is a real Lie algebra homomorphism, namely, for $x,y \in \mathfrak{M}$ and $\lambda \in \mathbb{R}$, we have $\delta(x)\delta(y) - \delta(y)\delta(x) = \delta(xy - yx)$ and $\lambda\delta(x) = \delta(\lambda x)$.

Proof. The result follows from the fact that ρ is multiplicative. However, we have already known that $\delta(x)$ corresponds to $x + J_0xJ_0$ through the isometry Ξ from $\mathcal{H}_{\mathfrak{M}}$ onto $\mathcal{H}_0$ so that we show it by direct computation.

$$\begin{aligned} \lambda x + J_0(\lambda x)J_0 &= \lambda x + \lambda J_0xJ_0 \qquad \text{(since } \lambda \text{ is real)} \\ &= \lambda(x + J_0xJ_0) , \end{aligned}$$

$$\begin{aligned} &(x + J_0xJ_0)(y + J_0yJ_0) - (y + J_0yJ_0)(x + J_0xJ_0) \\ &\quad = xy + J_0xyJ_0 + xJ_0yJ + J_0xJ_0y - (yx + J_0yxJ_0 + yJ_0xJ_0 + JyJ_0x) \\ &\quad = (xy - yx) + J_0(xy - yx)J_0 . \qquad \text{(since } x,y \in \mathfrak{M};\ J_0xJ_0, J_0yJ_0 \in \mathfrak{M}') \end{aligned}$$

(Q.E.D.)

Now we are at the position to define a representation of $\mathfrak{M}$ on the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$.

Definition 2.2.8. For each $x \in \mathfrak{M}$, we set

$$\pi(x) = \tfrac{1}{2}(\delta(x) - i\delta(ix))$$

$$\pi'(x) = \tfrac{1}{2}(\delta(x^*) + i\delta(ix^*)) .$$

Theorem 2.2.9. The above π (resp. π') gives rise to a faithful normal representation (resp. anti-representation) of $\mathfrak{M}$ on the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$. Furthermore, the quadruple $(\pi(\mathfrak{M}),\mathcal{H}_{\mathfrak{M}},J_{\mathfrak{M}},(\mathcal{H}_{\mathfrak{M}})_+)$ is a standard form and $J_{\mathfrak{M}}\pi(x)J_{\mathfrak{M}} = \pi'(x)^*$.

Proof. Through the isometry Ξ from $\mathcal{H}_{\mathfrak{M}}$ onto $\mathcal{H}_0$, $\pi(x)$ corresponds to

$$\tfrac{1}{2}\{(x + J_0xJ_0) - i(ix + J_0ixJ_0)\} = \tfrac{1}{2}(x + J_0xJ_0 + x - J_0xJ_0) = x ,$$

while $\pi'(x)$ corresponds to

$$\begin{aligned}\tfrac{1}{2}\{(x^* + J_0x^*J_0) + i(ix^* + J_0ix^*J_0)\} &= \tfrac{1}{2}(x^* + J_0x^*J_0 - x^* + J_0x^*J_0)\\ &= J_0x^*J_0 .\end{aligned}$$

Thus, the isometry Ξ sends $(\pi(\mathfrak{M}),\mathcal{H}_{\mathfrak{M}},J_{\mathfrak{M}},(\mathcal{H}_{\mathfrak{M}})_+)$ exactly to $(\mathfrak{M},\mathcal{H}_0,J_0,P_0^{\natural})$. (Q.E.D.)

Definition 2.2.10. By identifying $\mathfrak{M}$ with $\pi(\mathfrak{M})$ in the theorem, we obtain the standard form $(\mathfrak{M},\mathcal{H}_{\mathfrak{M}},J_{\mathfrak{M}},(\mathcal{H}_{\mathfrak{M}})_+)$, which we call the canonical standard form (associated with $\mathfrak{M}$).

Remark 2.2.11. As soon as a functional $\varphi \in \mathfrak{M}_*^+$ is given, a "vector" $\sqrt{\varphi}$ in $(\mathcal{H}_{\mathfrak{M}})_+$ is assigned. From our construction it is a unique implementing vector in the self-dual cone $(\mathcal{H}_{\mathfrak{M}})_+$ for φ, that is, $\varphi(x) = (x\sqrt{\varphi}|\sqrt{\varphi})$, $x \in \mathfrak{M}$. In particular, whenever two $\varphi, \psi \in \mathfrak{M}_*^+$ (ψ faithful) are given, one can construct the relative modular operator $\Delta_{\varphi\psi}$ as an operator on the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$ as in Chapter I. This operator is attached to φ, ψ canonically so that we shall make use of these "canonical" relative modular operators to construct canonical L^p-spaces in the next chapter.

Remark 2.2.12. It is also possible to construct a relative modular operator $\Delta_{\varphi\psi}$ on $\mathcal{H}_{\mathfrak{M}}$ from given two semi-finite normal weights φ, ψ on $\mathfrak{M}$ (ψ faithful). Let j be the anti-automorphism from $\mathfrak{M}$ (acting on $\mathcal{H}_{\mathfrak{M}}$) onto $\mathfrak{M}'$ given by $j(x) = J_{\mathfrak{M}}x^*J_{\mathfrak{M}}$, $x \in \mathfrak{M}$. By using notion of a spatial derivative, [7], a relative modular operator $\Delta_{\varphi\psi}$ is given by $\Delta_{\varphi\psi} = d\varphi/d(\psi \circ j)$ as an operator on the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$.

We now consider a functorial property. When a normal *-isomorphism θ from $\mathfrak{M}$ onto $\mathfrak{N}$ is given, θ naturally induces the surjective isometry u_θ from $\mathcal{H}_{\mathfrak{N}}$ onto $\mathcal{H}_{\mathfrak{M}}$ ($u_\theta((\mathcal{H}_{\mathfrak{N}})_+) = (\mathcal{H}_{\mathfrak{M}})_+$) by Proposition 2.1.11.

Proposition 2.2.13. The surjective isometry u_θ in Proposition 2.1.11 satisfies $\theta(x) = u_\theta^* x u_\theta$, $x \in \mathfrak{M}$, that is, $\theta = \mathrm{Ad}u_\theta^*$.

Proof. For $\varphi \in \mathfrak{N}_*^+$ with $\psi = \theta_*\varphi \in \mathfrak{M}_*^+$, we know, [41], $\theta \circ \sigma_t^\psi \circ \theta^{-1} = \sigma_t^\varphi$, $t \in \mathbb{R}$. The function f introduced at the beginning of this section satisfies $(x,y \in \mathfrak{M})$

$$f_y^{\psi,x}(t) = \psi(\sigma_{-t}^\psi(x^*)xyx\sigma_t^\psi(x^*))$$

$$= \varphi(\theta(\sigma_{-t}^\psi(x^*))\theta(x)\theta(y)\theta(x)\theta(\sigma_t^\psi(x^*)))$$

$$= \varphi(\sigma^{\varphi}_{-t}(\theta(x)^*)\theta(x)\theta(y)\theta(x)\sigma^{\varphi}_{t}(\theta(x)^*))$$
$$= f^{\varphi,\theta(x)}_{\theta(y)}(t) .$$

Thus, we have

$$\begin{aligned}(\theta_*(\rho_{\mathfrak{n}}(\theta(x))\varphi))(y) &= (\rho_{\mathfrak{n}}(\theta(x))\varphi)(\theta(y)) \\ &= f^{\varphi,\theta(x)}_{\theta(y)}(t) \\ &= f^{\psi,x}_{y}(t) \\ &= (\rho_{\mathfrak{m}}(x)\psi)(y) \\ &= (\rho_{\mathfrak{m}}(x)(\theta_*\varphi))(y) ,\end{aligned}$$

where $\rho_{\mathfrak{m}}$ and $\rho_{\mathfrak{n}}$ should be understood in obvious ways. Since $\rho_{\mathfrak{n}}$ and $\rho_{\mathfrak{m}}$ are linear and $\mathfrak{m}^+_* = (\mathcal{H}_{\mathfrak{m}})_+$ and $\mathfrak{n}^+_* = (\mathcal{H}_{\mathfrak{n}})_+$ span $\mathcal{H}_{\mathfrak{m}}$ and $\mathcal{H}_{\mathfrak{n}}$ respectively, we have $u_\theta \circ \rho_{\mathfrak{n}}(\theta(x)) = \rho_{\mathfrak{m}}(x) \circ u_\theta$, $x \in \mathfrak{m}$, that is, $\rho_{\mathfrak{n}}(\theta(x)) = u^*_\theta \rho_{\mathfrak{m}}(x) u_\theta$. We thus conclude that $u^*_\theta \pi_{\mathfrak{m}}(x) u_\theta = \pi_{\mathfrak{n}}(\theta(x))$, $x \in \mathfrak{m}$, that is, $\theta = \mathrm{Ad}u^*_\theta$. (Q.E.D.)

Remark 2.2.14. Proposition 2.1.11, 2.2.13 assert that, when a normal *-isomorphism θ from $\mathfrak{m}$ onto $\mathfrak{n}$ is given, we have the surjective isometry u_θ from $\mathcal{H}_{\mathfrak{n}}$ onto $\mathcal{H}_{\mathfrak{m}}$ with $u_\theta((\mathcal{H}_{\mathfrak{n}})_+) = (\mathcal{H}_{\mathfrak{m}})_+$ and $\theta = \mathrm{Ad}u^*_\theta$. Therefore, this u_θ is a canonical implementation for θ, [2], [6], [17]. We thus showed that, for a given θ, there exists the canonical implementation for θ.

Remark 2.2.15. Our construction of the canonical standard form looks to require the existence of a faithful normal state φ_0 on $\mathfrak{m}$. But this restriction is superficial. Using the fact that any countable family of σ-finite projections is dominated by a σ-finite projection, we can remove the assumption on the existence of a faithful normal state. Namely, if $\{p_\iota\}$ is an increasing net of σ-finite projections with $\lim_\iota p_\iota = 1$, then we have the natural inductive system $\{\mathcal{H}_{\mathfrak{M}p_\iota}\}$ of the canonical Hilbert spaces associated with $\mathfrak{M}p_\iota$. Then the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$ associated with $\mathfrak{M}$ is defined by

$$\mathcal{H}_{\mathfrak{M}} = \varinjlim \mathcal{H}_{\mathfrak{M}p_\iota} .$$

It is also possible to let $\mathfrak{M}$ act on $\mathcal{H}_{\mathfrak{M}}$ canonically and to obtain the canonical standard form $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$ in this way.

Chapter III Canonical L^p-spaces

In this chapter, we give a construction of canonical L^p-spaces from a given von Neumann algebra.

Throughout the chapter, we fix an arbitrary abstract von Neumann algebra $\mathfrak{M}$ (except in §3.1). We have already had the canonical standard form $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$. Furthermore, whenever a pair of two elements in $\mathfrak{M}_*^+$ is given, we have the "canonical" relative modular operator on $\mathcal{H}_{\mathfrak{M}}$. (See Remark 2.2.11.) We shall construct L^p-spaces by using these relative modular operators (and polar decompositions of elements in $\mathfrak{M}_*^+$) so that our L^p-spaces will be constructed in a canonical fashion.

§3.1 Preliminaries.

In this section, we collect some standard results on (unbounded) measurable operators, [31], [39], [40], crossed products and dual weights [10], [19], [42], and Haagerup's L^p-spaces, [20].

Measurable Operators

Let $\mathfrak{R}$ be a semi-finite von Neumann algebra on a Hilbert space $\mathcal{K}$ with a faithful semi-finite normal trace τ. The following concept, whose origin is von Neumann's T-theorem (see also [15], [39]), is one of the most important concepts in the theory of operator algebras:

Definition 3.1.1 ([31], [39]). Let T be a closed operator affiliated with $\mathfrak{R}$, which is not necessarily bounded. We say that T is τ-measurable if there exists a sequence $\{p_n\}$ of projections in $\mathfrak{R}$ satisfying

(i) $\|Tp_n\| < \infty$ for each n,

(ii) $\tau(1 - p_n)$ is finite for each n,

(iii) $p_n \uparrow 1$ as $n \to \infty$.

The following criterion due to Nelson is useful:

<u>Proposition 3.1.2</u> ([31]). Let T be a closed operator affiliated with $\mathfrak{R}$, with the polar decomposition $T = u|T|$, and the spectral decomposition $|T| = \int_0^\infty \lambda de(\lambda)$. The operator T is τ-measurable if and only if $\tau(1 - e(\lambda)) < \infty$ for λ sufficiently large and $u \in \mathfrak{R}$.

Let T and S be τ-measurable operators. Their adjoint operators T^* and S^* are τ-measurable. Furthermore, their algebraic sum $T + S$ (with $\mathcal{D}(T + S) = \mathcal{D}(T) \cap \mathcal{D}(S)$) and their algebraic product TS (with $\mathcal{D}(TS) = \{\xi \in \mathcal{D}(S); S\xi \in \mathcal{D}(T)\}$) are known to be densely defined and closable, and the closures $(T + S)^-$ and $(TS)^-$ are again τ-measurable. In the literature, $(T + S)^-$ and $(TS)^-$ are called the strong sum and strong product respectively. We shall simply write $T + S$, TS respectively by omitting the closure signs which will never make confusion due to the following:

<u>Theorem 3.1.3</u> [39]. (i) The set of all τ-measurable operators is a *-algebra relative to the above mentioned operations.

(ii) If τ-measurable operators T and S satisfy $T \subseteq S$, then T and S are identical.

It is convenient to equip the set of all τ-measurable operators with the following topology:

Definition 3.1.4 ([31], [40]). The measure topology (on the set of all τ-measurable operators) is a linear topology (not necessarily locally convex) whose fundamental system of neighborhoods around 0 is given by

$$\mathfrak{O}_{\varepsilon,\delta} = \{\tau\text{-measurable operator } T;\ \|Tp\| \leq \varepsilon,$$
$$\tau(1 - p) \leq \delta \text{ with some projection } p \in \mathcal{R}\}$$

where ε, δ are arbitrary positive numbers.

It is known that the set of all τ-measurable operators is complete under the measure topology.

Crossed Products

Let $(\mathfrak{M},\mathfrak{H},J,\mathfrak{P}^{\natural})$ be a standard form. Although crossed products are defined for a continuous automorphism action of a locally compact group on $\mathfrak{M}$, we shall use the ones given by modular automorphism groups. Let $\{\sigma_t^{\varphi_0}\}_{t\in R}$ be the modular automorphism group associated with a distinguished faithful $\varphi_0 \in \mathfrak{M}_*^+$. Objects which we shall consider shortly do not depend on a choice of φ_0 we shall simply write σ_t, $t \in R$, Δ without indicating φ_0, unless any confusion occurs (see Theorem 3.1.8).

Definition 3.1.5 ([42]). Let $\mathcal{K}$ be the tensor product $\mathfrak{H} \otimes L^2(R)$ of $\mathfrak{H}$ and the Hilbert space $L^2(R)$ consisting of all square integrable functions on R with respect to the Lebesgue measure dx. We sometimes identify $\mathcal{K}$ with the Hilbert space $L^2(R;\mathfrak{H})$ consisting of all $\mathfrak{H}$-valued square integrable functions. Let

π be the faithful normal representation of $\mathfrak{M}$ on $\mathcal{K}$ given by $\pi(x) = x \otimes 1$, $x \in \mathfrak{M}$, where 1 denotes the identity operator on $L^2(\mathbb{R})$. We also define a continuous unitary representation u $(= u_{\varphi_0})$ of $\mathbb{R}$ on $\mathcal{K}$ given by $u(t) = \Delta^{it} \otimes \lambda(t)$ $(= \Delta_{\varphi_0}^{it} \otimes \lambda(t))$, $t \in \mathbb{R}$. Here, $\lambda(t)$ is the left translation on $L^2(\mathbb{R})$, that is, $(\lambda(t)f)(s) = f(s - t)$, $f \in L^2(\mathbb{R})$, $t,s \in \mathbb{R}$. The <u>crossed product</u> $\mathfrak{R} = (\mathfrak{M} \times_{\sigma^{\varphi_0}} \mathbb{R})$ is the von Neumann algebra on $\mathcal{K}$ generated by $\pi(\mathfrak{M})$ and $u(\mathbb{R})$.

<u>Remark 3.1.6</u>. (i) Let L be the unique positive self-adjoint operator on $L^2(\mathbb{R})$ satisfying $\lambda(t) = L^{it}$, $t \in \mathbb{R}$. (Formally, L is $\exp(\frac{1}{i} \frac{d}{dt})$.) We then remark that

$$u(t) = \Delta^{it} \otimes L^{it} = (\Delta \otimes L)^{it} ,$$

so that $\Delta \otimes L$ is (the exponential of) the infinitesimal generator of the unitary representation $u(t)$. Here, we also remark that tensor products of two closed operators are treated nicely in [38].

(ii) Our definition of the crossed product $\mathfrak{R}$ is slightly different from the usual one, [42]. However, they are spatially isomorphic to each other via a unitary operator w on $\mathcal{K}$ given by $(wf)(t) = \Delta^{it}f(t)$, $f \in L^2(\mathbb{R};\mathcal{H})$.

(iii) The algebra $\mathfrak{M}$ can be imbedded into $\mathfrak{R}$ by $\pi : x \in \mathfrak{M} \mapsto \pi(x) = x \otimes 1 \in \mathfrak{R}$, which is exactly an <u>amplification</u>.

(iv) Two representations π and u enjoy the following <u>covariance relation</u>:

$$\sigma_t(x) = u(t)xu(t)^*, \quad t \in \mathbb{R}, \quad x \in \mathfrak{M}$$

$$(\text{or} \quad \pi(\sigma_t(x)) = u(t)\pi(x)u(t)^*) .$$

We consider a unitary representation v of $\mathbb{R}$ on $\mathcal{K}$ given by

$$(v(s)f)(t) = e^{-ist}f(t), \quad f \in L^2(\mathbb{R};\mathcal{H}), \quad t,s \in \mathbb{R} .$$

Straight-forward calculation yields

$$v(s)u(t)v(s)^* = e^{-ist}u(t) ,$$

$$v(s)\pi(x)v(s)^* = \pi(x) .$$

Definition 3.1.7 ([42]). By the above two relations, $\mathrm{Ad}v(s)$, $s \in \mathbb{R}$, gives rise to a one parameter automorphism group $\{\theta_s\}_{s\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathfrak{R}$, which we call the dual action (of $\sigma = \sigma^{\varphi_0}$).

It is known that $\mathfrak{M}$ (imbedded in $\mathfrak{R}$) is exactly the fixed point subalgebra $\mathfrak{R}^\theta$ of $\mathfrak{R}$ under the dual action, [10], [42]. The following result due to Takesaki follows from the existence of Radon-Nikodym cocycles.

Theorem 3.1.8 ([42]). The pair $(\mathfrak{R},\theta)$ does not depend on a choice of φ_0 in the sense that for any two faithful $\varphi_0, \psi_0 \in \mathfrak{M}_*^+$, there exists a spatial isomorphism from $\mathfrak{M} \times_{\sigma^{\varphi_0}} \mathbb{R}$ onto $\mathfrak{M} \times_{\sigma^{\psi_0}} \mathbb{R}$ which intertwines the respective dual actions.

Dual Weights

We shall deal with only semi-finite normal weights on a von

Neumann algebra $\mathfrak{M}$, and denote the set of all such weights simply by $P(\mathfrak{M})$.

Although the theory of dual weights was initiated by [42], we shall take an approach due to Haagerup, [18], [19]. For $x \in \mathfrak{R}_+$, we shall consider an integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} \theta_s(x)dx$ taking a value in the extended positive part $\hat{\mathfrak{M}}_+$ of $\mathfrak{M}$, [18]. We notice that the map:

$$x \in \mathfrak{R}_+ \mapsto \varepsilon(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta_s(x)ds \in \hat{\mathfrak{M}}_+$$

gives rise to a (faithful semi-finite normal) operator valued weight from $\mathfrak{R}$ onto $\mathfrak{M}$, [18].

<u>Definition 3.1.9</u> ([10], [19], [42]). For a weight $\varphi \in P(\mathfrak{M})$, we denote a weight $\varphi \circ \varepsilon \in P(\mathfrak{R})$ by $\hat{\varphi}_0$, and call it the <u>dual weight</u> of φ.

We remark that, even if φ belongs to $\mathfrak{M}_*^+$, the dual weight $\hat{\varphi}$ on $\mathfrak{R}$ is infinite, that is, $\hat{\varphi}(1) = \infty$.

We consider the dual weight $\hat{\varphi}_0$ of the distinguished $\varphi_0 \in \mathfrak{M}_*^+$, which was used to define the crossed product $\mathfrak{R}$. The modular automorphism group determined by $(\mathfrak{R}, \hat{\varphi}_0)$ is exactly $Ad(u(t))$, $t \in R$, with $u(t) = (\Delta \otimes L)^{it}$. (Remark 3.1.6,(i)) In particular, since it is inner, $\mathfrak{R}$ is semi-finite, [34], [41].

<u>Definition 3.1.10</u>. We denote the faithful weight $\hat{\varphi}_0((\Delta \otimes L)^{-1}.)$ on $\mathfrak{R}$ ([34]) by τ. Then, by the above remark, τ is a faithful (semi-finite normal) trace on $\mathfrak{R}$. For any $\varphi \in P(\mathfrak{M})$, we denote the

Radon-Nikodym derivative $d\hat{\varphi}/d\tau$ of the dual weight $\hat{\varphi} \in P(\mathcal{R})$ with respect to the trace τ by h_φ, that is, h_φ is a unique positive self-adjoint operator affiliated with $\mathcal{R}$ satisfying $\hat{\varphi} = \tau(h_\varphi \cdot)$.

Since $\theta_s(u(t)) = v(s)u(t)v(s)^* = e^{-ist}u(t)$, that is, $\theta_s(\Delta \otimes L) = e^{-s}(\Delta \otimes L)$, and $\hat{\varphi}_0$ is invariant under θ, we conclude that

$$\tau \circ \theta_s = e^{-s}\tau, \quad s \in \mathbb{R} .$$

We also remark that

$$h_0 \; (= h_{\varphi_0}) = \Delta \otimes L \; (= \Delta_{\varphi_0} \otimes L = \Delta_{\varphi_0 \varphi_0} \otimes L) .$$

More generally, we have:

<u>Lemma 3.1.11</u>. For each $\varphi \in P(\mathfrak{M})$, we have

$$h_\varphi = \Delta_{\varphi\varphi_0} \otimes L .$$

Proof. For $t \in \mathbb{R}$, we simply compute

$$\begin{aligned} h_\varphi^{it} &= (D\hat{\varphi} : D\tau)_t = (D\hat{\varphi} : D\hat{\varphi}_0)_t (D\hat{\varphi}_0 : D\tau)_t \\ &= (D(\varphi \circ \varepsilon) : D(\varphi_0 \circ \varepsilon))_t h_0^{it} \\ &= (D(\varphi \circ \varepsilon) : D(\varphi_0 \circ \varepsilon))_t (\Delta^{it} \otimes L^{it}) . \end{aligned}$$

Since [18] and Remark 3.1.6,(ii) yield:

$$\begin{aligned} (D(\varphi \circ \varepsilon) : D(\varphi_0 \circ \varepsilon))_t &= (D\varphi : D\varphi_0)_t \\ &= (D\varphi : D\varphi_0)_t \otimes 1 , \end{aligned}$$

we compute

$$\begin{aligned} h_\varphi^{it} &= ((D\varphi : D\varphi_0) \otimes 1)(\Delta^{it} \otimes L^{it}) \\ &= (\Delta_{\varphi\varphi_0}^{it} \Delta^{-it} \otimes 1)(\Delta^{it} \otimes L^{it}) \\ &= \Delta_{\varphi\varphi_0}^{it} \otimes L^{it} \\ &= (\Delta_{\varphi\varphi_0} \otimes L)^{it} . \end{aligned}$$

(Q.E.D.)

<u>Lemma 3.1.12</u>. The correspondence: $\varphi \mapsto h_\varphi$ gives rise to a bijection from $P(\mathfrak{M})$ onto the set of all positive self-adjoint operators h affiliated with $\mathfrak{R}$ satisfying $\theta_s(h) = e^{-s}h$, $s \in \mathbf{R}$.

Proof. In the proof of the previous lemma, we observed that $h_\varphi^{it} = (D\varphi : D\varphi_0)_t h_0^{it}$. Since $(D\varphi : D\varphi_0)_t$ belongs to $\mathfrak{M} = \mathfrak{R}^\theta$, one gets, for each $s \in \mathbf{R}$,

$$\begin{aligned} \theta_s(h_\varphi^{it}) &= (D\varphi : D\varphi_0)_t \theta_s(h_0^{it}) \\ &= (D\varphi : D\varphi_0)_t e^{-ist} h_0^{it} \\ &= e^{-ist} h_\varphi^{it} . \end{aligned}$$

Conversely, assume that a positive self-adjoint operator h affiliated with $\mathfrak{R}$ satisfies $\theta_s(h) = e^{-s}h$. Then a weight $\psi = \tau(h\cdot)$ on $\mathfrak{R}$ is invariant under θ so that $\psi = \hat{\varphi}$ with unique $\varphi \in P(\mathfrak{M})$ (see [19]), that is, $h = h_\varphi$. (Q.E.D.)

<u>Haagerup's L^p-spaces</u>

The following observation is crucial in Haagerup's theory of L^p-spaces:

<u>Theorem 3.1.13</u> ([20]). Let $h_\varphi = \int_0^\infty \lambda de_\varphi(\lambda)$ denote the spectral decomposition of h_φ, $\varphi \in P(\mathfrak{M})$, then

$$\tau(1 - e_\varphi(\lambda)) = \varphi(1)/\lambda, \qquad \lambda > 0 .$$

Thus, by Proposition 3.1.2, h_φ is τ-measurable if and only if $\varphi \in \mathfrak{M}_*^+$.

Proof. By Theorem 3.1.8, we may assume that $\varphi = \varphi_0$, that is, $h_\varphi = h_0$. Let f be a function in $\mathcal{S}$, the set of all rapidly decreasing functions on $\mathbb{R}$. The Fourier transform $\hat{f}$ belongs to $\mathcal{S}$ again, and, by making use of the expression $\log h_0 = \int_0^\infty \log \lambda de(\lambda)$, we compute

$$\begin{aligned}
\hat{f}(\log h_0) &= \int_0^\infty \hat{f}(\log \lambda) de(\lambda) \\
&= \int_0^\infty \left(\int_{-\infty}^\infty f(t)\lambda^{it} dt \right) de(\lambda) \\
&= \int_{-\infty}^\infty f(t) \left(\int_0^\infty \lambda^{it} de(\lambda) \right) dt \qquad \text{(Fubini's theorem)} \\
&= \int_{-\infty}^\infty f(t) h_0^{it} dt ,
\end{aligned}$$

so that the Fourier inversion formula and the definition of the dual weight $\hat{\varphi}_0$ yield, for each $f \in \mathcal{S}$,

$$\begin{aligned}
\hat{\varphi}_0(f(\log h_0)) &= \varphi_0(\hat{f}(0)1) = \hat{f}(0)\varphi_0(1) \\
&= \left(\int_{-\infty}^\infty f(t) dt \right) \varphi_0(1) .
\end{aligned}$$

Let $\{f_n\}$ be an increasing sequence of positive functions in $\mathcal{S}$ such that $\lim_{n\to\infty} f_n(t) = g(t) = e^{-t}\chi_{[\log \lambda,\infty)}(t)$, where $\chi_{[\log \lambda,\infty)}$ denotes the characteristic function of $[\log \lambda,\infty)$. Since we compute

$$\hat{\varphi}_0(g(\log h_0)) = \hat{\varphi}_0(h_0^{-1}\chi_{[\log \lambda,\infty)}(\log h_0))$$

$$= \hat{\varphi}_0(h_0^{-1}\chi_{[\lambda,\infty)}(h_0))$$

$$= \hat{\varphi}_0(h_0^{-1}(1 - e(\lambda)))$$

$$= \tau(1 - e(\lambda)) \qquad \text{(see Definition 3.1.10)},$$

the first half of the proof, the normality of $\hat{\varphi}_0$, and monotone convergence theorem imply:

$$\tau(1 - e(\lambda)) = \lim_{n\to\infty} \varphi_0(f_n(\log h_0))$$

$$= \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} f_n(t)dt\right) \varphi_0(1)$$

$$= \left(\int_{-\infty}^{\infty} g(t)dt\right) \varphi_0(1)$$

$$= \varphi_0(1)/\lambda . \qquad \text{(Q.E.D.)}$$

This theorem means that one can regard the set of all τ-measurable operators h satisfying $\theta_s(h) = e^{-s}h$, $s \in R$, as a "copy" of $\mathfrak{M}_*$, by identifying $\varphi = u|\varphi|$ (the polar decomposition) in $\mathfrak{M}_*$ with $uh_{|\varphi|}$. We also notice that this set has the natural linear structure due to Theorem 3.1.3,(i).

<u>Definition 3.1.14</u> ([20]). For the above τ-measurable $k = uh_{|\varphi|}$, we set

$$\mathrm{tr}(k) = |\varphi|(u) = \varphi(1) .$$

Obviously, tr is a linear functional.

Theorem 3.1.8 and the above definition yield the following:

<u>Proposition 3.1.15</u>. The triple $(\mathfrak{R},\theta,\mathrm{tr})$ does not depend on the choice of φ_0.

<u>Definition 3.1.16</u> ([20]). Let $L^p(\mathfrak{m};\varphi_0)$, $0 < p \leq \infty$, be the set of all τ-measurable operators k (affiliated with $\mathfrak{R} = \mathfrak{m} \times_{\sigma^{\varphi_0}} R$) satisfying $\theta_s(k) = e^{-s/p}k$, $s \in R$. On $L^{\varphi}(\mathfrak{m} : \varphi_0)$, we set $\|k\|_p = (\mathrm{tr}(|k|^p))^{1/p}$.

By Proposition 3.1.15, the isomorphism class of the normed space $\{L^p(\mathfrak{m};\varphi_0),\|\cdot\|_p\}$ does not depend on a choice of φ_0. (Thus, in [18], it was simply denoted by $L^p(\mathfrak{m})$.) However, we shall write it in this way to clarify which crossed product we are dealing with.

It follows from Theorem 3.1.3,(i) that $L^p(\mathfrak{m};\varphi_0)$ has the natural linear structure and that we can freely multiply elements in different spaces, namely $hk \in L^r(\mathfrak{m};\varphi_0)$ if $h \in L^p(\mathfrak{m};\varphi_0)$ and $k \in L^q(\mathfrak{m};\varphi_0)$, $1/p + 1/q = 1/r$. Furthermore, $\{L^1(\mathfrak{m};\varphi_0),\|\cdot\|_1\}$ is isomorphic to $\mathfrak{m}_*$ from the construction so that $L^1(\mathfrak{m};\varphi_0)$ is complete. (Woronowicz, [45], also considered this space in a slightly different method.)

We now list some properties for later reference.

<u>Proposition 3.1.17</u> ([20]). Let h, k be positive (as operators) elements in $L^1(\mathfrak{M};\varphi_0)$. Then, the Banach space valued function: $z \mapsto h^z k^{1-z} \in L^1(\mathfrak{M};\varphi_0)$ is bounded and continuous (resp. analytic) on $0 \leq \mathrm{Re}\, z \leq 1$ (resp. $0 < \mathrm{Re}\, z < 1$).

We remark that this result and the next result are slightly different forms of the relative K.M.S. condition (Theorem 1.4.4).

<u>Proposition 3.1.18</u> ([20]). For $h \in L^p(\mathfrak{M};\varphi_0)$ and $k \in L^q(\mathfrak{M};\varphi_0)$, $1/p + 1/q = 1$, we have $\mathrm{tr}(hk) = \mathrm{tr}(kh)$.

This justifies the notation "tr." Finally, the completeness of $L^p(\mathfrak{M};\varphi_0)$ $1 \leq p \leq \infty$, is a consequence of the following result and the completeness of the set of all τ-measurable operators under the measure topology.

<u>Proposition 3.1.19</u> ([20]. Let $\{k_n\}$ be a sequence in $L^p(\mathfrak{M};\varphi_0)$, $1 \leq p < \infty$, such that $\{\|k_n\|_p\}$ tends to 0 as $n \to \infty$, then $\{k_n\}$ converges to 0 in the measure topology.

Proof. Since each k_n belongs to $L^p(\mathfrak{M};\varphi_0)$, the polar decomposition has a form $k_n = u_n h_{\varphi_n}^{1/p}$ with $\varphi_n \in \mathfrak{M}_*^+$ and the assumption means that $\{\varphi_n\}$ tends to 0 in the norm topology of $\mathfrak{M}_*$. For a given small $\varepsilon > 0$, we can choose a positive number N such that $\|\varphi_n\| \leq \varepsilon^{p+1}$ for $n \geq N$. Let $\int_0^\infty \lambda de_n(\lambda)$ be the spectral decomposition of h_{φ_n}. Then, the projections $e_n(\varepsilon^p)$, $n \geq N$, in $\mathfrak{R}$ satisfy

$$\|k_n e_n(\varepsilon^p)\| = \|u_n h_{\varphi_n}^{1/p} e_n(\varepsilon^p)\| \leq \varepsilon$$

$$\tau(1 - e_n(\varepsilon^p)) = \frac{\varphi(1)}{\varepsilon^p} \leq \varepsilon \qquad \text{(Theorem 3.1.13)},$$

so that $k_n \in \mathcal{O}(\varepsilon,\varepsilon)$, $n \geq N$. (Q.E.D.)

§3.2 Homogeneous Operators.

Henceforth, we fix the canonical standard form $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$ and the relative modular operators $\Delta_{\varphi\varphi_0}$ are always the canonical ones described in Remark 2.2.11.

These relative modular operators are (-1)-homogeneous and integrable in the following sense:

<u>Definition 3.2.1</u> ([7], [26]). Let T be a closed operator on $\mathcal{H}_{\mathfrak{M}}$ whose polar decomposition is $u|T|$, and $\alpha \leq 0$ (for convenience). We say that T is <u>α-homogeneous</u> (relative to φ_0') if

(i) the phase part u belongs to $\mathfrak{M}$,

(ii) $|T|^{it}x' = \sigma'_{\alpha t}(x')|T|^{it}$, $t \in \mathbb{R}$, $x' \in \mathfrak{M}'$. Here, σ'_t is the modular automorphism group on $\mathfrak{M}'$ determined by a faithful vector functional $\varphi_0'(x') = (x'\sqrt{\varphi_0}|\sqrt{\varphi_0})$.

A (-1)-homogeneous positive self-adjoint operator T is said to be $(\varphi_0'$-) <u>integrable</u> if $\sqrt{\varphi_0}$ belongs to the domain of $\sqrt{T}$.

By considering the crossed product $\mathcal{R} = \mathfrak{M} \times_{\sigma^{\varphi_0}} \mathbb{R}$, we obtain the following relationship between α-homogeneous operators on $\mathcal{H}_{\mathfrak{M}}$ and the operators on $\mathcal{H}_{\mathfrak{M}} \otimes L^2(\mathbb{R})$, which were considered in the last part of the previous section:

<u>Lemma 3.2.2</u>. Let $T = u|T|$ be a closed operator on $\mathcal{H}_{\mathfrak{M}}$. If

T is α-homogeneous, then $h = T \otimes L^{-\alpha}$ is affiliated with $\mathcal{R}$ and satisfies $\theta_s(h) = e^{\alpha s}h$, $s \in \mathbb{R}$. Conversely, every closed operator h affiliated with $\mathcal{R}$ satisfying $\theta_s(h) = e^{\alpha s}h$, $s \in \mathbb{R}$, arises in this way. Furthermore, for the above T, the (-1)-homogeneous $|T|^{-1/\alpha}$ is integrable if and only if $h = T \otimes L^{-\alpha}$ is τ-measurable.

Proof. At first, we assume that T is α-homogeneous, that is, $u \in \mathfrak{M}$ and $|T|^{it}x' = \sigma'_{\alpha t}(x')|T|^{it}$. We notice

$$\Delta_{\varphi_0}^{it}x'\Delta_{\varphi_0}^{-it} = \sigma'_{-t}(x') = (|T|^{-1/\alpha})^{it}x'(|T|^{-1/\alpha})^{-it} ,$$

so that $(|T|^{-1/\alpha})^{it}\Delta_{\varphi_0}^{-it} = x_t$ belongs to $\mathfrak{M}'' = \mathfrak{M}$. Thus, we compute

$$\begin{aligned}(|T|^{-1/\alpha} \otimes L)^{it} &= (|T|^{-1/\alpha})^{it} \otimes L^{it} \\ &= x_t\Delta_{\varphi_0}^{it} \otimes L^{it} = x_t(\Delta_{\varphi_0}^{it} \otimes L^{it}) \\ &= x_t h_0^{it} \in \mathcal{R} ,\end{aligned}$$

$$\begin{aligned}\theta_s((|T|^{-1/\alpha} \otimes L)^{it}) &= \theta_s(x_t h_0^{it}) = x_t\theta_s(h_0^{it}) \\ &= x_t e^{-its}h_0^{it} \\ &= e^{-its}(|T|^{-1/\alpha} \otimes L)^{it}, \quad s \in \mathbb{R} .\end{aligned}$$

Conversely, assume that $h = v|h|$ satisfies $\theta_s(h) = e^{\alpha s}h$. We then notice $\theta_s(|h|) = e^{\alpha s}|h|$, $\theta_s(v) = v$, so that v belongs to the fixed point subalgebra $\mathcal{R}^{\theta} = \mathfrak{M}$. It follows from Lemma 3.1.11, 3.1.12, that there exists $\varphi \in P(\mathfrak{M})$ such that $|h|^{-1/\alpha} = h_\varphi = \Delta_{\varphi\varphi_0} \otimes L$. Thus, we conclude that $h = v(\Delta_{\varphi\varphi_0} \otimes L)^{-\alpha} = v\Delta_{\varphi\varphi_0}^{-\alpha} \otimes L^{-\alpha}$ with

α-homogeneous $v\Delta_{\varphi\varphi_0}^{-\alpha}$.

Finally, the last statement follows from Theorem 3.1.13. (Q.E.D.)

Definition 3.2.3 ([6], [23]). For each $0 < p \leq \infty$, the set of all $(-1/p)$-homogeneous $T = u|T|$ (relative to φ_0') on $\mathcal{H}_{\mathfrak{m}}$ with (φ_0'-) integrable $|T|^p$ is denoted by $L^p(\mathfrak{m};\varphi_0')$.

The proof of Lemma 3.2.2 shows that $L^p(\mathfrak{m};\varphi_0')$ is isomorphic to $L^p(\mathfrak{m};\varphi_0)$ via $T \mapsto T \otimes L^{1/p}$. The next result is useful:

Proposition 3.2.4 ([23]). (i) If S and T in $L^p(\mathfrak{m};\varphi_0')$ satisfy $S \subseteq T$, then $S = T$.

(ii) For $T_1, T_2, \ldots, T_n$ in $L^p(\mathfrak{m};\varphi_0')$, $T_1 + T_2 + \cdots + T_n$ is densely-defined and closable, and the closure $(T_1 + T_2 + \cdots + T_n)^-$ belongs to $L^p(\mathfrak{m};\varphi_0')$. Furthermore, we have $((T_1 + T_2)^- + T_3)^- = (T_1 + (T_2 + T_3)^-)^-$.

(iii) For $T_n \in L^{p_n}(\mathfrak{m};\varphi_0')$, $n = 1,2,\ldots,m$, with $\Sigma_{n=1}^m 1/p_n = 1/p$, $T_1T_2 \cdots T_m$ is densely defined and closable, and the closure $(T_1T_2 \cdots T_m)^-$ belongs to $L^p(\mathfrak{m};\varphi_0')$. Furthermore, we have $((T_1T_2)^-T_3)^- = (T_1(T_2T_3)^-)^-$.

To prove this, the following observation is necessary:

Lemma 3.2.5. For a closed operator T on $\mathcal{H}_{\mathfrak{m}}$, the domain of the closed operator $T \otimes L^{1/p}$ (on $\mathcal{H}_{\mathfrak{m}} \otimes L^2(R) = L^2(R;\mathcal{H}_{\mathfrak{m}})$) is exactly the set of all $\zeta \in L^2(R;\mathcal{H}_{\mathfrak{m}})$ satisfying

$$\int_{-\infty}^{\infty} e^{2t/p}\|T(\zeta(t))\|^2 dt < \infty .$$

Proof. Let $\mathfrak{F}$ be the Fourier transformation from $L^2(\mathbb{R})$ onto itself, and $\bar{\mathfrak{F}} = 1 \otimes \mathfrak{F}$ be the unitary operator from $\mathcal{H}_{\mathfrak{M}} \otimes L^2(\mathbb{R})$ onto itself. Since $L = \exp(\frac{1}{i} \frac{d}{dt})$, $\bar{\mathfrak{F}}^*(T \otimes L^{1/p})\mathfrak{F} = T \otimes E(p)$, where $E(p)$ is an operator on $L^2(\mathbb{R})$ given by $(E(p)f)(t) = e^{t/p}f(t)$, $t \in \mathbb{R}$, $f \in L^2(\mathbb{R})$. We notice that $\zeta \in L^2(\mathbb{R};\mathcal{H}_{\mathfrak{M}})$ belongs to the domain of $T \otimes L^{1/p}$ if and only if $\|(T \otimes E(p))\bar{\mathfrak{F}}^*\zeta\| < \infty$. However, since $\bar{\mathfrak{F}}^* = 1 \otimes \mathfrak{F}^*$ and $\mathfrak{F}^*$ is the inverse Fourier transformation on $L^2(\mathbb{R})$, we compute

$$\|(T \otimes E(p))\bar{\mathfrak{F}}^*\zeta\|^2 = \int_{-\infty}^{\infty} e^{2t/p}\|T((\bar{\mathfrak{F}}^*\zeta)(t))\|^2 dt$$

$$= \int_{-\infty}^{\infty} e^{2t/p}\|T(\zeta(t))\|^2 dt \, . \qquad \text{(Q.E.D.)}$$

Proof of Proposition 3.2.4 ([23]). (i) The assumption yields $S \otimes L^{1/p} \subseteq T \otimes L^{1/p}$ and both of $S \otimes L^{1/p}$, $T \otimes L^{1/p}$ are τ-measurable by Lemma 3.2.2. Thus it follows from Theorem 3.1.3,(ii), that $S \otimes L^{1/p} = T \otimes L^{1/p}$. For $\xi \in \mathcal{D}(T)$ and a characteristic function $\chi_{[0,1]} \in L^2(\mathbb{R})$, $\xi \otimes \chi_{[0,1]}$ belongs to $\mathcal{D}(T \otimes L^{1/p}) = \mathcal{D}(S \otimes L^{1/p})$ by Lemma 3.2.5, that is,

$$\int_0^1 e^{2t/p}\|S\xi\|^2 dt < \infty \, .$$

Thus, $\|S\xi\|$ must be finite, that is, $\xi \in \mathcal{D}(S)$.

(ii) Since each $T_i \otimes L^{1/p}$ belongs to $L^p(\mathfrak{M};\varphi_0')$, the strong sum $T_1 \otimes L^{1/p} + T_2 \otimes L^{1/p} + \cdots + T_n \otimes L^{1/p}$ belongs to $L^p(\mathfrak{M};\varphi_0')$. Clearly, for each i,

$$\mathfrak{D}(T_i \otimes L^{1/p}) = \left\{\zeta \in L^2(R;\mathcal{H}_{\mathfrak{m}}) : \int_{-\infty}^{\infty} e^{2t/p}\|T_i(\zeta(t))\|^2 dt < \infty\right\}$$

$$\subseteq \{\zeta \in L^2(R;\mathcal{H}_{\mathfrak{m}}) : \zeta(t) \in \mathfrak{D}(T_i) \text{ a.e. } t \in R\} ,$$

so that $\{\zeta \in L^2(R;\mathcal{H}_{\mathfrak{m}}); \zeta(t) \in \mathfrak{D}(T_1) \cap \mathfrak{D}(T_2) \cap \cdots \cap \mathfrak{D}(T_n) \text{ a.e. } t \in R\}$ is dense in $L^2(R;\mathcal{H}_{\mathfrak{m}})$. We thus conclude that $\mathfrak{D}(T_1) \cap \mathfrak{D}(T_2) \cap \cdots \cap \mathfrak{D}(T_n)$ is dense in $\mathcal{H}_{\mathfrak{m}}$. The same argument also shows that $T_1^* + T_2^* + \cdots + T_n^*$ is densely defined, that is, $T_1 + T_2 + \cdots + T_n$ is closable. Also $T_1 \otimes L^{1/p} + T_2 \otimes L^{1/p} + \cdots + T_n \otimes L^{1/p} = (T_1 + T_2 + \cdots + T_n)^- \otimes L^{1/p}$ and $(T_1 + T_2 + \cdots + T_n)^-$ belongs to $L^p(\mathfrak{m};\varphi_0')$ by Lemma 3.2.2.

We clearly have,

$$((T_1 + T_2)^- + T_3)^- \supseteq (T_1 + T_2)^- + T_3 \supseteq (T_1 + T_2) + T_3$$
$$(T_1 + (T_2 + T_3)^-)^- \supseteq T_1 + (T_2 + T_3)^- \supseteq T_1 + (T_2 + T_3) .$$

Since the associativity for algebraic sums of unbounded operators holds, we have

$$((T_1 + T_2)^- + T_3)^- \supseteq T_1 + T_2 + T_3 \subseteq (T_1 + (T_2 + T_3)^-)^- ,$$
$$((T_1 + T_2)^- + T_3)^- \supseteq (T_1 + T_2 + T_3)^- \subseteq (T_1 + (T_2 + T_3)^-)^- ,$$

so that (i) yields the desired associativity $((T_1 + T_2)^- + T_3)^- = (T_1 + (T_2 + T_3)^-)^-$. (iii) can be proved by a similar argument as (i). (Q.E.D.)

By Proposition 3.2.4, even if we omit the closure signs of sums and products of elements in $L^p(\mathfrak{m};\varphi_0')$, no confusion occurs. Thus, we shall omit them henceforth.

Now we are at the position to state the following result, which will be crucial in our construction of canonical L^p-spaces.

Theorem 3.2.6. Let φ_1, φ_2 be elements in $\mathfrak{M}_*$ with the polar decompositions $\varphi_1 = u_1|\varphi_1|$, $\varphi_2 = u_2|\varphi_2|$ respectively, and φ_0 be a faithful element in $\mathfrak{M}_*^+$. Let uT be the polar decomposition of a closed operator

$$u_1\Delta_{|\varphi_1|\varphi_0}^{1/p} + u_2\Delta_{|\varphi_2|\varphi_0}^{1/p} .$$

Then, u belongs to $\mathfrak{M}$ and there exists a unique χ in $\mathfrak{M}_*^+$ with $T = \Delta_{\chi\varphi_0}^{1/p}$. Furthermore, u and χ do not depend on a choice of φ_0. If φ_1 and φ_0 are positive, then $\Delta_{\varphi_1\varphi_0}^{1/p} + \Delta_{\varphi_2\varphi_0}^{1/p}$ is positive self-adjoint. $(0 < p < \infty)$.

Proof. Since both of $u_1\Delta_{|\varphi_1|\varphi_0}^{1/p}$ and $u_2\Delta_{|\varphi_2|\varphi_0}^{1/p}$ belong to $L^p(\mathfrak{M};\varphi_0)$, uT belongs $L^p(\mathfrak{M};\varphi_0)$ and u belongs to $\mathfrak{M}$ by Proposition 3.2.4,(i). Since $T^p \otimes L$ is a τ-measurable positive self-adjoint operator (affiliated with $\mathfrak{R}$) satisfying $\theta_s(T^p \otimes L) = e^{-s}T^p \otimes L$, $s \in \mathbb{R}$, it follows from Lemma 3.1.11, 3.1.12 that there exists a unique $\chi \in \mathfrak{M}_*^+$ satisfying $T^p \otimes L^{1/p} = h_\chi = \Delta_{\chi\varphi_0} \otimes L$, that is, $T = \Delta_{\chi\varphi_0}^{1/p}$. (Also, the above u is exactly the phast part of the polar decomposition $uh_\chi^{1/p} = u(\Delta_{\chi\varphi_0}^{1/p} \otimes L^{1/p})$.) In the above argument, u and χ do not depend on the choice of φ_0 due to Proposition 3.1.15. The last statement follows from Proposition 3.2.4,(i). (Q.E.D.)

By the same argument, the next result is also valid:

Proposition 3.2.7. Let φ be an element in $\mathfrak{M}_*$ with the polar decomposition $u|\varphi|$, and φ_0 be a faithful element in $\mathfrak{M}_*^+$. Let vT be the polar decomposition of $(u\Delta^{1/p}_{|\varphi|\varphi_0})^*$. Then, u belongs to $\mathfrak{M}$ and there exists a unique $\chi \in \mathfrak{M}_*^+$ with $T = \Delta^{1/p}_{\chi\varphi_0}$. Furthermore, u and χ do not depend on the choice of φ_0, and $\chi(1) = |\varphi|(1)$.

§3.3 Canonical L^p-spaces.

Based on the canonical standard form $(\mathfrak{M}, \mathcal{H}_{\mathfrak{M}}, J_{\mathfrak{M}}, (\mathcal{H}_{\mathfrak{M}})_+)$ and the "canonical" relative modular operators on $\mathcal{H}_{\mathfrak{M}}$ (Remark 2.2.11), we shall give a construction of canonical L^p-spaces, $1 \leq p < \infty$. ($L^\infty(\mathfrak{M}) = \mathfrak{M}$). Although our construction is canonical, it is convenient for proofs to have a distinguished faithful state φ_0 on $\mathfrak{M}$. (See Remark 2.2.15.)

We begin with introducing new addition, scalor multiplication (by complex numbers), and $*$-structure on the predual $\mathfrak{M}_*$ for each $1 \leq p < \infty$. To avoid confusion, we write $\varphi^{1/p}$ instead of φ ($\in \mathfrak{M}_*$) when we deal with the new operations.

Definition 3.3.1. For $\varphi_1, \varphi_2 \in \mathfrak{M}_*$ with the polar decompositions $\varphi_1 = u_1|\varphi_1|$ and $\varphi_2 = u_2|\varphi_2|$, and a complex number $\lambda = e^{i\theta}|\lambda|$, we set

(i) $\varphi_1^{1/p} + \varphi_2^{1/p} = (u\chi)^{1/p}$ where $u\chi \in \mathfrak{M}_*$ is given by

$$u_1\Delta^{1/p}_{|\varphi_1|,|\varphi_1|+|\varphi_2|} + u_2\Delta^{1/p}_{|\varphi_2|,|\varphi_1|+|\varphi_2|} = u\Delta^{1/p}_{\chi,|\varphi_1|+|\varphi_2|},$$

(see Theorem 3.2.6).

(ii) $\lambda\varphi_1^{1/p} = ((e^{i\theta}u)(|\lambda|^p|\varphi_1|))^{1/p}$.

(iii) $(\varphi_1^{1/p})^* = (u\chi)^{1/p}$ where $u\chi \in \mathfrak{m}_*$ is given by

$$(u_1\Delta_{|\varphi_1|}^{1/p})^* = u\Delta_{\chi|\varphi_1|}^{1/p} . \qquad \text{(See Proposition 3.2.7)}$$

Lemma 3.3.2. With the structures just defined, $\mathfrak{m}_*$ is actually a vector space with an involution.

Proof. If we replace $|\varphi_1| + |\varphi_2|$ by $|\varphi_0|$ in Definition 3.3.1,(i) and $|\varphi_1|$ by $|\varphi_0|$ in Definition 3.3.1,(iii), then the definitions of $\varphi_1^{1/p} + \varphi_2^{1/p}$ and $(\varphi_1^{1/p})^*$ are not affected due to Theorem 3.2.6 and Proposition 3.2.7. (Although we lose the fact that $\mathfrak{m}_*^+$ with (i), (ii), (iii) is a canonical object.)

We identify $\varphi = u|\varphi|$ with $u\Delta_{\varphi\varphi_0}^{1/p} \in L^p(\mathfrak{m};\varphi_0)$. The addition (i) and the *-operation (iii) clearly correspond to the usual addition and the *-operation as operators, while the multiplication (ii) corresponds to the usual scalor multiplication as operators because the polar decomposition of $\lambda(u\Delta_{\varphi\varphi_0}^{1/p})$ is exactly

$$(e^{i\theta}u)(|\lambda|\Delta_{|\varphi|\varphi_0}^{1/p}) = (e^{i\theta}u)\Delta_{|\lambda|^p|\varphi|,\varphi_0}^{1/p} . \qquad \text{(Q.E.D.)}$$

Definition 3.3.3. We denote the vector space with the involution above described by $L^p(\mathfrak{m})$. We also introduce the non-negative function $\|\cdot\|_p$ on $L^p(\mathfrak{m})$ by

$$\|\varphi^{1/p}\|_p = (|\varphi|(1))^{1/p} = \|\varphi\|_{\mathfrak{m}_*}^{1/p}.$$

To investigate the above $\|\cdot\|_p$ (to be a norm), we need a sesquilinear form on $L^p(\mathfrak{m}) \times L^q(\mathfrak{m})$, $1/p + 1/q = 1$, which will also be indispensable to establish the expected duality.

For $\varphi_i = u_i|\varphi_i|$, $i = 1,2$, as in Definition 3.3.1, Theorem 3.2.6 $(p = 1/2)$ guarantees the existence of a unique $\psi \in \mathfrak{M}_*^+$ satisfying

$$\Delta^2_{|\varphi_1|,|\varphi_1|+|\varphi_2|} + \Delta^2_{|\varphi_2|,|\varphi_1|+|\varphi_2|} = \Delta^2_{\psi,|\varphi_1|+|\varphi_2|} .$$

We notice that

$$\Delta^2_{|\varphi_i|,|\varphi_1|+|\varphi_2|} \leq \Delta^2_{\psi,|\varphi_1|+|\varphi_2|}, \qquad i = 1,2 ,$$

that is, $|\varphi_i| \leq \psi(1)$, $i = 1,2$, in the sense of Connes-Takesaki, [9].

Definition 3.3.4. For $\varphi_1, \varphi_2 \in \mathfrak{M}_*$ as above, and $1/p + 1/q = 1$, we set

$$\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle = \psi((D|\varphi_2| : D\psi)^*_{-i/q} u_2^* u_1 (D|\varphi_1| : D\psi)_{-i/p}) .$$

Lemma 3.3.5. The above $\langle\ ,\ \rangle$ is a sesquilinear form on $L^p(\mathfrak{M}) \times L^q(\mathfrak{M})$. Also, the definition of $\langle\ ,\ \rangle$ does not depend on the choice of ψ whenever $(D|\varphi_2| : D\psi)_{-i/q}$ and $(D|\varphi_1| : D\psi)_{-i/p}$ make sense.

Proof. If we identify $L^p(\mathfrak{M})$ with $L^p(\mathfrak{M};\varphi_0')$ (hence with $L^p(\mathfrak{M};\varphi_0)$ as in the Proof of Lemma 3.3.2, then we compute

$$\psi((D|\varphi_2| : D\psi)^*_{-i/q} u_2^* u_1 (D|\varphi_1| : D\psi)_{-i/p})$$

$$= tr(h_\psi (h^{1/q}_{|\varphi_2|} h_\psi^{-1/q})^* u_2^* u_1 (h^{1/p}_{|\varphi_1|} h_\psi^{-1/p})^*)$$

$$= \mathrm{tr}(h_{|\varphi_2|}^{1/q} u_2^* u_1 h_{|\varphi_1|}^{1/p}) \quad \text{(Proposition 3.1.18)}$$

$$= \mathrm{tr}((u_2 h_{|\varphi_2|}^{1/q})^* (u_1 h_{|\varphi_1|}^{1/p})) ,$$

so that $\langle \, , \, \rangle$ is sesquilinear. The above computation and Proposition 3.1.15 yield that $\langle \, , \, \rangle$ does not depend on the choice of ψ. (Q.E.D.)

Corollary 3.3.6.

(i) $\langle \varphi^{1/p}, \varphi^{1/p} \rangle = |\varphi|(1)$ for $\varphi \in \mathfrak{M}_*$.

(ii) For $\varphi_n = u_n|\varphi_n|$, $n = 1,2$, $\langle \varphi_1^{1/2}, \varphi_2^{1/2} \rangle$ is exactly

$$(|\varphi_1| + |\varphi_2|)((D|\varphi_2| : D(|\varphi_1| + |\varphi_2|))^*_{-i/2} u_2^* u_1 (D|\varphi_1| : D(|\varphi_1| + |\varphi_2|))_{-i/2}).$$

Proof. (i) If $\varphi_1 = \varphi_2 = \varphi = u|\varphi|$, then we can choose $\psi = |\varphi|$ and $(D|\varphi| : D|\varphi|)_z = 1$ for all $z \in \mathbf{C}$ so that $\langle \varphi^{1/p}, \varphi^{1/q} \rangle = |\varphi|(u^*u) = |\varphi|(1)$. (ii) Since $|\varphi_n| \leq |\varphi_1| + |\varphi_2|$, $n = 1,2$, as functionals, it follows from Corollary 1.4.3 that $(D|\varphi_n| : D(|\varphi_1| + |\varphi_2|)_{-i/2}$, $n = 1,2$, make sense. Thus, by choosing $\psi = |\varphi_1| + |\varphi_2|$, we have the desired result. (Q.E.D.)

We also have the following expression:

Theorem 3.3.7. Assume that $1/p + 1/q = 1$, $p,q > 1$, and $\varphi_n = u_n|\varphi_n| \in \mathfrak{M}_*$ $(n = 1,2)$. If the support of $|\varphi_1|$ is majorized by the one of $|\varphi_2|$, then $\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle = f(-i/p)$. Here, $f(z)$ is the (relative K.M.S.) function, which is bounded and continuous (resp. analytic) on $-1 \leq \mathrm{Im}\, z \leq 0$ (resp. $-1 < \mathrm{Im}\, z < 0$), with boundary values:

$$f(t) = |\varphi_2|(u_2^* u_1 (D|\varphi_1| : D|\varphi_2|)_t), \quad t \in \mathbb{R},$$

$$f(t - i) = |\varphi_1|((D|\varphi_1| : D|\varphi_2|)_t u_2^* u_1), \quad t \in \mathbb{R}.$$

Proof. By the proof of Lemma 3.3.5, we have already known $\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle = \mathrm{tr}(h_{|\varphi_2|}^{1/q} u_2^* u_1 h_{|\varphi_1|}^{1/p})$. On the other hand, the relative K.M.S.-function $f(z)$ at $z = t \in \mathbb{R}$ is expressed by

$$\begin{aligned} f(t) &= |\varphi_2|(u_2^* u_1 (D|\varphi_1| : D|\varphi_2|)_t) \\ &= \mathrm{tr}(h_{|\varphi_2|} u_2^* u_1 h_{|\varphi_1|}^{it} h_{|\varphi_2|}^{-it}) \\ &= \mathrm{tr}(u_2^* u_1 h_{|\varphi_1|}^{it} h_{|\varphi_2|}^{1-it}), \end{aligned}$$

which is the boundary value of the function: $z \mapsto \mathrm{tr}(u_2^* u_1 h_{|\varphi_1|}^{iz} h_{|\varphi_2|}^{1-iz})$ (see Proposition 3.1.17). Thus, we conclude that

$$f(-i/p) = \mathrm{tr}(u_2^* u_1 h_{|\varphi_1|}^{1/p} h_{|\varphi_2|}^{1/q}) = \langle \varphi^{1/p}, \varphi^{1/q} \rangle. \qquad \text{(Q.E.D.)}$$

<u>Remark 3.3.8.</u> The above theorem means that the study of the sesquilinear form $\langle \ , \ \rangle$ on $L^p(\mathfrak{M}) \times L^q(\mathfrak{M})$ and the expected duality is nothing but the analysis of the behavior of relative K.M.S.-functions <u>inside</u> the strip.

<u>For</u> φ_1 <u>and</u> φ_2 <u>in Theorem 3.3.7,</u> the theorem of three lines yields that

$$\begin{aligned} |\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle| &\le \exp\Big(\frac{1}{q} \log(\sup_{t \in \mathbb{R}} |f(t)|) + \frac{1}{p} \log(\sup_{t \in \mathbb{R}} |f(t - i)|)\Big) \\ &\le \exp\Big(\frac{1}{q} \log \| |\varphi_2| \|_{\mathfrak{M}_*} + \frac{1}{p} \log \| |\varphi_1| \|_{\mathfrak{M}_*}\Big) \end{aligned}$$

$$= \|\varphi_1\|_{\mathfrak{M}_*}^{1/p}\|\varphi_2\|_{\mathfrak{M}_*}^{1/q} = \|\varphi_1^{1/p}\|_p\|\varphi_2^{1/q}\|_q ,$$

which is exactly Hölder's inequality. To show the inequality for arbitrary pairs (φ_1, φ_2), we prepare two lemmas.

Lemma 3.3.9. If $\{\varphi_n\}$ is a monotone increasing sequence in $\mathfrak{M}_*^+$ satisfying $\sup_n \|\varphi_n\| < \infty$, then there exists a (unique) φ in $\mathfrak{M}_*^+$ such that $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$.

Proof. Set $\varphi(x) = \sup_n \varphi_n(x)$, $x \in \mathfrak{M}$. By the σ-weak semi-continuity, φ is linear. The assumption yields $\varphi(1) < \infty$, that is, $\varphi \in \mathfrak{M}_*^+$. Since $\varphi_n \leq \varphi$, we conclude that

$$\|\varphi - \varphi_n\| = (\varphi - \varphi_n)(1) \to 0 \quad \text{as} \quad n \to \infty . \qquad \text{(Q.E.D.)}$$

Lemma 3.3.10. Let $|\varphi_n|$, $n = 1,2$, be elements in $\mathfrak{M}_*$. Let χ_n, $n = 1,2,\ldots$, denote functionals in $\mathfrak{M}_*^+$ determined by

$$\left(\frac{1}{n}\Delta_{|\varphi_1|\psi}\right)^2 + \Delta^2_{|\varphi_2|\psi} = \Delta^2_{\chi_n\psi} .$$

(See remarks before Definition 3.3.4.) Then, the sequence $\{\chi_n\}$ converges to $|\varphi_2|$ in norm.

Proof. Considering the tensor product with L^2, one gets $\left(\frac{1}{n}\Delta_{|\varphi_1|,\psi} \otimes L\right)^2 + (\Delta_{|\varphi_2|,\psi} \otimes L)^2 = (\Delta_{\chi_n\psi} \otimes L)^2$, that is, $\frac{1}{n^2} h^2_{|\varphi_1|} + h^2_{|\varphi_2|} = h^2_{\chi_n}$ (in $L^{1/2}(\mathfrak{M};\psi)$). Clearly, $\{h^2_{\chi_n}\}$ converges to $h^2_{|\varphi_2|}$ in the measure topology (Definition 3.1.4). Furthermore, the sequence is monotone decreasing. Since the square root operation is operator monotone ([33]), $\{h_{\chi_n}\}$ is a montone decreasing sequence, that is, $\{\chi_n\}$ is a monotone decreasing sequence in $\mathfrak{m}_*^+$ so that Lemma 3.3.9 guarantees the existence of a unique $\varphi \in \mathfrak{m}_*^+$ such that $\|\chi_n - \varphi\| \to 0$ as $n \to \infty$. It now suffices to show that $\varphi = |\varphi_2|$. By Proposition 3.1.19, $\{h_{\chi_n}\}$ converges to h_φ in the measure topology so that $\{h_{\chi_n}^2\}$ converges to h_φ^2 in the measure topology ([31]). We thus conclude that $h_\varphi^2 = h_{|\varphi_2|}^2$, that is, $\varphi = |\varphi_2|$.

(Q.E.D.)

<u>Theorem 3.3.11</u> (Hölder's inequality). For any $\varphi_1, \varphi_2 \in \mathfrak{m}_*^+$, and $1/p + 1/q = 1$, $p,q > 1$, we have

$$|\langle \varphi_1^{1/p}, \varphi_2^{1/q}\rangle| \leq \|\varphi_1^{1/p}\|_p \|\varphi_2^{1/q}\|_q .$$

Proof. For the above χ_n, $n = 1,2,\ldots$, clearly we have $\frac{1}{n^2}\Delta^2_{|\varphi_1|\psi}, \Delta^2_{|\varphi_2|\psi} \leq \Delta^2_{\chi_n\psi}$, that is, $|\varphi_1| \leq n\chi_n(1)$ and $|\varphi_2| \leq \chi_n(1)$. It follows from Lemma 3.3.5 that, for each n,

$$\langle \varphi_1^{1/p}, \varphi_2^{1/q}\rangle = \chi_n((D|\varphi_2| : D\chi_n)^*_{-i/q} u_2^* u_1 (D|\varphi_1| : D\chi_n)_{-i/p}) .$$

For each n, we consider the function

$$f_n(z) = \chi_n((d|\varphi_2| : D\chi_n)^*_{-i/q} u_2^* u_1 (D|\varphi_1| : D\chi_n)_z) .$$

This is bounded and continuous (resp. analytic) on $-1 \leq \mathrm{Im}\, z \leq 0$ (resp. $-1 < \mathrm{Im}\, z < 0$) by the relative K.M.S. condition (or by $|\varphi_1| \leq n\chi_n(1)$). Also, $f_n(-i/p) = \langle \varphi_1^{1/p}, \varphi_2^{1/q}\rangle$. For each n, the theorem of three lines yield the following estimate:

$$|\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle| \leq \exp\left(\frac{1}{q} \log(\sup_{t\in R} |f_n(t)|) + \frac{1}{p} \log(\sup_{t\in R} |f_n(t - i)|)\right).$$

Since $|\varphi_2| \leq \chi_n(1)$, $|f_n(t)|$ is majorized by $\|\chi_n\|$. On the other hand, by Theorem 1.4.4, we have

$$f_n(t - i) = |\varphi_1|((D|\varphi_1| : D\chi_n)_t (D|\varphi_2| : D\chi_n)^*_{-i/q} u_2^* u_1),$$

so that $f_n(t - i)$ is majorized by $\||\varphi_1|\| = \|\varphi_1\|$. Hence,

$$|\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle| \leq \|\varphi_1\|^{1/p} \|\chi_n\|^{1/q}$$

for each n. By Lemma 3.3.9, $\{\|\chi_n\|^{1/q}\}$ tends to $\|\varphi_2\|^{1/q}$ as $n \to \infty$ so that we estimate

$$|\langle \varphi_1^{1/p}, \varphi_2^{1/q} \rangle| \leq \|\varphi_1\|^{1/p} \|\varphi_2\|^{1/q} = \|\varphi_1^{1/p}\|_p \|\varphi_2^{1/q}\|_q. \qquad \text{(Q.E.D.)}$$

The next result is a consequence of the theorem and Corollary 3.3.6,(ii).

<u>Corollary 3.3.12</u>. For each $\varphi \in \mathfrak{M}_*$, $\|\varphi^{1/p}\|_p$ is the supremum of $|\langle \varphi^{1/p}, \psi^{1/q} \rangle|$ over $\psi \in \mathfrak{M}_*$ satisfying $\|\psi^{1/q}\|_q$ $(= \|\psi\|^{1/q}) \leq 1$, and the supremum is actually attained. In particular, $\|\cdot\|_p$ $(1 \leq p < \infty)$ is a norm on $L^p(\mathfrak{M})$.

<u>Theorem 3.3.13</u>. For each $1 \leq p < \infty$, $\{L^p(\mathfrak{M}), \|\cdot\|_p\}$ is a Banach space. Among them, $L^1(\mathfrak{M})$ is exactly the predual $\mathfrak{M}_*$, while $L^2(\mathfrak{M})$ is isomorphic to the canonical Hilbert space $\mathcal{H}_{\mathfrak{M}}$. In particular, $\mathfrak{M}$ acts on $L^2(\mathfrak{M})$.

Proof. The first statement is obtained from Proposition 3.1.19

and the completeness of the set of all τ-measurable operator affiliated with $\mathcal{R} = \mathfrak{M} \times_{\sigma^{\varphi_0}} \mathbf{R}$ because $\|\cdot\|_p$ on $L^p(\mathfrak{M})$ corresponds to the norm on $L^p(\mathfrak{M};\varphi_0)$ introduced in Definition 3.1.16 when we identify $L^p(\mathfrak{M})$ with $L^p(\mathfrak{M};\varphi_0)$.

We now consider the case $p = 1$. In this case, Definition 3.3.1 and 3.3.4 reduce to the usual linear structure and the predual norm on $\mathfrak{M}_*$. To show this, it is sufficient to prove

$$u_1\Delta_{|\varphi_1|,|\varphi_1|+|\varphi_2|} + u_2\Delta_{|\varphi_2|,|\varphi_1|+|\varphi_2|} = u\Delta_{\chi,|\varphi_1|+|\varphi_2|}$$

with $u_1|\varphi_1| + u_2|\varphi_2| = u\chi$, or by taking the tensor product with L,

$$u_1h_{|\varphi_1|} + u_2h_{|\varphi_2|} = uh_\chi \,.$$

However, this is obvious because $\mathfrak{M}_*$ is isomorphic to $L^1(\mathfrak{M},|\varphi_1| + |\varphi_2|)$ (if $|\varphi_1| + |\varphi_2|$ is faithful).

Finally, we consider the case $p = 2$. Corollary 3.3.6,(ii) shows that $L^2(\mathfrak{M})_+ = \{\varphi^{1/2};\varphi \in \mathfrak{M}_*^+\} \ni \varphi^{1/2} \mapsto \sqrt{\varphi} \in (\mathcal{H}_{\mathfrak{M}})_+$ is a surjective isometry, which extends to a surjective isometry from $L^2(\mathfrak{M})$ onto $\mathcal{H}_{\mathfrak{M}}$. (Q.E.D.)

Due to the above theorem, it is reasonable to call $L^p(\mathfrak{M})$, $1 \leq p < \infty$, the canonical L^p-space (associated with $\mathfrak{M}$). We now consider a functorial property of the correspondence: $\mathfrak{M} \mapsto \mathcal{H}_{\mathfrak{M}}$. It follows from Proposition 2.1.11, 2.2.12 that a normal $*$-isomorphism from $\mathfrak{M}$ onto another von Neumann algebra $\mathfrak{N}$ naturally induces the unitary operator u_θ from $\mathcal{H}_{\mathfrak{N}}$ onto $\mathcal{H}_{\mathfrak{M}}$ satisfying $u_\theta((\mathcal{H}_{\mathfrak{N}})_+) = (\mathcal{H}_{\mathfrak{M}})_+$

and $\theta = \mathrm{Ad}u_\theta^*$, the canonical implementation of θ.

Lemma 3.3.14. For $\varphi, \psi \in \mathfrak{N}_*^+$ (ψ faithful), we have

$$\Delta_{\theta_*\varphi, \theta_*\psi} = u_\theta \Delta_{\varphi,\psi} u_\theta^* .$$

Here $\theta_*\varphi$ (resp. $\theta_*\psi$) in $\mathfrak{M}_*^+$ means $(\theta_*\varphi)(x) = \varphi(\theta(x))$ (resp. $(\theta_*\psi)(x) = \psi(\theta(x))$), $x \in \mathfrak{M}$.

Proof. For $x \in \mathfrak{M}$, $\chi \in \mathfrak{N}_*^+$, one gets $(\theta_*\chi)(x) = \chi(\theta(x)) = \chi(u_\theta^* x u_\theta) = (x u_\theta \sqrt{\chi} \mid u_\theta \sqrt{\chi})$. Since $u_\theta \sqrt{\chi}$ belongs to $(\mathcal{H}_{\mathfrak{M}})_+$, we conclude that $u_\theta \sqrt{\chi} = \sqrt{\theta_*\chi}$.

For each $x \in \mathfrak{M}$, we compute

$$\begin{aligned} u_\theta^* J_{\mathfrak{M}} \Delta_{\theta_*\varphi, \theta_*\psi}^{1/2} x \sqrt{\theta_*\psi} &= u_\theta^* x^* \sqrt{\theta_*\varphi} \\ &= u_\theta^* x^* u_\theta \sqrt{\varphi} = \theta(x^*) \sqrt{\varphi} = \theta(x)^* \sqrt{\varphi} \\ &= J_{\mathfrak{N}} \Delta_{\varphi\psi}^{1/2} \theta(x) \sqrt{\psi} = J_{\mathfrak{N}} \Delta_{\varphi\psi}^{1/2} u_\theta^* x u_\theta \sqrt{\psi} \\ &= J_{\mathfrak{N}} \Delta_{\varphi\psi}^{1/2} u_\theta^* x \sqrt{\theta_*\psi} . \end{aligned}$$

Because $\mathfrak{M}\sqrt{\theta_*\psi}$ (resp. $u_\theta^* \mathfrak{M} \sqrt{\theta_*\psi} = \mathfrak{N}\sqrt{\psi}$) is a core for $\Delta_{\theta_*\varphi, \theta_*\psi}^{1/2}$ (resp. $\Delta_{\varphi\psi}^{1/2}$), the above calculation yields

$$J_{\mathfrak{N}} \Delta_{\varphi\psi}^{1/2} u_\theta^* = u_\theta^* J_{\mathfrak{M}} \Delta_{\theta_*\varphi, \theta_*\psi}^{1/2} = J_{\mathfrak{M}} u_\theta^* \Delta_{\theta_*\varphi, \theta_*\psi}^{1/2} ,$$

so that the uniqueness of the polar decomposition gives the result.

(Q.E.D.)

Proposition 3.3.15. For a normal *-isomorphism from $\mathfrak{M}$ onto another von Neumann algebra $\mathfrak{N}$, the map θ_* from $\mathfrak{N}_*$ onto $\mathfrak{M}_*$

given by $(\theta_* \chi)(x) = \chi(\theta(x))$, $x \in \mathfrak{m}$, $\chi \in \mathfrak{n}_*$, gives rise to a surjective isometry from $L^p(\mathfrak{m})$ onto $L^p(\mathfrak{n})$ for each p.

Proof. Clearly, θ_* is surjective and norm-preserving because

$$\|(\theta_*\chi)^{1/p}\|_p = \|\chi \circ \theta\|_{\mathfrak{m}_*}^{1/p} = \|\chi\|_{\mathfrak{m}_*}^{1/p} = \|\chi^{1/p}\|_p \,.$$

Thus, it suffices to show that θ_* is linear, that is, θ_* preserves the addition and scalor multiplications introduced in Definition 3.3.1. For $\varphi_1 = u_1|\varphi_1|$, $\varphi_2 = u_2|\varphi_2|$ in $\mathfrak{n}_*$, the polar decompositions of $\theta_*\varphi_1$, $\theta_*\varphi_2$ are $\theta^{-1}(u_1)\theta_*(|\varphi_1|)$, $\theta^{-1}(u_2)\theta_*(|\varphi_2|)$ respectively.

We assume that

$$u_1\Delta^{1/p}_{|\varphi_1|,|\varphi_1|+|\varphi_2|} + u_2\Delta^{1/p}_{|\varphi_2|,|\varphi_1|+|\varphi_2|} = u\Delta^{1/p}_{\chi,|\varphi_1|+|\varphi_2|}$$

as in Definition 3.3.1,(i). We then have

$$u_\theta u_1 u_\theta^* u_\theta \Delta^{1/p}_{|\varphi_1|;|\varphi_1|+|\varphi_2|} u_\theta^* + u_\theta u_2 u_\theta^* u_\theta \Delta^{1/p}_{|\varphi_2|,|\varphi_1|+|\varphi_2|} u_\theta^*$$
$$= u_\theta u \Delta^{1/p}_{\chi,|\varphi_1|+|\varphi_2|} u_\theta^* \,.$$

It follows from the previous lemma that

$$\theta^{-1}(u_1)\Delta^{1/p}_{\theta_*|\varphi_1|_*,\theta_*|\varphi_1|+\theta_*|\varphi_2|} + \theta^{-1}(u_2)\Delta^{1/p}_{\theta_*|\varphi_2|,\theta_*|\varphi_1|+\theta_*|\varphi_2|}$$
$$= \theta^{-1}(u)\Delta^{1/p}_{\theta_*\chi,\theta_*|\varphi_1|+\theta_*|\varphi_2|} \,,$$

that is, $(\theta_*\varphi_1)^{1/p} + (\theta_*\varphi_2)^{1/p} = (\theta_*(u\chi))^{1/p}$. Similarly, the other operations are preserved under θ_*. (Q.E.D.)

§3.4 Duality.

In this section, we obtain inequalities concerning the $\|\cdot\|_p$-norms. We then establish the duality between $L^p(\mathfrak{M})$ and $L^q(\mathfrak{M})$, $1/p + 1/q = 1$.

Lemma 3.4.1. Let $L^p(\mathfrak{M};\varphi_0)$ and $L^q(\mathfrak{M};\varphi_0)$, $1/p + 1/q = 1$, be the Haagerup's L^p-spaces constructed from φ_0. For $a,b \in L^p(\mathfrak{M};\varphi_0)$ and $c,d \in L^q(\mathfrak{M};\varphi_0)$, we have the following inequalities:

(i) $|\mathrm{tr}((a + b)c + (a - b)d)| \leq 2^{1/q}\{\|a\|_p^p + \|b\|_p^p\}^{1/p}\{\|c\|_q^q + \|d\|_q^q\}^{1/q}$ for $2 \leq p < \infty$.

(ii) $|\mathrm{tr}((a + b)c + (a - b)d)| \leq 2^{1/p}\{\|a\|_p^p + \|b\|_p^p\}^{1/p}\{\|c\|_q^q + \|d\|_q^q\}^{1/q}$ for $1 < p \leq 2$.

Proof. At first, we show the required inequality in the special case $p = q = 2$. Since the sesquilinear form $(a,b) \mapsto \mathrm{tr}(b^*a)$ gives rise to an inner product on $L^2(\mathfrak{M};\varphi_0)$, we compute

$$|\mathrm{tr}((a + b)c + (a - b)d)| \leq |\mathrm{tr}((a + b)c)| + |\mathrm{tr}((a - b)c)|$$

$$\leq \|a + b\|_2\|c\|_2 + \|a - b\|_2\|d\|_2 \quad \text{(Cauchy-Schwarz inequality)}$$

$$\leq (\|a + b\|_2^2 + \|a - b\|_2^2)^{1/2}(\|c\|_2^2 + \|d\|_2^2)^{1/2}$$

$$\leq 2^{1/2}(\|a\|_2^2 + \|b\|_2^2)^{1/2}(\|c\|_2^2 + \|d\|_2^2)^{1/2} \quad \text{(The parallelogram law)}.$$

To consider the general case, we assume that $a = u\tilde{a}^{1/p}$ and $b = v\tilde{b}^{1/p}$ (resp. $c = \tilde{c}^{1/q}t$ and $d = \tilde{d}^{1/q}w$) be left (resp. right) polar decompositions. For convenience, we write

$$x = \|\tilde{a}\|_1 + \|\tilde{b}\|_1 = \|\tilde{a}^{1/r}\|_r^r + \|\tilde{b}^{1/r}\|_r^r, \quad 1 \leq r < \infty,$$

$$y = \|\tilde{c}\|_1 + \|\tilde{d}\|_1 = \|\tilde{c}^{1/r}\|_r^r + \|\tilde{d}^{1/r}\|_r^r, \qquad 1 \leq r < \infty .$$

We consider the function

$$f(z) = \operatorname{tr}((u\tilde{a}^z + v\tilde{b}^z)\tilde{c}^{1-z}t + (u\tilde{a}^z + v\tilde{b}^z)\tilde{d}^{1-z}w) ,$$

which is bounded and continuous (resp. analytic) on $0 \leq \operatorname{Re} z \leq 1$ (resp. $0 < \operatorname{Re} z < 1$).

(i) For $z = is \in i\mathbb{R}$, we estimate

$$|f(is)| = |\operatorname{tr}((u\tilde{a}^{is} + v\tilde{b}^{is})\tilde{c}^{1-is}t + (u\tilde{a}^{is} + v\tilde{b}^{is})\tilde{d}^{1-is}w))|$$
$$\leq 2(\|\tilde{c}\|_1 + \|\tilde{d}\|_1) = 2y .$$

For $z = 1/2 + is$, $s \in \mathbb{R}$, we estimate

$$|f(1/2 + is)| \leq 2^{1/2}(\|\tilde{a}^{1/2}\|_2^2 + \|\tilde{b}^{1/2}\|_2^2)^{1/2}(\|\tilde{c}^{1/2}\|_2^2 + \|\tilde{d}^{1/2}\|_2^2)^{1/2}$$
$$= 2^{1/2}x^{1/2}y^{1/2} ,$$

by the first part of the proof. When $p \geq 2$, that is, $0 < 1/p \leq 1/2$, the theorem of three lines yields

$$|\operatorname{tr}((a + b)c + (a - b)d)| = |f(1/p)|$$
$$\leq \exp((1 - 2/p)\log(\sup_{s\in\mathbb{R}} |f(is)|) + 2/p \log(\sup_{s\in\mathbb{R}} |f(1/2 + is)|))$$
$$\leq \exp((1 - 2/p)\log(2y) + 2/p \log(2^{1/2}x^{1/2}y^{1/2})$$
$$= \exp(1/q \log 2 + 1/p \log x + 1/q \log y)$$
$$= 2^{1/q}(\|a\|_p^p + \|b\|_p^p)^{1/p}(\|c\|_q^q + \|d\|_q^q)^{1/q} .$$

(ii) For $s \in \mathbb{R}$, we estimate

$$|f(1 + is)| = |tr((u\tilde{a}^{1+is} + v\tilde{b}^{1+is})\tilde{c}^{-is}t + (u\tilde{a}^{1+is} + v\tilde{b}^{1+is})\tilde{d}^{-is}w)|$$

$$\leq 2(\|\tilde{a}\|_1 + \|\tilde{b}\|_1) = 2x .$$

When $1 < p \leq 2$, that is, $1/2 \leq 1/p < 1$, we estimate

$$|tr((a + b)c + (a - d)c)| = |f(1/p)|$$

$$\leq \exp((2 - 2/p)\log(2^{1/2}x^{1/2}y^{1/2}) + (2/p - 1)\log(2x))$$

$$= \exp(1/p \log 2 + 1/p \log x + 1/q \log y)$$

$$= 2^{1/p}(\|a\|_p^p + \|b\|_p^p)^{1/p}(\|c\|_q^q + \|d\|_q^q)^{1/q} . \qquad \text{(Q.E.D.)}$$

<u>Proposition 3.4.2</u> ([12], [20], [23], [44]). Let A, B be elements in $L^p(\mathfrak{m})$.

(i) $\|A + B\|_p^p + \|A - B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p)$ for $2 \leq p < \infty$.

(ii) $\|A + B\|_p^p + \|A - B\|_p^p \geq 2^{p-1}(\|A\|_p^p + \|B\|_p^p)$ for $1 < p \leq 2$.

Proof. It is sufficient to show the above inequalities in $L^p(\mathfrak{m};\varphi_0)$. (i) For $a,b \in L^p(\mathfrak{m};\varphi_0)$, there exist two positive numbers x, y such that $(\|a + b\|_p^p + \|a - b\|_p^p)^{1/p} = x\|a + b\|_p + y\|a - b\|_p$, $x^q + y^q = 1$. By Corollary 3.3.12, it is possible to choose $c,d \in L^q(\mathfrak{m};\varphi_0)$ satisfying

$$\|a + b\|_p\|c\|_q = |tr((a + b)c)|, \qquad \|c\|_q = x ,$$

$$\|a - b\|_p\|d\|_q = |tr((a - b)d)|, \qquad \|d\|_q = y .$$

Here, we may assume that both of $tr((a + b)c)$ and $tr((a - b)d)$ are non-negative by considering multiples of c and d. We then estimate

$$(\|a + b\|_p^p + \|a - b\|_p^p)^{1/p} = x\|a + b\|_p + y\|a - b\|_p$$
$$= \|a + b\|_p\|c\|_q + \|a - b\|_p\|d\|_q$$
$$= tr((a + b)c + (a - b)d)$$
$$\leq 2^{1/q}(\|a\|_p^p + \|b\|_p^p)^{1/p}(\|c\|_q^q + \|d\|_q^q)^{1/q} ,$$

by Lemma 3.4.1,(i). Since $x^q + y^q = \|c\|_q^q + \|d\|_q^q = 1$, we have

$$(\|a + b\|_p^p + \|a - b\|_p^p)^{1/p} \leq 2^{1/q}(\|a\|_p^p + \|b\|_p^p)^{1/p} .$$

(ii) The above argument (together with Lemma 3.4.1,(ii)) shows

$$(\|a + b\|_p^p + \|a - b\|_p^p)^{1/p} \leq 2^{1/p}(\|a\|_p^p + \|b\|_p^p)^{1/p} ,$$

that is, $\|a + b\|_p^p + \|a - b\|_p^p \leq 2(\|a\|_p^p + \|b\|_p^p)$. By replacing a, b by $(a + b)/2$, $(a - b)/2$ respectively, we obtain

$$\|a\|_p^p + \|b\|_p^p \leq 2^{1-p}(\|a + b\|_p^p + \|a - b\|_p^p) . \qquad \text{(Q.E.D.)}$$

We recall that a Banach space X is uniformly convex if it always follows from $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, and $\lim_{n\to\infty} \|(x_n + y_n)/2\| = 1$ $(x_n, y_n \in X)$ that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. ([28]) A uniformly convex Banach space is reflexive, that is, the double dual X^{**} is isomorphic to X.

<u>Theorem 3.4.3</u>. For $1/p + 1/q = 1$, $1 \leq p < \infty$, $L^q(\mathfrak{M})$ is the dual space of $L^p(\mathfrak{M})$. Here, the duality is given by the sesquilinear from introduced in Definition 3.3.4.

Proof ([12]). Since $L^\infty(\mathfrak{M}) = \mathfrak{M}$ is the dual space of $L^1(\mathfrak{M}) = \mathfrak{M}_*$, we may assume $1 < p < \infty$.

It follows from Lemma 3.4.2,(i) that $L^p(\mathfrak{m})$, $2 \leq p < \infty$, is uniformly convex, hence, reflexive. In fact, for $A, B \in L^p(\mathfrak{m})$, we have

$$\|A - B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p) - \|A + B\|_p^p$$

$$= 2^{p-1}(\|A\|_p^p + \|B\|_p^p) - 2^p\|(A + B)/2\|_p^p$$

$$= 2^{p-1}\{(\|A\|_p^p + \|B\|_p^p) - 2\|(A + B)/2\|_p^p\} .$$

Due to Corollary 3.3.12, $L^q(\mathfrak{m})$, $1/p + 1/q = 1$, $2 \leq p < \infty$, can be isometrically imbedded into $L^p(\mathfrak{m})^*$ through the sesquilinear form. In particular, $L^q(\mathfrak{m})$ is a closed subspace of $L^p(\mathfrak{m})^*$. To show $L^q(\mathfrak{m}) = L^p(\mathfrak{m})^*$ by contradiction, we assume that $L^q(\mathfrak{m}) \subsetneq L^p(\mathfrak{m})^*$. It follows from the Hahn-Banach theorem that there exists a non-zero functional f belonging to $L^p(\mathfrak{m})^{**}$ such that f vanishes on $L^q(\mathfrak{m})$. However, since $L^p(\mathfrak{m})$ is reflexive, f belongs to $L^p(\mathfrak{m})$ so that Corollary 3.3.12,(i) implies that f is zero, which is a contradiction. We thus conclude that $L^q(\mathfrak{m}) = L^p(\mathfrak{m})^*$, $2 \leq p < \infty$. Also, the reflexivity of $L^p(\mathfrak{m})$ implies $L^q(\mathfrak{m})^* = L^p(\mathfrak{m})^{**} = L^p(\mathfrak{m})$.

(Q. E. D.)

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