TOPOS THEORETIC METHODS
IN GEOMETRY

A collection of articles

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PREFACE

The present collection of articles is part of a trend towards use of topos theory in "basic" mathematics, in particular in geometry including differential geometry, but also in elementary analysis and algebra.

As such, the collection can be seen as work on the points one and two in the three-point program outlined in the article "Categorical Dynamics" by Lawvere, which is included as the first article in the collection. The lectures, of which this article is a summary, date back to 1967 and have deeply influenced the subject.

Thus the articles all "do" basic mathematics in toposes, or in some specific topos, but hopefylly it will be apparent that they do that as part of a program, whose aim is not just to "do", but to guide the learning development and use of mathematics.

It is appropriate here to give some historical remarks on how and when the remaining articles in the present volume were collected.

In the period May 10-24, 1978, an arrangement* took place at the Mathematics Institute at Aarhus, with the title

OPEN HOUSE ON
TOPOS THEORETIC METHODS
IN GEOMETRY AND ANALYSIS

*partially supported by the Danish Natural Science Research Council

The idea was to have a work session on the specific program mentioned but at the same time to provide an open forum for talks or discussions on mathematical and category theoretical topics in general. I used this meeting to make proposals to some of the speakers (namely those whose contributions I deemed were inside the specific program) to print reports on their talks in a quick informal way - together with some other, previously unpublished, but relevant material.

This is what is being done here (except for the quickness). Some authors needed longer time than others for having their contributions ready, so that I had to postpone the deadline several times; for which I apologize to the contributors who obeyed the first or second dead-line.

It is clear from the foregoing that the present collection is not a "Proceedings of the Open House Arrangement", but that, on the other hand, the collection to a large extent is an off-spring of the Open House.

A brief report of the Open House is included as item 10.

I shall attempt to give a few comments on the contents and mutual relationship between the articles. The numbers refer to the "table of contents" above.

In 1., the whole program of synthethic differential geometry and categorical dynamics, is presented (as well as, implicitely, the importance of topos theory). The articles 6. and 7. are contributions to development of differential geometry on such an axiomatic or synthetic basis, whereas the articles 2. and 4. are concerned with the question of models for these axioms. These papers contain two different proofs of the affirmative answer to a question I raised in Nov. 1977 in the "Peripatetic Seminar on

Sheaves and Logic", namely whether the generic ring for an ε-stable coherent theory of rings would provide a model for synthetic differential geometry. The notion of ε-stability was introduced in 5. in an attempt to understand those coherent ring theoretic properties that seem to be relevant to both real-number-objects in spatial toposes, like in 8., and to line-type rings in synthetic differential geomety. The study of both these kinds of rings leads to the study of real-algebraic geometry, which even in the <u>set</u>-case is not too well understood. The article 3. is a contribution towards that, using sheaf theoretic methods. Finally, 8. and 9. deal with complex numbers and complex analysis in toposes: in 8. the problem of how to adjoin a square root of -1 to a field object in a topos is solved by means of Artin glueing; in 9, the notion of complex structure on a ("classical") manifold is studied using the internal language of the topos it defines, and two distinguished real-number objects therein.

The articles were available in their present form (except for minor revisions) at the following dates: 1.: May 79, 2.: Aug.78, 3.: Aug.78, 4.: Feb.79, 5.: June 77, 5b.: March 78, 6.: Oct.78, 7.: Dec.77, 8.: Aug.78, 9.: May 78.

Anders Kock

CATEGORICAL DYNAMICS

F. William Lawvere

[The following is intended as a summary of some lectures which I gave at several places in 1967. In these lectures, I offered some preliminary calculations in support of a program to (3) axiomatize the foundations of continuum mechanics in the spirit of Walter Noll on the basis of (2) a direct axiomatization of the essence of differential topology using results and methods of the French work in algebraic geometry (some of which I had learned from Gabriel); but I further maintained that this requires (1) axiomatic study of categories of smooth sets, similar to the topos of Grothendieck, since the most natural form of (2) is incompatible with "usual" set theory. Now, since my joint work with Tierney in 1969-1970, several conferences, many articles, and even one published book (by Johnstone) have been devoted to carrying out part (1) of this program. Meanwhile, a serious start on part (2) by Wraith and Kock has been followed by several further contributions, and in particular Dubuc in August 1978 explicitly demonstrated the consistency of part (2) by constructing a category in which ordinary differential topology is fully embedded but which moreover, satisfies the set-theoretically outragious axioms suggested by algebraic geometry. Work on (2) is far from complete (for example, it now seems that an approach in this spirit to differential forms involves still further divergence from "usual" set-theoretical logic). However, the growth of confidence in the program engendered by these developments has also led to a growth of interest in the origin of the program itself. I am taking advantage of this current interest to publish this summary, along with the observation that seriously taking up part (3) of the program will surely lead in particular to further illumination of parts

(1) and (2). Of course, the framework of "ordinary" set-theory has not succeeded to prevent Noll's own work from advancing; two fundamental works from the early 1970's are included in his selected papers published by Springer. My main external sources for the following summary have been page 937 of volume 14 of the Notices of the AMS and especially notes taken by Saunders MacLane on May 19, 1967 at Chicago and on November 25, 1967 at Urbana, which he very kindly sent to me in summer 1978. Some remarks based on more recent developments have been inserted into the summary between brackets [ ].]

I hope that categorical methods can be used to give a simple axiomatic basis for parts of mathematics which arose from physics (particle mechanics, fluid mechanics, differential geometry, harmonic analysis, etc). Some physicists and engineers seem in effect to have the insight that geometrical and physical constructions can be performed, with almost as much freedom as sets can be defined in naive set theory, without ever leaving the realm of smooth objects and smooth maps. But usual mathematical models, such as the category of smooth manifolds, on the one hand presuppose a long intricate purely mathematical construction (there does not seem to be an intrinsic description of that category which could reasonably be taken as a "simple" starting point) and on the other hand are poor in regard to closure properties since even something so fundamental (for calculus of variations etc) as the smooth space of smooth maps between two smooth spaces is ambiguous and difficult, and pullbacks in general don't exist.

[As I emphasized in my 1971 - 72 Aarhus lectures, not only the function space but also the smooth space of smooth subspaces and the smooth space of representations of a given smooth group "should" have clear meanings according to such insight] But rather than scoffing at insight (which some seemed to have considered the only healthy public response in recent decades) we can try to axiomatically express what some aspects of it might mean precisely and also to construct mathematically acceptable models of such axioms, in the hope ultimately of actually clarifying the learning, development, and use of these branches of mathematics. From 1966 Oberwolfach lectures by M. Demazure and P. Gabriel I learned some facts and methods which seem important both for the axiomatics and for the construction of models, essentially the Cartier-Grothendieck functorial approach to algebraic groups [since published in Springer Lecture Notes # 151 (1970) and a 1969 North-Holland book by Demazure-Gabriel]*

Consider a category $\mathcal{X}$ in which we have a given ring object R . About $\mathcal{X}$ we will assume that it has a terminal object 1, pullbacks, and for each $X \xrightarrow{f} Y$, a right adjoint

$$(1) \qquad \mathcal{X}/X \xrightarrow{\Pi_f} \mathcal{X}/Y$$

to the functor f^* of pulling back along f. This implies that each $\mathcal{X}/X$ has an internal hom right adjoint to product over X, denoted by exponentiation. [Thus $\mathcal{X}$ is what came to be called, after the work of Penon, a locally cartesian-closed category.]*

(Later we will need one construction which is most easily guaranteed by assuming $\mathcal{X}$ has countable coproducts and coequalizers.) Objects of $\mathcal{X}$ are to be thought of as smooth spaces, and morphisms $X \longrightarrow R$ are to be thought of as quantities smoothly varying over X. Note that for example $Hom_R(A,B)$ for two R-modules has a well-defined meaning as a subobject of B^A. R-modules are to be thought of as vector spaces (with a smooth structure) even though we do <u>not</u> assume R is a field. The geometric origin of R is roughly as follows. In $\mathcal{X}$ there are Euclidean spaces E_1, E_2, E_3 whose structure (=basic geometric constructions) are given by morphisms of $\mathcal{X}$. In particular there are <u>abelian</u> subgroups

$$V_n = \text{Trans}\ (E_n) \subset E_n^{E_n}$$

of translations and hence rings

$$R_n = \text{Hom}\ (V_n,\ V_n).$$

$R = R_1$ is <u>commutative</u> because of two facts: E_1 is <u>one</u> - dimensional, and every homomorphism $V_1 \longrightarrow V_1$ is a homothety because it, like every map in $\mathcal{X}$, is smooth. Of course from analytic geometry we know essentially how to use cartesian products to construct co-ordinatized models of E_n, imagining in inverted fashion that we start with the datum R.

The second axiom will permit an intrinsic theory of differentiation to be developed. We assume given a subobject $D \subset R$ which contains the zero quantity $1 \xrightarrow{0} R$ and which is to be thought of as the space of first-order infinitesimal quantities.

For any object X, the object X^D will be thought of as the <u>tangent bundle</u> of X, with projection $X^D \longrightarrow X$ induced by $1 \xrightarrow{0} D$, and for any morphism $X \xrightarrow{f} Y$ in $\mathcal{X}_0$, f^D will be thought of as the derivative of f. Thus a tangent vector $D \longrightarrow X$ to X is at the point $1 \xrightarrow{0} D \longrightarrow X$ of X, and the derivative f^D of f takes any tangent vector $D \xrightarrow{v} X$ at a point x to the tangent vector $D \xrightarrow{v} X \xrightarrow{f} Y$ at the point fx. The functoriality of exponentiation $(\ )^D$ is thus essentially the chain rule for differentiation. To prove 1) the Leibniz rule (for differentiation of variable quantities) as well as that 2) there are precisely the right amount of tangent vectors for R and related spaces, we assume our second axiom (2): We need that D is closed under the action of the multiplicative monoid R, and that the composites

(a) $\quad D \rightarrowtail R \overset{0}{\underset{(\ )^2}{\rightrightarrows}} R$

are equal, where $(\ )^2$ denotes the squaring map from the ring structure of R and 0 denotes the constantly 0 map $R \longrightarrow 1 \xrightarrow{0} R$, and we also need that there is an isomorphism

(b) $\quad R^D \cong R \times R$

In fact, we may as well define D by requiring that (a) be an equalizer, and assume (b). [However, as Massimo Galuzzi and Gian-Carlo Meloni calculated in July 1978, (a) follows from (b) if we assume that $\frac{1}{2} \in R$ and interpret (b), as in the meantime had been done in several papers by Anders Kock, to mean that the canonical morphism $R^D \longleftarrow R \times R$ is invertible]. Though there are many morphisms $R \longrightarrow R$ (there are at least all the polynomials), upon restricting to D they all become <u>linear</u>; but on the other hand D is large enough so that distinct linear (i.e. affine) maps $R \rightrightarrows R$ have distinct restrictions $D \rightrightarrows R$. We need category theory for this axiom, since it seems no such ring could exist in classical set theory [as was proved in considerable generality by calculations in the mid-70's by Kock, Schanuel, and Lawvere]. The condition (b) is not restricted to "line-like" R, since it follows that for any R-module V

$$(V^*)^D = V^* \times V^*$$

canonically, since

$$\mathrm{Hom}_R(V,R)^D = \mathrm{Hom}_R(V,R^D) = \mathrm{Hom}_R(V,R\times R) = V^* \times V^*$$

However, many vector spaces are not dual modules and it is less clear how to compute their tangent bundles. But it is trivial that for any X, Y in $\mathcal{X}$

$$(Y^X)^D = (Y^D)^X$$

showing how "easy" the smooth structure of infinite-dimensional objects really is. Using (b) we can define the pre-gradient of any variable quantity $X \xrightarrow{f} R$ to be the composite

$$X^D \xrightarrow{f^D} R^D \cong R \times R \xrightarrow{\pi} R$$

where π is the other projection, the one not corresponding to the map induced by $1 \xrightarrow{0} D$. Also the interpretation of tangent vectors as distributions ("of compact support") is given by the morphism

$$X^D \longrightarrow \mathrm{Hom}_R(R^X, R)$$

corresponding to

$$X^D \times R^X \longrightarrow R^D \xrightarrow{\pi} R.$$

Note that differentiation is itself a smooth map

$$Y^X \longrightarrow (Y^D)^{(X^D)}$$

as is the pre-gradient

$$R^X \longrightarrow \mathrm{Hom}_R^{\bullet}(X^D, R)$$

where $\mathrm{Hom}_R^{\bullet}$ denotes morphisms which are homogeneous with respect to the action of the <u>multiplicative</u> <u>monoid</u> R. This monoid acts on D, hence on X^D. On the other hand, <u>addition</u> of tangent vectors

$$X^D \times_X X^D \xrightarrow{+} X^D$$

only exists under the assumption on X, that the functor $X^{()}$ takes certain non-pushout squares of D=like objects into products in $\mathcal{X}/X$, [essentially what is called "condition E" in SGA3 as I noticed in April 1979]. On the other hand, since C^∞ maps which are everywhere defined on a vector space and homogeneous of degree 1 are automatically additive, we may expect that

$$\mathrm{Hom}_R\ (V,V') \hookrightarrow \mathrm{Hom}_R^{\bullet}\ (V,V')$$

has a strong tendency to be an isomorphism in our $\mathcal{X}$, and that in particular

$$\mathrm{Hom}_R^{\bullet}\ (R^D,\ R)$$

may serve as a reasonable surrogate for

$$\mathrm{Hom}_{R \times X}\ (X^D,\ R \times X)$$

even when X^D is not additive over X.(Here we imagine that Hom has been given some rational definition using the rich supply of additive relations induced by those not-necessarily-pullback squares of multiple tangents over X).

[(After reading Kock's exposés on Synthetic Differential Geometry from the Benabou Seminar Jan. 1979) The natural extension of the axiom (b) itself to multiple and higher tangents seems to be to consider the category $\mathcal{W}$ of all commutative R-algebras W in $\mathcal{X}$ with the following properties

$$W = R \oplus H$$
$$H \cong R^k \text{ as R-modules, some } k \in N$$
$$H \underset{(\;)^{p+1}}{\overset{0}{\rightrightarrows}} H \text{ equal for some } p \in N$$

and to define

$$D(W) = \underline{Alg_R}(W,R)$$

and then require that the natural map

$$W \longrightarrow R^{D(W)}$$

into the double dual be an isomorphism for all W in $\mathcal{W}$. This implies again the same statement for any dual vector space

$$\begin{array}{ccc} V^* \underset{R}{\otimes} W & \overset{\approx}{\longrightarrow} & (V^*)^{D(W)} \\ & \searrow_{\cong} \quad \nearrow_{\cong} & \\ & \mathrm{Hom}_R(V,W) & \end{array}$$

]

We define a <u>vector field</u> on an object X in $\mathcal{X}$ to be any section v of $X^D \longrightarrow X$, and a <u>morphism of vector fields</u> $X,v \longrightarrow X',v'$ to be any f in $\mathcal{X}$ such that

$$\begin{array}{ccc} X^D & \xrightarrow{f^D} & X'^D \\ \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle v'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

is commutative. We thus get a category $\mathrm{Vect}_R(\mathfrak{X})$. Because our tangent concept is representable by a single generic object D, the notion of vector field can be equivalently expressed in the simpler form

$$\begin{array}{ccc} X \times D & \xrightarrow{\bar{v}} & X \\ {\scriptstyle 0}\nwarrow & & \nearrow{\scriptstyle 1_X} \\ & X & \end{array}$$

with a corresponding form of the notion of morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \uparrow{\scriptstyle \bar{v}} & & \uparrow{\scriptstyle \bar{v}'} \\ X \times D & \xrightarrow[f \times D]{} & X' \times D \end{array}$$

When convenient, the notion of vector field can be equivalently expresse in a third way:

$$\begin{array}{ccc} D & \xrightarrow{\bar{\bar{v}}} & X^X \\ {\scriptstyle 0}\nwarrow & & \nearrow{\scriptstyle \ulcorner 1_X \urcorner} \\ & 1 & \end{array}$$

The object R carries a canonical vector field (essentially the derivative of the identity) so that for any path $R \longrightarrow X$, its derivative can be composed with it to yield a path of tangent vectors.

The notion of vector field is usually taken to be the basic notion of "differentiable dynamical system", in infinitesimal form. The corresponding integrated form, is a <u>flow</u> or action of the additive group R; in the continuous case the study of such is called "topological dynamics". The narrow meaning of the term "categorical dynamics" is thus analogous to the use of "cat" as a variable which can take values like cat=top, cat = diff, cat = PL, etc., i.e. the study of $\mathcal{X}$-flows, where $\mathcal{X}$ denotes a pair $\mathcal{X}$, R satisfying our two axioms and where a flow is a pair X, $X \times R \longrightarrow X$ in $\mathcal{X}$ satisfying the usual axioms

$$xe^{0} = x$$

$$xe^{t_1+t_2} = (xe^{t_1})e^{t_2}$$

where this use of the symbol e is solely for notational harmony. A morphism f of flows satisfies

$$f(xe^{t}) = (fx)e^{t}.$$

Thus we have a category $\mathrm{Flow}_R(\mathcal{X})$ of $\mathcal{X}$-dynamical objects.

Now since $D \subset R$, every flow $X \times R \longrightarrow X$ restricts to a vector field $X \times D \longrightarrow X$ by considering only those time-lapses infinitesimally close to 0, yielding a functor

$$\mathrm{Flow}_R(\mathcal{X}) \xrightarrow{(\ )^{\cdot}} \mathrm{Vect}_R(\mathcal{X})$$

which preserves underlying space. The problem of integrating a system of ordinary differential equations could thus be viewed as having two parts, namely applying an adjoint to the functor $(\ )^{\cdot}$ and then studying to what extent the underlying space has been changed by such "integration". Actually the above functor has two adjoints, which might fancifully be called the "upper and lower integrals of a vector field". The right adjoint can be seen to exist without further ado as

$$\mathrm{Hom}_D(R,X)$$

the subspace (of the space of all complete paths $R \longrightarrow X$) consisting of morphisms from the canonical vector field on R to given one v on X, or briefly the subspace consisting of solution curves for the infinitesimal flow v. This solution space carries a natural flow, induced by translations on R itself, whose corresponding infinitesimal flow is mapped morphically back to X,v by the evaluation at 0

$$\mathrm{Hom}_D(R,X) \xrightarrow{\ \epsilon\ } X$$

which to every solution curve assigns its underlying initial-value at 0. The properties of injectivity or surjectivity of ϵ express exactly the uniqueness or existence theorem for the initial-value problem for the ODE system determined by v.

To calculate the left adjoint

$$\mathrm{Vect}_R(\mathcal{X}) \xrightarrow[\ (\ )\otimes_{D} R\ ]{} \mathrm{Flow}_R(\mathcal{X})$$

to $(\ )^{\bullet}$, we need the existence of coequalizers

$$X \times D \times R \rightrightarrows X \times R \longrightarrow X \otimes_D R$$

$$x, h, t \rightsquigarrow x, t+h$$
$$\rightsquigarrow xe^h, t$$

where we have written $xe^h = v(x,h)$ and where of course R acts on $X \otimes_D R$ by $\langle x,s\rangle e^t = \langle x, s+t\rangle$. To <u>compute</u> $X \otimes_D R$ in a particular case is more difficult, as it depends not only on detailed knowledge of v but also on detailed knowledge of coequalizers in $\mathcal{X}$. An approximation to such a computation may be useful along the following lines. Let $\mathcal{X}$, R be as appropriate for algebraic or analytic geometry over a field of characteristic 0 (see below) and suppose X = spec(A) for a commutative algebra A. Then a vector field on X can be identified with a derivation d_v (Leibniz rule) on A since elements of A are identified with morphisms $X \xrightarrow{f} R$ and we can always form

$$\underbrace{X \xrightarrow{v} X^D \xrightarrow{f^D} R^D \cong R \times R \xrightarrow{\pi} R}_{d_v f}$$

Now define (in sets $\mathcal{S}$)

$$A_v = \left\{ f \in A \;\middle|\; \exists\, n \geq 0 \;\left[d_v^{n+1}(f) = 0 \right] \right\}$$

in terms of iterates of d_v, a subalgebra of A, for which a flow on spec (A_v) can be explicitly defined by

$$\begin{array}{ccc} A_v & \longrightarrow & A_v[t] \\ f & \rightsquigarrow & \sum \dfrac{d_v^n(f)}{n!} \; t^n \end{array}$$

Then there is a unique morphism of flows such that the following diagram of morphisms of vector fields commutes

$$\begin{array}{ccc} X & \longrightarrow & X \otimes_D R \\ & \searrow & \vdots \\ & & \downarrow \\ & & \text{spec}\ (A_v) \end{array}$$

Note that A_v is filtered into quantities invariant under the flow "on X", quantities whose time dependence is linear along the flow, etc. The value of such approximation seems limited, however.

If we also have countable coproducts in $\mathcal{X}$, then the two "integrals" for $(\ )^{\bullet}$ can be viewed as special cases of the very general adjoints $\mathrm{Hom}_S(R,-)$, $(\ )\otimes_S R$ associated to a homomorphism $S \longrightarrow R$ of any two monoids in $\mathcal{X}$. For we can define

$$e^D = \sum_{n=0} D^n/n!$$

the free commutative monoid on the object D, where $(\ )^n/n!$ denotes the orbit space for the natural action of the symmetric group, and find a natural homomorphism $e^D \longrightarrow R$ induced by the inclusion $D \subset R$, and whose image is the ideal of R generated by D, consisting of all sums of elements of square 0. If S is defined as the quotient of e^D modulo the congruence relation determined by the condition that $1 \xrightarrow{0} D$ be congruent to the neutral element of e^D, then $e^D \longrightarrow S \longrightarrow R$ and we have

$$\mathrm{Flow}_R(\mathcal{X}) = \mathcal{X}^R \xrightarrow{(\ )^{\circ}} \mathcal{X}^S \cong \mathrm{Vect}_R(\mathcal{X}).$$

<u>provided</u> D is so small that every vector field in $\mathcal{X}$ also satisfies <u>infinitesimal commutativity</u>

$$\begin{array}{ccccc} X \times D \times D & \xrightarrow{v \times D} & X \times D & \searrow^{v} & \\ \Big\downarrow X \times \tau & & & & X \\ X \times D \times D & \xrightarrow[v \times D]{} & X \times D & \nearrow_{v} & \end{array}$$

However, if it turns out that the latter is a special condition on X and/or v (i.e. does not follow from the identity axiom for vector-fields by itself) then one should consider its companion condition of infinitesimal invertibility

$$X \times D \xrightarrow{\langle v, 1_D \rangle} X \times D \xrightarrow{X \times (-1)} X \times D \xrightarrow{v} X \qquad (\text{composite } = 1_X)$$

and even consider strengthening the concept of vector field by adding the following still stronger axiom of infinitesimal associativity to the definition:

$$\begin{array}{ccccc} X \times D_1 & \xrightarrow{X \times (+)} & X \times D & & \\ \downarrow & & & \searrow^{v} & \\ & & & & X \\ & & & \nearrow_{v} & \\ X \times D \times D & \xrightarrow[v \times D]{} & X \times D & & \end{array}$$

Here D_1 is the infinitesimal neighborhood of the diagonal in D x D defined by the pullback with multiplication

$$\begin{array}{ccc} D_1 & \longrightarrow & 1 \\ \downarrow & & \downarrow 0 \\ D \times D & \xrightarrow{\cdot} & D \end{array}$$

whose importance here is that it is equally well defined by the pullback.

$$\begin{array}{ccc} D_1 & \hookrightarrow & D \times D \\ (+) \, \vdots & & \downarrow + \\ D & \hookrightarrow & R \end{array}$$

since we assume $\frac{1}{2} \in R$. (Even if the above additional laws are added to the definition of S, it is not clear whether $S \longrightarrow R$ becomes monic).

Vector fields are just the simplest kind of models of a differential-algebraic theory, where the latter refers to a concept more general than algebraic theories in $\mathcal{X}$, whose arities are natural numbers, but significantly less general than general monads (= triples) in $\mathcal{X}$, whose arities are arbitrary objects, namely we consider theories in $\mathcal{X}$ whose arities and co-arities are objects like D {i.e. more generally D(W) for $W \in \mathcal{W}$} where in general an operation of arity A and co-arity C on X means a map $C \times X^A \longrightarrow X$. The hope would be that more refined theorems as to coequalizers, etc. could be proved for such limited theories than could be true for arbitrary monads in X. Thus for example in ordinary algebraic theories we can deal with commutative algebras X with an additional unary operation f satisfying $f(x_1 + x_2) = f(x_1) \cdot f(x_2)$, but only with differential-algebraic theories as modeled in such $\mathcal{X}$ does it become "algebraic" to require also f' = f where ( )' is the <u>intrinsic</u> derivative for the underlying object of X (preceeded by $X \xrightarrow{\langle X,1\rangle} X \times X \cong X^D$ and followed by $X^D \cong X \times X \xrightarrow{\pi} X$). Even ordinary "abstract" algebraic theories, e.g. groups or Lie algebras, when extended naturally to "trivial" differential-algebraic theories, may have non-trivial morphisms of differential-algebraic theories between them.

As is well known, if G is a model of an algebraic theory in a category with exponentiation and if I is an object then G^I is a model of the same theory, and moreover maps $I' \longrightarrow I$ induce homomorphisms $G^I \longrightarrow G^{I'}$. For example if G is a monoid then the projection $G^D \longrightarrow G$ is a homomorphism of monoids (of groups if G is a group) and the kernel of that homomorphism is Lie(G). For example, $\mathrm{Lie}(X^X) = \mathrm{Vect}(X)$, the object whose elements are all the vector fields on X, which is thus seen to always carry an "addition" (maybe non-commutative even for the X which are so bad that addition on X^D in $\mathcal{X}/X$ does not exist; of course, if addition of tangent vectors does exist then the Eckmann-Hilton Lemma shows that the "addition" must be commutative since it must agree with addition by naturality. Since Lie is functorial for monoid homomorphisms, an associative action of G on a space X induces an "infinitesimal action"

$$X \times \mathrm{Lie}(G) \times D \longrightarrow X$$

of Lie(G) on vector fields on X. What is explicitly the monoid M obtained by dividing the free monoid generated by Lie(G) x D by all relations which are true in all actions induced from a global G-action? It is again clear in principle that there are left and right "integration" adjoints.

Now the functor

$$\mathrm{Gr}(\mathcal{X}) \xrightarrow{\mathrm{Lie}} \mathcal{L}ie(\mathcal{X})$$

is itself representable, in fact by the S previously discussed. However we don't know exactly what $\mathcal{L}ie(\mathcal{X})$ is; with respect to which doctrine of theories should the costructure of S be computed - partial differential-algebraic theories? {In the first circulated article following the synthetic approach suggested in the lectures here summarized, Gavin Wraith in the early 70's showed how the pullback conditions on multiple tangents of G needed to get the Lie-algebra structure on Lie(G) could be expressed and used in the axiomatic setting } For any definite interpretation of $\mathcal{L}ie(\mathcal{X})$ general principles say that Lie will have a <u>left adjoint</u>, and hence in particular for each $G \in \mathrm{Gr}(\mathcal{X})$ a co-adjunction homomorphism $\hat{G} \longrightarrow G$, whose kernel and cokernel are further definite groups which could be called $\pi_1(G)$ and $\pi_0(G)$....But whatever may be the complications which may lurk in "arbitrary" group objects, the above definitions and axioms are sufficient to calculate explicitly in $\mathcal{X}$ the Lie algebra of classical algebraic groups, e.g.

Lie(GL(n)) = R_n with commutator

Lie(SO(3)) = V_3 with cross product.

[See not only the writings of Demazure and Gabriel but also J.P. Serre's Benjamin book on Lie groups and Lie algebras.(1965)]

The physical study of a dynamical system involves not only a state space X equipped with a dynamical vector field, but actually a more specific construction of such in terms of simpler objects. Frequently there is a space Q of configurations and a given map $X \longrightarrow Q$ expressing that each state has an underlying configuration, but in general must involve more. For particle mechanic rigid body mechanics, and hydrodynamics one can define

$$X = Q^D$$

but this actually amounts to the very restrictive hypothesis that the response of the material depends only on the infinitesimal history of its motion, where motions are interpreted to mean paths $R \xrightarrow{q} Q$ in configuration space. In the "simple" cases just mentioned, the analysis of the required vector field on X is often associated with the study of a "Lagrangian"

$$\mathcal{L} : X \longrightarrow R$$

which induces a functional

$$Q^R \longrightarrow R^{R \times R}$$

called "action" by applying to

$$R \xrightarrow{\text{canon}} R \times R \cong R^D \xrightarrow{q^D} Q^D = X \xrightarrow{\mathcal{L}} R$$

$$\text{"}\mathcal{L}(q,\dot{q})\text{"}$$

the integration process

$$R^R \xrightarrow{\int} R^{R \times R}$$

$$f \rightsquigarrow (\langle a,b\rangle \rightsquigarrow \int_a^b f(t)dt)$$

A possible (physically motivated) addition to our two axioms would be the existence of the morphism $\int$, but it is not clear what condition on it would be both desirable and possible. [one of the desirable ones would be

$$\int_a^{a+h} f(t)dt = f(a)h$$

for any h such that $h^2 = 0$. This would seem to yield an algebraic proof of the fundamental theorem of calculus, in conjunction with the additivity of $\int$ in each of its two kinds of argument.]

But more fundamentally, even if the rather abstract Lagrangian is useful, its construction and the construction of the vector field on states in a particular case involves the knowledge of <u>forces</u> and more particularly of an analysis of forces into three kinds; inertial, external, and internal mutual response. Such an analysis depends in turn on a more specific construction of the configuration space Q, which (even when X is more general than Q^D) is usually realized as a given subspace

$$Q \subseteq E^B$$

{of "placements"} where $E = E_3$ is the actual space and where B is the space of "particles" of the material body in question.

In particle mechanics, B is a finite discrete set, but in continuum mechanics it is usually a three-dimensional manifold [although in the theory of rods, cords, plates, and shells, B is perhaps a lower-dimensional object for which the fibers of $B^D \longrightarrow B$ are nonetheless three dimensional]. One of the motivations for the axiomatic theory of $\mathcal{X}$,R is to give simple expression to the old idea that the theory of the infinite-dimensional Q with dim(E)>0 should be in some respects "just like" the particle case [which was also a motivation for K.T. Chen's Urbana (1978?) notes on the calculus of variations, in which a category with some properties in common to our $\mathcal{X}$ is independently constructed].

A reasonable condition on

$$Q \subseteq E^B$$

would be that Q is mapped into itself by the induced action of the group of rigid motions of E. The group G(E,Q) of all those invertible endomorphisms of E which map Q into Q might thus serve as a crude measure of the distinction between very rigid bodies (G minimal) and rarefied gases (G maximal); however a more serious measure of the distinction of the kinds of material B is made of should involve infinitesimal symmetry of the internal mutual response functional, not discussed here.

When the simple definition of state space suffices, we have

$$X = Q^D \subseteq (E^B)^D = (E^D)^B = (E \times V)^B$$

where $E^D = E \times V$ with $V = V_3$, the translation vector space of the affine space $E = E_3$, and hence

$$X = Q^D \subseteq Q \times V^B$$

where V^B is the space of velocity fields on B. Inertial forces, momentum and kinetic energy involve not only velocity fields and a metric on V for their computation, but also a further given structure of a mass distribution on the body. Using the total mass of the body as unit so that the mass of parts can be measured in terms of pure quantities R, such distribution can be considered as an R-linear morphism

$$R^B \xrightarrow{\ m\ } R$$

which preserves constants and which is positive. [But what is the best way to account axiomatically for positivity? Do the elements of D support a notion of positivity or not? Unpublished lecture of André Joyal at Columbia University, December 1975 on "real algebraic geometry" gives some indications.] Integration with respect to m can then be applied to functions with values in convex sets such as E, yielding in particular a "center of mass" map

$$Q \longrightarrow E.$$

The mass distribution and the metric on E are the main ingredients in the analysis of one kind of external force and internal mutual response, namely gravitation. For more details on more subtle internal mutual response which material bodies may have, see papers of Walter Noll in the Archive of Rational Mechanics and Analysis, late 1950s [and especially Noll's Selected Papers published by Springer 1974] the main physical and mathematical ideas of which can hopefully be expressed in categories like our $\mathcal{X}$.

An important virtue of the categorical axiomatics we have indicated is that if there is one model $\mathcal{X}$ then there are immediately infinitely many other interesting and useful models for the whole theory, in fact at least two classes of such. If G is any group object in $\mathcal{X}$ (e.g. the Galilean or Lorentz group??) then the category $\mathcal{X}^G$ of G-actions and equivariant morphims is again a model for our axioms if we interpret R to mean R with trivial action. Also if M is any ("parameter") object in $\mathcal{X}$ then the category $\mathcal{X}/M$ of objects over M is also again a model for our axioms, interpreting R as R x M; theory of dynamical systems in $\mathcal{X}/M$ is the theory of families of dynamical systems in $\mathcal{X}$ parameterized by M [as in bifurcation theory, see Marsden BAMS vol. 84, Nov. 1978]. [It was, as briefly indicated in paper for the Eilenberg volume, qualitative and unpublished considerations of the kind just mentioned, as much or more than published problems of independence, etc. in abstract set theory and logic, which were an important impetus toward the 1969 -70 Lawvere-Tierney development of essentially algebraic axioms for topos theory.]

Now we consider three general categorical constructions which are useful in showing the existence of models for our axioms as well as for suggesting possible stronger axioms. All our models $\mathcal{X}$ are subgenerated by the algebraic theory $\underline{A}$ whose n-ary operations are by definition

$$\underline{A}\,(R^n,R) = \mathcal{X}\,(R^n,R)$$

where by "subgenerate" we mean (strongly) generated by the full subcategory $\underline{C}$ of $\mathcal{X}$ determined by those objects X which occur as equalizers

$$X \longrightarrow R^n \rightrightarrows R^m$$

Thus conversely we can construct such $\mathcal{X}$ by starting with a suitable algebraic theory $\underline{A}$ and considering the category $\underline{C}^{op}$ of finitely presented $\underline{A}$-algebras, i.e. those that occur as coequalizers of finitely generated free $\underline{A}$-algebras in the category $\mathrm{Alg}(\underline{A}) = \mathrm{Lex}(\underline{C},\mathcal{S})$ of $\underline{A}$-algebras. Then $\mathcal{X}$ is to be sought as a full subcategory of $\mathcal{S}^{\underline{C}^{op}}$ whose inclusion has a left-exact left adjoint, for then thanks to work of Giraud and Verdier in SGA4, we can conclude that the category $\mathcal{X}$, called a "topos" will satisfy our first axiom on the existence of $\prod$ and in fact have further useful exactness properties. As a matter of fact, the basic duality between algebra and geometry is just the restriction of an adjoint pair called "conjugacy" by Isbell:

$$\begin{array}{ccc}
(\mathcal{S}^{\underline{C}})^{op} & \rightleftarrows & \mathcal{S}^{\underline{C}^{op}} \\
\uparrow & & \uparrow \\
\mathrm{Lex}(\underline{C},\mathcal{S})^{op} & & \mathrm{sh}(\underline{C},\mathcal{S}) \\
\| & & \| \\
\mathrm{Alg}(\underline{A})^{op} & \underset{\text{spec}}{\overset{\text{function algebra}}{\rightleftarrows}} & \mathcal{X} = \text{"}\mathrm{Geom}(\underline{A})\text{"}
\end{array}$$

where both conjugates are defined by the same formula

$$\text{conj}(\ )(C) = \text{Nat}(-,C)$$

where the C on the right denotes the representable functor of the appropriate variance, and Nat refers to natural transformations of functors of the same variance. Thus

$$(\text{function algebra of } X)(C) = \mathcal{X}(X,C)$$

and

$$\text{spec}(A)(C^{op}) = \text{Alg}(\underline{A})(A,C^{op}).$$

Those $A \in \text{Alg}(\underline{A})$ which are <u>inverse</u> limits of finitely-presented $\underline{A}$-algebras will satisfy

$$A = \text{function algebra of spec}(A).$$

There is still the choice of <u>which</u> subtopos $\mathcal{X}$ of $\mathcal{S}^{C^{op}}$ is more appropriate but note that the conditions " $\underline{C} \subseteq \mathcal{X}$ " and "$\text{spec}(A) \in \mathcal{X}$ for all $A \in \text{Alg}(\underline{A})$" are equivalent and provide a minimum restriction on this choice.

In order to satisfy the second axiom, we define

$$R \in \mathcal{X} \subseteq \mathcal{S}^{\underline{C}^{op}}$$

to be the underlying set functor on $\underline{C}^{op} \subseteq \text{Alg}(\underline{A})$, or equivalently the functor represented by the free $\underline{A}$-algebra on one generator. Thus the spectrum of any algebra in $\underline{C}^{op}$ is a space form $C \in \underline{C}$ and any object $X \in \mathcal{X}$ is determined by a discrete fibration $\underline{C}/X$ over $\underline{C}$ whose fibers consist of all figures in X of a given form (possibly singular figures) and whose morphisms are "incidence relations" between such figures. But X also determines, by mapping into R, R^2, in general into all the equalizers $C \longrightarrow R^n \rightrightarrows R^m$ in $\mathcal{X}$, a discrete op-fibration $X/\underline{C}$ over $\underline{C}$ whose objects are all variable quantities on X satisfying various given equations, and whose morphisms are $\underline{A}$-algebraic operations on such quantities. The requirement that R be a commutative ring in $\mathcal{X}$ will be met if the theory $\underline{A}$ contains the algebraic theory of commutative rings as a subtheory, and then $D \in \underline{C}$ will be forced to be the spectrum of the $\underline{A}$-algebra obtained by dividing the free $\underline{A}$-algebra on one generator t by the $\underline{A}$-congruence relation generated by the one relation $t^2 = 0$, i.e.

$$D \in \mathcal{X} \subseteq \mathcal{S}^{\underline{C}^{op}}$$

is the covariant set-valued functor on the category $\underline{C}^{op}$ of algebras which assigns to each algebra $C^{op} \in \underline{C}^{op}$ its <u>subset</u> of elements of square 0. Since the full inclusion

$$\underline{C} \subseteq \mathcal{X} \subseteq \mathcal{S}^{\underline{C}^{op}}$$

preserves products and whatever exponentials may exist (for any small category $\underline{C}$), in order to verify our second axiom for $\mathcal{R}$, namely

$$R^D \cong R \times R$$

it suffices to know that $(\ )^D$, right adjoint to $(\ ) \times D$ the free or "tensor" product in Alg($\underline{A}$), exists in $\underline{C}$ and that the axiom holds there. In the case of <u>algebraic</u> geometry, where $\underline{A}$ consists <u>only</u> of polynomials with coefficients in some ground field k, this is indeed the case, in fact any algebra which is <u>finite-dimensional</u> as a <u>k-vector space</u> can be applied as an exponent in Alg $(\underline{A}_k)^{op}$, and $D = \mathrm{spec}(k[d])$, where $k[d] = k[t]/_{t^2}$ is two-dimensional. But it should be possible to take $\underline{A}$ as the algebraic theory of all real- analytic functions or of all C^∞ real functions of n variables. [In 1978 Eduardo Dubuc succeeded in constructing an $\mathcal{X}$ satisfying both of our two axioms and containing as a full subcategory the category of all real C^∞ manifolds; the extent to which it is generated by (the dual of) the category of all finitely - presented C^∞-algebras is still unclear to me at this writing] There should be many algebraic theories $\underline{A}$ intermediate between only polynomials as operations and all C^∞ functions as operations, perhaps satisfying some suitable closure conditions, in particular the $\underline{A}$ generated by cos, sin, exp, e^{-1/x^2} [Anders Kock has studied the closure condition that with each f ($\underline{x}$,t) in $\underline{A}$ there is also contained in $\underline{A}$ the unique continuous $f_1(\underline{x},t,h)$ such that $f(\underline{x},t+h) = f(\underline{x}) + f_1(\underline{x},t,h).h$ for all real $\underline{x}$, t,h. The inverse of this condition would also seem interesting].

[Some feel that a geometrical category $\mathcal{X}$ should not require a category as big as $\underline{C}$ to generate it, nor should it satisfy the topos exactness condition that monic epics are isomorphisms, but rather should be generated (weakly) more nearly by points in a narrow sense. If

$$\mathcal{D} \subset \underline{C} \subset \mathcal{X}$$

are categories where $\mathcal{D}$ (weakly) generates $\underline{C}$, $\underline{C}$ generates $\mathcal{X}$, and $\mathcal{X}$ is complete cartesian closed, then the full subcategory $\mathcal{Y}_\mathcal{D}$ of $\mathcal{X}$ weakly generated by $\mathcal{D}$, can be defined to consist of all Y such that for any distinct $C \rightrightarrows Y$ with C in $\underline{C}$, there is $E \longrightarrow C$, with $E \in \mathcal{D}$ such that $E \longrightarrow C \rightrightarrows Y$ are different. $\mathcal{Y}_\mathcal{D}$ is closed under inverse limits and $\mathcal{X}$ -subobjects, so is epi-reflective in $\mathcal{X}$. In fact $\mathcal{Y}_\mathcal{D}$ is closed under exponentiation and contains $\underline{C}$ as a full subcategory, as well as being weakly generated by $\mathcal{D}$. The interest of this general construction for us is that an appropriate Nullstellensatz for $\underline{A}$ would tell us that the dual $\mathcal{D}$ of the special category $\mathcal{W}$ of finite dimensional algebras defined earlier,

$$\mathcal{W}^{op} \xrightarrow[\approx]{D} \mathcal{D}$$

actually does weakly generate $\underline{C}$. Since $\underline{C} \subset \mathcal{Y}_{\mathcal{D}} \subset \mathcal{X}$ are full, $\mathcal{Y}_{\mathcal{D}}$ is more nearly a geometrical category than other cartesian closed categories generated (strongly) by $\mathcal{D}$ such as $\mathcal{S}^{\mathcal{W}}$. As pointed out in my 1972 Aarhus lectures, such categories as the latter retain only the formal aspects of the groups, spaces, etc from $\mathcal{X}$]

For any category $\underline{C}$ having finite products and split idempotents, an object $C \in \mathcal{S}^{\underline{C}^{op}}$ is representable iff the functor $(\ )^C$ has a right ajoint

$$\mathcal{S}^{\underline{C}^{op}} \xrightarrow[(\ )_C]{} \mathcal{S}^{\underline{C}^{op}}$$

In fact for any Y in $\mathcal{S}^{\underline{C}^{op}}$ we have for any S in $\underline{C}$

$$Y_C(S) = \mathrm{Nat}(S^C, Y)$$

and in particular if $S^C \in \underline{C}$ for all $S \in \underline{C}$ then

$$Y_C(S) = Y(S^C).$$

A subtopos $\mathcal{X}$ of $S^{\underline{C}^{op}}$ will be closed under $(\ )_C$ provided $(\ )^C$ preserves coverings. [Several people have recently pointed out that the foregoing is true (in the models) for $C = D(W)$, $W \in \mathcal{W}$. Thus for example we have the rule

$$\frac{X^D \longrightarrow Y}{X \longrightarrow Y_D}$$

In particular if $Y = R$ we find that there is a subobject L of R_D defined by the condition that the two induced actions of the multiplicative monoid of R agree. Then the gradient of a map $X \longrightarrow R$ can be interpreted not only as an element of

$$\mathrm{Hom}^{\bullet}_R(X^D, R) \subset R^{(X^D)}$$

but also as a map

$$X \longrightarrow L \subset R_D.$$

In fact, there is a canonical map $R \xrightarrow{d} R_D$ which factors through L, and the gradient of any $X \xrightarrow{f} R$ can be computed as the composite

$$\mathrm{grad}(f) = df$$

Something like this feature exists also in the cartesian-closed category constructed by K.T. Chen in BAMS vol. 83, September 1977, even though the objects D and L do not exist since his category is weakly generated by 1 .]

THE GENERIC MODEL OF AN ε-STABLE GEOMETRIC EXTENSION OF THE THEORY OF RINGS IS OF LINE TYPE.

Marie-Françoise Coste and Michel Coste

Anders Kock introduced in [5] the notion of an ε-stable theory: let T be a geometric extension of the theory of (commutative and unitary) rings, G its generic model; T is said to be ε-stable when $G[\varepsilon]$ (the ring of dual numbers over G) is again a model of T. If T is given by axioms involving at most denumerable disjunctions, T is ε-stable iff for any model A of T in the category of sets $A[\varepsilon]$ is again a model of T (this is due to a general result of Makkai and Reyes [2]).

The theories of local rings and of strictly henselian local rings are ε-stable [5] and it is proved in [3] that their generic models are of line type. In this paper we generalise this result. The proofs in [3] rely on the fact that the Zariski topology (for local rings) and the étale topology (for strictly henselian local rings) are subcanonical. In general the topology associated to the extension T is not subcanonical. The idea of our proof is to construct first a site with a subcanonical topology which allows us to use the same methods than in the particular cases above.

This work is actually a joint work with Anders Köck: his letters have led us to modify completely our original proof which gave the result only in the coherent case. He pointed out to us that the main fact is that $G[\varepsilon]$ is classified by a geometric morphism which has $(-)^D$ (where D is the object of infinitesimals of G) as inverse image. We follow in the second part his proof of this fact.

Gonzalo Reyes has a different proof of the same result [4]. Actually our proof gives a slightly more general result but up to now we don't see any application of this further generality. The reading of his proof has influenced the presentation of ours.

I) Standard extensions

In the following L_0 is a language, T_0 a $\underleftarrow{\lim}$-theory [1] and T a geometric (finitary or infinitary) extension of T_0 both in L_0.

a) It is well known that to the extension T of T_0 corresponds a Grothendieck topology $\mathfrak{T}$ on the category $\underline{FPModT_0}^{op}$ (where $\underline{FPModT_0}$ is the category of finitely presented models of T_0) : see for instance [2] or [1] chapter IV.

Definition : T is a standard extension of T_0 when $\mathfrak{T}$ is subcanonical (i.e. every representable presheaf is a sheaf for $\mathfrak{T}$).

There is a syntactical characterization of this situation ([1], IV.3.2.) :

Proposition 1 : T is a standard extension of T_0 iff both following conditions are satisfied :

i) every sequent $\Phi(\vec{x}) \vdash \exists \vec{y}\, \Psi(\vec{x},\vec{y})$ where Φ and Ψ are conjunctions of atomic formulas and $\Psi(\vec{x},\vec{y}), \Psi(\vec{x},\vec{z}) \vdash_{T_0} \vec{y} = \vec{z}$ which is a theorem of T is a theorem of T_0.

ii) every geometric formula $\Theta(\vec{x},\vec{y})$ which is in T a functional relation $\vec{x} \mapsto \vec{y}$ (i.e. $\Theta(\vec{x},\vec{y}), \Theta(\vec{x},\vec{z}) \vdash_T \vec{y} = \vec{z}$) with domain a conjunction of atomic formulas $\Phi(x)$ (i.e.

$\exists \vec{v}\, \Theta(\vec{x},\vec{v}) \underset{T}{\models\!\dashv} \Phi(\vec{x})$) is equivalent in T to a formula $\exists \vec{t}\, \Psi(\vec{x},\vec{v},\vec{t})$ where Ψ is a conjunction of atomic formulas and

$$\Psi(\vec{x},\vec{v},\vec{t})\ ,\ \Psi(\vec{x},\vec{y},\vec{u}) \vdash_{T_0} \vec{t} = \vec{u} \quad .$$

b) In the case where T is not a standard extension of T_0 , is there a $\underleftarrow{\lim}$-theory T_1 between T_0 and T such that T is a standard extension of T_1 ? The answer is "yes , but possibly with a change of language" :

Proposition 2 : There exist a language L_1 containing L_0 and having the same sorts of variables , a $\underleftarrow{\lim}$-theory T_1 and a geometric extension T' of T_1 both in L_1 such that :

i) T_1 is an extension of T_0 ,

ii) T' is an extension of T which is equivalent to T ; precisely , the canonical geometric morphism from the classifying topos for T' to the classifying topos for T is an equivalence .

iii) T' is a standard extension of T_1 .

Proof : Let L_0^0 be L_0 and T^0 be T . Suppose that L_0^n and T^n have already been constructed with L_0^n extension of L_0 and T^n a geometric theory in L_0^n extension of T and equivalent to T . L_0^{n+1} is constructed in the following way :

For any geometric formula $\Theta(\vec{x},\vec{y})$ of L_0^n which is in T^n a functional relation $\vec{x} \mapsto \vec{y}$ with domain a conjunction of atomic formulas of L_0^n , add to L_0^n a new predicate $R_\Theta(\vec{x},\vec{y})$. T^{n+1} is T^n plus the axioms $R_\Theta(\vec{x},\vec{y}) \models\!\dashv \Theta(\vec{x},\vec{y})$. T^{n+1} is equivalent to T^n , hence to T .

Let $L_1 = \bigcup_n L_0^n$ and $T' = \bigcup_n T^n$. No new sort of variables has been added , and T' is of course equivalent to T . Let T_1 be the $\underleftarrow{\lim}$-theory in L_1 with axioms all the theorems of T' like $\Phi(\vec{x}) \vdash \exists \vec{y}\, \Psi(\vec{x},\vec{y})$ where Φ and Ψ are conjunctions of atomic formulas of L_1 and $\Psi(\vec{x},\vec{y})$, $\Psi(\vec{x},\vec{z}) \vdash_{T'} \vec{y} = \vec{z}$. T_1 is an extension of T_0 and by proposition 1 T' is a standard extension of T_1 . ∎

Let $\mathcal{E}$ be a topos .

Corollary 1 : Let U be the restriction functor from $\underline{Mod}(T_1,\mathcal{E})$ to $\underline{Mod}(T_0,\mathcal{E})$. U is faithful .

Proof : Clear , since L_1 has no more sort of variables than L_0 . ∎

Corollary 2 : Let A be a model of T in $\mathcal{E}$. It may canonically be considered as a model A' of T' . Then for any model C of T_1 in $\mathcal{E}$ the application

$$\mathrm{Hom}_{L_1}(A',C) \longrightarrow \mathrm{Hom}_{L_0}(A,UC)$$

is bijective .

Proof : Since A = UA' the application is injective by corollary 1 . Now let $f : A \longrightarrow UC$ be a L_0-morphism . We have to show that it preserves the new predicates of L_1 . Let $R(\vec{x})$ be one of these new predicates . It is equivalent in T' to a geometric formula $\Phi(\vec{x})$ of L_0 , so we have $\{\vec{x} \in A' \mid R(\vec{x})\} = \{\vec{x} \in A \mid \Phi(\vec{x})\}$. Since $\Phi(\vec{x}) \vdash_{T'} R(\vec{x})$ we have $\Phi(\vec{x}) \vdash_{T_1} R(\vec{x})$ and $\{\vec{x} \in C \mid R(\vec{x})\} \supset \{\vec{x} \in UC \mid \Phi(\vec{x})\}$. So since f preserves Φ , it preserves also R . ∎

II) Generic models of $\mathfrak{S}$-stable theories

L_0 , T_0 , T , L_1 , T_1 and T' are as in proposition 2 .

a) Let $\mathcal{E}[T_1]$ be the classifying topos for T_1. We know that $\mathcal{E}[T_1] = \underline{\text{FPMod}T_1}^{op}$. The generic model G_1 of T_1 is the inclusion $\underline{\text{FPMod}T_1} \hookrightarrow \underline{\text{Mod}T_1}$.

$\mathcal{E}[T]$, the classifying topos for T, is equivalent to $\mathcal{E}[T']$ so we may suppose it is the topos of sheaves on $\underline{\text{FPMod}T_1}^{op}$ for the subcanonical topology associated to the standard extension T' of T_1. We have :

$$\mathcal{E}[T] = \mathcal{E}[T'] \underset{a}{\overset{i}{\rightleftarrows}} \longrightarrow \mathcal{E}[T_1] \quad .$$

The generic model G' of T' is aG_1 and the generic model G of T is UG', that is aUG_1. Since G_1 and UG_1 are represented by models of T_1 and T_0 in $\underline{\text{FPMod}T_1}^{op}$ and the topology is subcanonical we have $iG' = G_1$ and $iG = UG_1$.

b) Let I be a finite presentation in L_0 (i.e. a finite set of generators $\{a_1,\dots,a_n\}$ together with a finite set of closed atomic formulas of L_0 $\{\Phi_1(\vec{a}),\dots,\Phi_p(\vec{a})\}$ with parameters in $a_1,\dots,a_n$. If A is a model of T_0, (I,A) will be $\{(x_1,\dots,x_n)\in A^n \mid \Phi_1(\vec{x})\wedge\dots\wedge\Phi_p(\vec{x})\}$. Let $[I]$ (resp. $\langle I\rangle$) be the model of T_0 (resp. T_1) of presentation I. If A is a model of T_0 (resp. T_1) $A[I]$ (resp. $A\langle I\rangle$) will denote the sum $A \amalg [I]$ (resp. $A \amalg \langle I\rangle$) in the category of models of T_0 (resp. T_1); this makes sense even if A lives in a topos which is not the topos of sets by replacing $[I]$ (resp. $\langle I\rangle$) by the constant sheaf. We have the following isomorphisms :

$$\text{Hom}_{L_0}([I],B) \simeq \text{Hom}(1,(I,B))$$

$$\text{Hom}_{L_0}(A[I],B) \simeq \text{Hom}_{L_0}(A,B) \times \text{Hom}(1,(I,B))$$

where A and B are models of T_0, and

$$\text{Hom}_{L_1}(\langle I\rangle,D) \simeq \text{Hom}(1,(I,UD))$$

$$\text{Hom}_{L_1}(C\langle I\rangle,D) \simeq \text{Hom}_{L_1}(C,D) \times \text{Hom}(1,(I,UD))$$

where C and D are models of T_1.

c) We are going to prove the following result :

Theorem 1 : If $G[I]$ is a model of T , it is classified by a geometric morphism with $(-)^{(I,G)} : \mathcal{E}[T] \longrightarrow \mathcal{E}[T]$ as inverse image .

Lemma 1 : The functor $(-)^{(I,UG_1)} : \mathcal{E}[T_1] \longrightarrow \mathcal{E}[T_1]$ preserves the colimits , and it is the inverse image functor of a geometric morphism δ_1 .

Proof : The presheaf (I,UG_1) is represented by $\langle I \rangle$: For any $P \in \underline{FPModT_1}$ we have $(I,UG_1)(P) = (I,UP) = \mathrm{Hom}_{L_1}(\langle I \rangle ,P)$. So if X is any presheaf on $\underline{FPModT_1}^{op}$,

$$X^{(I,UG_1)}(P) \simeq \mathrm{Nat}(\mathrm{Hom}_{L_1}(P,-),X^{(I,UG_1)}) \simeq \mathrm{Nat}(\mathrm{Hom}_{L_1}(P,-)\times(I,UG_1),X)$$
$$\simeq \mathrm{Nat}(\mathrm{Hom}_{L_1}(P,-)\times\mathrm{Hom}_{L_1}(\langle I \rangle,-),X) \simeq \mathrm{Nat}(\mathrm{Hom}_{L_1}(P\langle I \rangle,-),X)$$
$$\simeq X(P\langle I \rangle) \quad .$$

This shows that $(-)^{(I,UG_1)}$ preserves colimits , and we already know that it preserves limits since it is a right adjoint .

Lemma 2 : Suppose $G[I]$ is a model of T , classified by δ . Then the diagram

$$\begin{array}{ccc} \mathcal{E}[T_1] & \xrightarrow{\delta_1} & \mathcal{E}[T_1] \\ i\uparrow & & \uparrow i \\ \mathcal{E}[T] & \xrightarrow{\delta} & \mathcal{E}[T] \end{array}$$

commutes (up to iso) .

Proof : We have to show that $\delta_1 i$ and $i\delta$ classify isomorphic models of T_1 . We know that $\delta_1{}^*G_1(P) = G_1{}^{(I,UG_1)}(P) \simeq G_1(P\langle I \rangle) = P\langle I \rangle$; hence $\delta_1{}^*G_1 \simeq G_1\langle I \rangle$ and $a\delta_1{}^*G_1 \simeq a(G_1\langle I \rangle) \simeq (aG_1)\langle I \rangle \simeq G'\langle I \rangle$. Now $\delta^*aG_1 = \delta^*G' = (G[I])'$ (i.e. $G[I]$ considered as a model of T') . $G[I]'$ has the same universal property that $G'\langle I \rangle$: For any model A of T_1 in $\mathcal{E}[T]$ we have :

$$\mathrm{Hom}_{L_1}(G[I]',A) \simeq \mathrm{Hom}_{L_0}(G[I],UA) \quad \text{(by corollary 2)}$$

$$\simeq \mathrm{Hom}_{L_0}(G,UA) \times \mathrm{Hom}(1,(I,UA)) \quad \text{(cf §b)}$$

$$\mathrm{Hom}_{L_1}(G',A) \times \mathrm{Hom}(1,(I,UA)) \quad \text{(corollary 2)} .$$

So $G'\langle I\rangle$ and $G[I]'$ are isomorphic .

Lemma 3 : With the hypothesis and notations of lemma 2 , δ^* is $(-)^{(I,G)}$.

Proof : Let X be an object of $\mathfrak{E}[T]$. we have :
$\delta^* X \simeq \delta^* aiX \simeq a\delta_1^* iX \simeq a(iX^{(I,UG_1)})$. Since $UG_1 = iG$ (cf §a) we have $(I,UG_1) = i(I,G)$ and $\delta^* X \simeq a(iX^{i(I,G)}) \simeq ai(X^{(I,G)}) \simeq X^{(I,G)}$.

this completes the proof of the theorem . ∎

d) We suppose now that T_0 is the theory of rings . Let I be the presentation given by a generator ε and the relation $\varepsilon^2 = 0$. $G[I]$ is $G[\varepsilon]$ and (I,G) is D . The theorem 1 gives immediately:

Theorem 2 : If T is ε-stable , $G[\varepsilon]$ is classified by a geometric morphism whith $(-)^D$ as inverse image . In particular G is of line type (i.e. $G[\varepsilon] \simeq G^D$) and D is internally projective (i.e. $(-)^D$ preserves finite colimits [3]) .

More generally let I_n be the presentation given by generators $\varepsilon_1,\dots,\varepsilon_n$ and the relations $\varepsilon_i\varepsilon_j = 0$ for all couples (i,j) . $G[I_n]$ is denoted by $G[\varepsilon_1,\dots,\varepsilon_n]$ and (I_n,G) by $D(n)$ (cf [3]) . Following G. Reyes [4] we say that T is 1-stable if for every n $G[\varepsilon_1,\dots,\varepsilon_n]$ is a model of T . We have :

Theorem 3 : If T is 1-stable , $G[\varepsilon_1,\dots,\varepsilon_n]$ is classified by a geometric morphism with $(-)^{D(n)}$ as inverse image (for all n) . In particular G is of line type , 1-small objects (i.e. finite products of D(n)'s) are internally projective , and G is infinitesimally linear (i.e. $G^{D(n)}$ is the n-fold pullback of G^D over G [3]) .

Proof : The only point which is not obvious is the last remark . We have to use the fact that $G[\varepsilon_1,\dots,\varepsilon_n]$ is the n-fold pullback of $G[\varepsilon]$ over G (cf [3] p. 26) .

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(Ed.)

TOPOLOGIES FOR REAL ALGEBRAIC GEOMETRY

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The following paper is a redaction of our lectures in the open house in topos theory in Aarhus in May 78.

We want to thank all the participants and especially the organiser A.Kock.The very interesting discussions we had there have been a great help for us.

A)REAL ZARISKI TOPOLOGY

Real ideals , i.e. those satisfying $x_1^2+\ldots+x_n^2 \in I \Longrightarrow x_1 \in I$ play an important role in real algebraic geometry,and have been used for the "real nullstellensatz" (Dubois,Risler,Efroymson).

The main idea of this first part is to replace prime ideals by real prime ideals.

In the case of a ring of polynomial functions on an algebraic variety in $\mathbb{R}^n$,real maximal ideals correspond exactly to real points of the variety.

This leads to replace local rings by real local rings,obtained by localisation at real primes.

We define the real analogues for the Zariski spectrum of a ring and its structural sheaf :they are just restrictions of the ordinary ones.Nevertheless a new problem arises :what rings are isomorphic to the ring of global sections of their real Zariski spectrum?

As G.Wraith suggested ,the proof we gave in Aarhus of the compactness of the real Zariski spectrum was very simplified by the study of real rings.Unfortunately this complicated proof will be necessary in part B) for the compactness of the real étale spectrum.

I) Real ideal,real local ring,real ring

Definition 1: I is a real ideal iff I is an ideal such that $x_1^2+\dots+x_n^2 \in I \Rightarrow \forall\, i=1,\dots,n \quad x_i \in I$

Example:

1) $\{0\}$ is real in $\mathbb{R}$,not in $\mathbb{C}$.

2) $(X-a)$ is real in $\mathbb{R}[X]$,not (X^2+1).

3) The ideal generated by $(X-1)$ and $(Y-1)$ is real in $\mathbb{R}[X,Y]/(XY-1)$,not the ideal generated by $(X+Y)$.

Proposition 1: Let I be a real ideal then $I=\sqrt{I}$.

proof:

$$\sqrt{I} = \bigcap_{\mathfrak{p}\ \text{prime} \supset I} \mathfrak{p} = \{a \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$$

Let n be the smallest integer such that $a^n \in I$

-If n is even $a^{\frac{n}{2}} \in I$:contradiction

-If n is odd $a^{n+1} \in I, a^{\frac{n+1}{2}} \in I, \frac{n+1}{2} = n$ and $n=1$.

Definition 2: A is a real local ring (formally real local ring in (1)) iff A is local and its maximal ideal $\mathfrak{m}_A$ is real.

Remark:

1) Real local rings are models of the following finitary geometric theory:

a) $0=1 \vdash$

b) $\vdash \exists\, y \quad xy=1 \vee \exists\, y' \quad (1-x)y'=1$

c) for each n in $\mathbb{N}$ $\vdash \exists\, y \quad (1+\sum_{i=1}^{i=n} x_i^2)y=1$

A ring is local iff a) and b) are verified (well known). If $\mathfrak{m}_A$ is real $1+\sum_{i=1}^{i=n} x_i^2$ cannot be in $\mathfrak{m}_A$ and hence is invertible. Conversely $\mathfrak{m}_A$ is real. If not we can find $(x_1,\dots,x_n)$ such that $x_1^2+\dots+x_n^2 \in \mathfrak{m}_A$ and $x_1 \notin \mathfrak{m}_A$. x_1 is invertible (A is local) hence $1+\sum_{i=2}^{i=n}\left(\frac{x_i}{x_1}\right)^2 \in \mathfrak{m}_A$: contradiction.

2) A ring is real local iff it is local and its residue field is real (-1 is not a sum of squares).

Proposition 2: Let I be a prime ideal. A_I is a real local ring iff I is real.

proof: trivial.

Proposition 3: Real ideal generated by an ideal (Risler)

Let I be an ideal

Let $I' = \{ a \mid \exists n \in \mathbb{N} \exists (x_1,\ldots,x_n)\ a^2+x_1^2+\ldots+x_n^2 \in I \}$

$\mathbb{R}(I)=\sqrt{I'}$ is the smallest real ideal containing I.

proof:

-It is clear that every real ideal containing I contains R(I).

-I' is an ideal. The only thing to prove is that $a \in I'; b \in I' \Rightarrow a-b \in I'$

We have $a^2+x_1^2+\ldots+x_n^2 \in I$, $b^2+y_1^2+\ldots+y_m^2 \in I$

$(a-b)^2+(a+b)^2+2(x_1^2+\ldots+x_n^2+y_1^2+\ldots+y_m^2) \in I$ hence $a-b \in I'$.

-$\sqrt{I'}$ is real. Let $a_1^2+\ldots+a_n^2$ be an element of $\sqrt{I'}$: there exists $m \geq 1$ and elements $(b_1,\ldots,b_k)$ such that $(a_1^2+\ldots+a_n^2)^{2m}+b_1^2+\ldots+b_k^2 \in I$ so there exist elements $c_1,\ldots,c_l$ such that $(a_1^{2m})^2+c_1^2+\ldots+c_l^2 \in I$ hence $a_1 \in \sqrt{I'}$.

Proposition 4: An ideal maximal among the real ideals is prime.

proof: Let I be maximal among the real ideals and such that $xy \in I$, $x \notin I$ and $y \notin I$.

$R(I+Ax)=A$, $R(I+Ay)=A$ hence we can find $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_m)$ such that $1+x_1^2+\ldots+x_n^2 \in I+Ax$, $1+y_1^2+\ldots+y_m^2 \in I+Ay$ hence $(1+x_1^2+\ldots+x_n^2)(1+y_1^2+\ldots+y_m^2) \in I$ hence $1 \in R(I)=I$: contradiction.

Definition 3: A ring is real (formally real in (1)) iff for every $n \in \mathbb{N}$ and every $(x_1,\ldots,x_n)$ $1+x_1^2+\ldots+x_n^2$ is invertible.

Remark: The theory of real rings is a $\overleftarrow{\lim}$-theory (2).

Proposition 5: In a real ring A the real ideal generated by a proper ideal is a proper ideal.

proof: Let us suppose that R(I)=A; $1 \in R(I)$ hence there exist $(x_1,\ldots,x_n)$ such that $1+x_1^2+\ldots+x_n^2 \in I$, that is I=A.

Proposition 6: A is a real ring iff its maximal ideals are real.

proof: Let $\mathfrak{m}$ be a maximal ideal in A, $R(\mathfrak{m})=\mathfrak{m}$ by proposition 5 hence $\mathfrak{m}$ is real.

Conversely $1+x_1^2+\ldots+x_n^2$ cannot belong to a maximal ideal hence is invertible.

Proposition 7: free real ring associated to a ring A.

$\Sigma_1=\{1+x_1^2+\ldots+x_n^2 \mid \forall\ n\in\mathbb{N}\ \forall\ (x_1,\ldots,x_n)\in A^n\}$ is a multiplicative subset of A.

$A(\Sigma_1^{-1})$ is a real ring and every ring-homomorphism between A and a real ring B factors by $A(\Sigma_1^{-1})$.

proof: Let $s_1,\ldots,s_n$ be elements of Σ_1. When we reduce $1+\left(\frac{x_1}{s_1}\right)^2+\ldots+\left(\frac{x_n}{s_n}\right)^2$ to the same denominator we find an expression of the form $\frac{s'}{s}$ with s and s' in Σ_1.

It is clear that $A(\Sigma_1^{-1})$ has the required universal property.

Proposition 8: free real ring with an element a of A inverted

$\Sigma_a=\{a^{2n}+x_1^2+\ldots+x_n^2 \mid \forall\ n\in\mathbb{N}\ \forall\ (x_1,\ldots,x_n)\in A^n\}$ is a multiplicative subset of A.

$A(\Sigma_a^{-1})$ is a real ring and every ring homomorphism f from A to a real ring B with f(a) invertible factors by $A(\Sigma_a^{-1})$.

proof: as in proposition 7.

Proposition 9: Let S be a multiplicative subset in A.

The application $\mathfrak{p} \longrightarrow \mathfrak{p}A(S^{-1})$ is a bijection of real prime ideals with no elements in S and real prime ideals in $A(S^{-1})$.

proof: no difficulties.

Corollary: The application $\mathfrak{p} \longrightarrow \mathfrak{p} A(\Sigma_1^{-1})$ is a bijection between real primes in A and in $A(\Sigma_1^{-1})$.

proof: Real prime ideals in A have no elements in $A(\Sigma_1^{-1})$.

Remark:

Prime ideals in a real ring are not always real. For example in $\mathbb{R}[X,Y]$, (X^2+Y^2) is prime, not real and does not contain elements of Σ_1. It defines a prime not real ideal in $\mathbb{R}[X,Y](\Sigma_1^{-1})$.

II) Real Zariski spectrum of a ring.

The real Zariski spectrum of a ring A is the following topological space:

-elements are real prime ideals

-basic opens are of the form $D_a = \{\mathfrak{p} \mid \mathfrak{p}$ real prime ideal $a \notin \mathfrak{p}\}$

This topology is the restriction of the Zariski topology to real prime ideals.

We note $\mathrm{Spec}_{RZar}(A)$ the real Zariski spectrum of A.

Let $f: A \longrightarrow B$ be a ring homomorphism and $\mathrm{Spec}_{RZar}(f)$ be the application $\mathfrak{p} \longrightarrow f^{-1}(\mathfrak{p})$. $\mathrm{Spec}_{RZar}(f)$ is a continuous map from $\mathrm{Spec}_{RZar}(B)$ to $\mathrm{Spec}_{RZar}(A)$.

Remark:

1) The elements of $\mathrm{Spec}_{RZar}(\mathbb{R}[X])$ are

-points of $\mathbb{R}$ (prime ideals of the form $(X-a)$)

-the $\{0\}$ ideal (the generic point)

2) The elements of $\mathrm{Spec}_{RZar}(\mathbb{R}[X,Y]/XY-1)$ are

-real points of the hyperbola (prime ideals generated by $(X-a)$ and $(Y-b)$ with $ab=1$, a and b real numbers)

- the generic point (the $\{0\}$ ideal).

Proposition 1: $\mathrm{Spec}_{RZar}(A(a^{-1}))$ and D_a are isomorphic topological spaces.

proof: cf I) proposition 9.

Proposition 2: $\mathrm{Spec}_{RZar}(A)$ and $\mathrm{Spec}_{RZar}(A(\Sigma_1^{-1}))$ are isomorphic topological spaces.

proof: cf I) corollary of proposition 9.

Definition: The family $(a_i)_{i\in I}$ covers $\mathrm{Spec}_{RZar}(A)$ iff $\bigcup_{i\in I} D_{a_i} = \mathrm{Spec}_{RZar}(A)$.

Proposition 3: Let A be a real ring. The family $(a_i)_{i\in I}$ covers Spec(A) iff it covers $\mathrm{Spec}_{RZar}(A)$.

proof: Let $\mathfrak{p}$ be a prime ideal of A. It is contained in a maximal ideal of A $\mathfrak{m}$. $\mathfrak{m}$ is real (I), proposition 6). So there exist a_{i_0} which does not belong to $\mathfrak{m}$, hence to $\mathfrak{p}$ in the family $(a_i)_{i\in I}$.

Theorem: The real Zariski spectrum of a ring is a compact topological space.

proof: Proposition 2, proposition 3 and compactness of Spec(A).

Corollary: For all a in A, D_a is compact.

proof: theorem and proposition 1.

Remark:

The condition "$(a_1,\ldots,a_n)$ covers $\mathrm{Spec}_{RZar}(A)$" is equivalent to "1 is a linear combination of $a_1,\ldots,a_n$ in $A(\Sigma_1^{-1})$", which can be easily expressed by a denumerable disjunction of finitary geometric formulae of the theory of rings with parameters $(a_1,\ldots,a_n)$.

III) Structural sheaf of the real Zariski spectrum of a ring.

Let $\mathcal{S}pec_{RZar}(A)$ be the following sheaf on $Spec_{RZar}(A)$:

-the stalk at the real prime ideal $\mathfrak{p}$ is $A_{\mathfrak{p}}$

-a basis of open sets is given by the $\left\{\frac{x}{s} \in A_{\mathfrak{q}} \,\middle|\, x \text{ and } s\right.$ fixed in A and $\mathfrak{q}$ varying in $\left.D_s\right\}$.

$\mathcal{S}pec_{RZar}(A)$ is called the structural sheaf of the real Zariski spectrum of A,or simply the real Zariski spectrum of A.It is just the restriction of the structural sheaf of the Zariski spectrum to $Spec_{RZar}(A)$.

$\mathcal{S}pec_{RZar}(A)$ is in some sense "the best real local ring over A":

Proposition: Let A be a ring and B a real local ring in the topos of sheaves over the topological space X.There a one-to-one correspondance between ring-homomorphisms f from A to $\Gamma(B)$ (the global sections of B) and the couples (φ, g) with φ continuous map from X to $Spec_{RZar}(A)$ and g local morphism from $\varphi^{*}\left(\mathcal{S}pec_{RZar}(A)\right)$ to B.

proof: The ring homomorphism $f: A \to \Gamma(B)$ defines for all x in X $f_x: A \to B_x$. B_x is a real local ring and f_x factors through an unique $A_{\mathfrak{p}}$ with $\mathfrak{p}$ real prime ideal. Define $\varphi(x)=\mathfrak{p}$. It is clear that φ is continuous. Now define $g\left(\frac{y}{s}\right)=\frac{f_x(y)}{f_x(s)}$ for $\frac{y}{s}$ in $\varphi(x)$.
This gives us the required one-to-one correspondance.

Remark:

We shall have later a stronger version of this "spectrum-property" of $\mathcal{S}pec_{RZar}(A)$.

IV) Real Zariski topos.

We are going to define a Grothendieck topology on the dual of the category FPRings of finitely presented rings, called the real Zariski topology.

Let us define first a Grothendieck pretopology: covering families of A are of the form $\left(A \longrightarrow A(a_i^{-1})\right)_{i\in I}$ with $(a_i)_{i\in I}$ covering $\mathrm{Spec}_{RZar}(A)$.

- $\left\{A \xrightarrow{1_A} A\right\}$ is a covering family
- local character is trivial
- stability under change of basis: given $f:A \longrightarrow B$ if the family $(a_i)_{i\in I}$ covers $\mathrm{Spec}_{RZar}(A)$ (by II) theorem, I may be supposed finite) the family $(f(a_i))_{i\in I}$ covers $\mathrm{Spec}_{RZar}(B)$ by last remark in II).

Now covering families for real Zariski topology are families of arrows of same source containing a family in the given pretopology.

Remark:

The Zariski topology is contained in the real Zariski topology.

Definition: The real Zariski topos, RZar is the topos of sheaves for the real Zariski topology. This topos is coherent since $\mathrm{Spec}_{RZar}(A)$ is for all A a compact topological space.

Theorem: The real Zariski topos is the classifying topos for real local rings.

proof: It is sufficent to show that C is a real local ring iff $\mathrm{Hom}_{Rings}(-,C):\mathrm{FPRings}^{op} \longrightarrow \mathrm{Sets}$ is continuous for the real Zariski topology.

-Let C be real local and $f:A\longrightarrow C$ be a ring homomorphism. $\mathfrak{m}_C$ is real, hence $f^{-1}(\mathfrak{m}_C)$ is a real prime ideal in A. Let $(a_i)_{i\in I}$ be a covering family for the real Zariski topology. There exists i_0 such that $f^{-1}(\mathfrak{m}_C)$ does not contain a_{i_0}. f factors through $A(a_{i_0}^{-1})$ hence $\mathrm{Hom}_{Rings}(-,C)$ is continuous for the real Zariski topology.

-Conversely Hom(-,C) is continuous for the real Zariski topology hence for the Zariski topology: C is a local ring. Also $\mathbb{Z}[X_1,\ldots,X_n] \longrightarrow \mathbb{Z}[X_1,\ldots,X_n]((1+X_1^2+\ldots+X_n^2)^{-1})$ is covering for the realZariski topology: a prime ideal in $\mathbb{Z}[X_1,\ldots,X_n]$ does not contain $1+X_1^2+\ldots+X_n^2$. $\mathrm{Hom}_{Rings}(-,C)$ being continuous for the real Zariski topology we have:

$\mathrm{Hom}_{Rings}(\mathbb{Z}[X_1,\ldots,X_n],C)=\mathrm{Hom}_{Rings}(\mathbb{Z}[X_1,\ldots,X_n],C)$

that is $1+x_1^2+\ldots+x_n^2$ always invertible in C.

V) Back to the real Zariski spectrum.

One can find in (2) a general construction of spectra.

Let us consider the following localisation triple:

T_0= theory of rings

T= theory of real local rings

V = $\{(\text{true}, \exists y\ xy=1)\}$

V-admissible morphisms are morphisms which reflect the fact of being invertible, that is, between local rings, local morphisms. A localisation of a ring A for this triple is a localisation of A at a real prime ideal.

The general construction of spectra gives in this case the sheaf $\mathcal{S}pec_{RZar}(A)$ described in III).

We have thus the universal property of spectra in the case of any Grothendieck topos:

Proposition: Let $\underline{Spec}_{RZar}(A)$ be the topos of sheaves over $Spec_{RZar}(A)$. Let $\mathcal{E}$ be a Grothendieck topos and B a real local ring in $\mathcal{E}$. There is a one-to-one correspondance between rings-homomorphisms from A to $\Gamma(\mathcal{E}, B)$ and local morphisms of real-local-ringed-toposes from $(\underline{Spec}_{RZar}(A), \mathcal{S}pec_{RZar}(A))$ to $(\mathcal{E}, B)$ (that is a geometric morphism φ from $\mathcal{E}$ to $\underline{Spec}_{RZar}(A)$ together with a local morphism from $\varphi^*(\mathcal{S}pec_{RZar}(A))$ to B).

A good algebraic approximation for the theory of real local rings?

Let us give a description of $\mathcal{S}pec_{RZar}(\mathbb{R}[X])$.

The stalk at the maximal ideal (X-r) is the real local ring of rational functions defined at r.

The stalk at $\{0\}$ is the real fiels $\mathbb{R}(X)$.

Global sections are rational functions defined at any real point, that is the ring $\mathbb{R}[X][\Sigma_1^{-1}]$ of rational functions with denominators of the form: positive real+sum of squares.

This leads us to the following question:

Is $A[\Sigma_1^{-1}]$ alxays the ring of global sections of $\mathcal{S}pec_{RZar}(A)$?

which means: is the theory of real rings a good algebraic approximation of the theory of real local rings? that is to say: is the theory of real local rings a standard extension of the theory of real rings?

When we try to answer "yes" we are led to the following problem: show that

$$\begin{array}{ccccc} & & A[\Sigma_a^{-1}] & & \\ & \nearrow & & \searrow & \\ A & & & & A[\Sigma_{a.1-a}^{-1}] \\ & \searrow & & \nearrow & \\ & & A[\Sigma_{1-a}^{-1}] & & \end{array}$$

push-out in the category of real rings is also a pull-back.

It is easy to see that if the image of x is null in $A[\Sigma_a^{-1}]$ and in $A[\Sigma_{1-a}^{-1}]$, x is null. But we are not able to prove that if two elements in $A[\Sigma_a^{-1}]$ and $A[\Sigma_{1-a}^{-1}]$ have same image in $A[\Sigma_{a.1-a}^{-1}]$ they are coming from the same element inA.

On the other hand the theory of real rings is rich enough to be a candidate for good algebraic approximation of real local rings. For example the axiom $x_1c_1+\dots+x_nc_n=1 \rightarrow \exists\ t\ (x_1^2+\dots+x_n^2)t=1$ obviously true in the theory of real local rings is also true in the theory of real rings:

If the ideal I generated by $x_1^2+\dots+x_n^2$ is different from A so is R(I) (I) proposition 5).The ideal generated by $(x_1,\dots,x_n)$ is contained in R(I) hence is different from A.

So that the question is still open.

B) REAL ETALE TOPOLOGY

We are here interested in the real analogue for etale topology . The etale topology is related to strictly henselian local rings . The corresponding real notion is the notion of real closed local rings (introduced by A. Kock in [1]) i.e. for rings in sets henselian local rings with real closed residue fields .

We are thus led to consider "localisations" of a ring A which are formally etale real closed local A-algebras , which correspond to prime real ideals $\mathfrak{p}$ of A together with an order on the residue field $k(\mathfrak{p})$. The set of these couples , with a natural topology , is the real etale spectrum of the ring .

When passing from Zariski topology to etale topology no new point is added (in both cases they are prime ideals of the ring) but in some sense automorphisms of the points are introduced (they correspond to the automorphisms of the separable closure of the residue field) . In the real case new points are added (since a residue field at a real prime may have several orders) and no automorphism is introduced (the real closure of a real field has no automorphism) . This suggests that the generic "localisation" of a ring A ought to be a sheaf over a topological space , precisely the real etale spectrum of A . We are able to prove that this is the case when A is the ring of polynomial functions over a real algebraic curve .

For all this we owe a great deal to Gavin Wraith . Actually it is one of his letters which motivated our work on this subject : In this letter he introduced the etale real spectrum of a ring (as its generic "localisation") and indicated this ought to be a sheaf over a topological space . He also remarked that the etale real spectrum of a ring of polynomial functions over a real algebraic variety should contain the sheaf of Nash functions as a restriction to the variety with its euclidian topology , and this is certainly the most interesting aspect of real etale topology .

The results in section I concerning real closed local rings are not original . The property of factorisation of a morphism into a real closed local ring was indicated to us by G. Wraith . It is obtained here from results on separably closed morphisms .

The "prime negideals" (or rather their complementaries) have also been considered by A. Joyal who is the first , we think , to have insisted on the interest of real closed local rings for real algebraic geometry .

I) Real closed local rings

a) These rings have been introduced by Anders Kock [1] under the name of separably real closed local rings .

Definition 1 : Let A be a real local ring . $A[i] = A[X]/_{X^2+1}$ is again a local ring . A is said to be a real closed local ring when A[i] is separably closed [3] (or strictly henselian) .

We know that the theory of separably closed local rings may be formulated as a finitary geometric theory in the language of rings ; this is due to Joyal and Wraith (see [3]) : Let P be an arbitrary monic polynomial of degree n . The elementary symetric functions of the $P'(\alpha_i)$ where $\alpha_1,\dots,\alpha_n$ are the virtual roots of P are polynomial expressions with coefficients in Z in the coefficients of P : these are the hyper discrimants $H_1,\dots,H_n$ of P (the n^{th} hyperdiscriminant is the usual discriminant) . In the case where P is a polynomial over a separably closed field it has a simple root iff one of its hyperdiscriminants is not zero . It follows that a separably closed local ring is a local ring which satisfy for each $n \geq 1$ the axioms

$$\exists z\ (H_i(P).z = 1) \vdash \exists y\ (P(y) = 0 \ \wedge \ /\dots$$

$$\dots/ \quad \exists t\ (P'(y).t = 1))$$

for $i = 1,\dots,n$ where P is the polynomial $Y^n + x_1Y^{n-1} + \dots + x_n$ whith $x_1,\dots,x_n$ variables .

So we can formulate the theory of real closed local rings as a finitary geometric theory :

Proposition 1 : The real closed local rings are the real local rings which satisfy the following axioms for each $n \geq 1$:

$$\exists z \ (\underline{R}(H_j(P))^2 + \underline{I}(H_j(P))^2).z = 1 \quad \vdash \ / \ldots$$

$$\ldots / \ \exists y \ \exists y' \ \left[\underline{R}(P(y+iy')) = 0 \ \wedge \ \underline{I}(P(y+iy')) = 0 \ \wedge \ / \ldots \right.$$

$$\left. \ldots / \ \exists t \ (\underline{R}(P'(y+iy'))^2 + \underline{I}(P'(y+iy'))^2).t = 1 \right]$$

for $j = 1,\ldots,n$ where

$P = Y^n + (x_1+ix_1')Y^{n-1} + \ldots + (x_n+ix_n')$,

$x_1,\ldots,x_n,x_1',\ldots,x_n'$ are variables and $\underline{R}$ and $\underline{I}$ denote respectively the real and imaginary part .

A real closed local ring in a topos will of course be a model of this geometric theory . For an ordinary ring we have the following equivalences :

Proposition 2 : Let A be a ring . The following are equivalent :

1) A is a real closed local ring ,

2) A is a local henselian ring and k_A is real closed ,

3) A is a real local ring and every real local-etale A-algebra (see [4]) is isomorphic to A .

Proof $1 \Rightarrow 2$: $k_{A[i]} \simeq k_A[i]$ is separably closed so k_A is real closed . We have to show that A is henselian .

Let P be a monic polynomial of $A[X]$ and suppose its image $\bar{P}$ in $k_A[X]$ has a simple root $\bar{a}$ in k_A . Since $A[i]$ is henselian $\bar{a}$ is the image of a root a+ib of P in $A[i]$. Since a-ib must also be a root of P and the lifting of a simple root is unique we have $b = 0$; this is what we wanted .

2 ⇒ 1 : A is real local so $A[i]$ is local . Since it is finite as an A-module , it is henselian . k_A is real closed so $k_{A[i]} \simeq k_A[i]$ is separably closed .

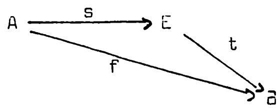
2 ⇒ 3 : Let B be a real local-etale A-algebra . k_B is real , and a separable extension of k_A , hence it is isomorphic to k_A . Using the fact that A is henselian iff every local-etale A-algebra with the same residue field is isomorphic to A , we have that B is isomorphic to A .

3 ⇒ 2 : A is surely henselian (use the equivalence mentioned above) . If K is a real field which is a finite extension of k_A , K is the residue field of some real local-etale A-algebra and hence is isomorphic to k_A . This proves that k_A is real closed . ∎

b) Factorisation of a morphism into a real closed local ring

We recall first some definitions and results of [5] .

Definition 2 : A morphism of rings $f : A \longrightarrow B$ is separably closed when for every commutative diagram



where s : A ⟶ E is an etale A-algebra , there is an unique $u : E \longrightarrow A$ such that $us = Id_A$ and $fu = t$. A morphism of rings $f : A \longrightarrow B$ in an arbitrary topos $\mathcal{E}$ is separably closed when for every X in $\mathcal{E}$ Hom(X,A) ⟶ Hom(X,B) is separably closed .

In [5] this definition applies only to local morphisms between local rings . Actually we shall consider separably closed morphisms only between local rings . We have :

<u>Proposition 3</u> : Let B be a local ring . A morphism $f : A \longrightarrow B$ is separably closed iff A is local , f is a local morphism and for every monic polynomial P in A[X] and every simple root b of P in B (P(b) = 0 and P'(b) invertible) there exists a root (necessarily unique and simple) a of P in A such that f(a) = b . This holds also for rings in a topos , with the convenient internalisation .

<u>Proof</u> : This is an easy consequence of the local structure of etale algebras (see [5]) . ∎

A typical example of separably closed morphism is the canonical morphism $A \longrightarrow k_A$ where A is local henselian . The main fact about separably closed morphisms is the following factorisation property which in the case of the morphism from a local ring to its residue field reduces to the henselisation :

<u>Proposition 4</u> : A morphism of rings $f : A \longrightarrow B$ has an initial factorisation $A \xrightarrow{g} C \xrightarrow{h} B$ whith h separably closed , and this factorisation is functorial : If

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \end{array}$$

is a commutative square with f' separably closed , there is an unique morphism $C \longrightarrow A'$ making everything commute . The factorisation property holds also for morphisms of rings in arbitrary toposes and the factorisation is stable under inverse image of geometric morphisms .

In the case of ordinary morphisms of rings , the factorisation is obtained by taking the filtered colimit of all factorisations $A \longrightarrow E \longrightarrow B$ where E is an etale A-algebra . Thus we have :

Proposition 5 : The factorisation of a morphism $f : A \longrightarrow B$ is isomorphic to $A \xrightarrow{f} B \xrightarrow{Id_B} B$ iff B is a formally etale A-algebra (i.e. a filtered colimit of etale A-algebras) . In the case of rings in an arbitrary topos the first property will serve as a definition of formally etale morphisms . So any morphism of rings admits an unique (up to iso) factorisation in a formally etale morphism followed by a separably closed morphism .

The fact of being a real closed local ring is reflected by separably closed morphisms (as the fact of being separably closed local) :

Proposition 6 : Let $f : A \longrightarrow B$ be separably closed and B be real closed local . Then A is real closed local (and f is a local morphism).

Proof (see [5]) : A and f are local by proposition 3 . Consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ k_A & \longrightarrow & k_B \end{array}$$

k_A is separably closed in k_B , and $A \longrightarrow k_A$ is separably closed (that is A is henselian) . Hence A is real closed . ∎

Corollary : A morphism $f : A \longrightarrow B$ in a real closed local ring B has an unique (up to iso) factorisation $A \xrightarrow{g} C \xrightarrow{h} B$ where C is real closed local , g is formally etale and h is separably closed (remark by the way that any local morphism between real closed local rings is separably closed) .

In the following , a formally etale real closed local A-algebra will be called a "localisation" of A (with the ") .

Proposition 7 : The isomorphism classes of "localisations" of A are in bijective correspondance with the couples composed of a real prime ideal of A , $\mathfrak{p}$, together with an order on the residue field $k(\mathfrak{p})$ (the orders on fields we consider are all total) .

Proof : The factorisation of $f : A \longrightarrow B$ with B real closed local may be obtained in the following way : Let $\mathfrak{p}$ be $f^{-1}(\mathfrak{m}_B)$. $\mathfrak{p}$ is real prime . The unique order on k_B induces an order on $k(\mathfrak{p})$. Let $\overline{k(\mathfrak{p})}$ be the real closure of $k(\mathfrak{p})$ with respect to this order , and $A_{\mathfrak{p}} \longrightarrow \overline{A_{\mathfrak{p}}} \longrightarrow \overline{k(\mathfrak{p})}$ the factorisation formally etale-separably closed of the composite

$A_{\mathfrak{p}} \longrightarrow k(\mathfrak{p}) \longrightarrow \overline{k(\mathfrak{p})}$

$$\begin{array}{ccccc} A & & \xrightarrow{\quad f \quad} & & B \\ \searrow & & & & \\ A_{\mathfrak{p}} & \longrightarrow & \overline{A_{\mathfrak{p}}} & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ k(\mathfrak{p}) & \longrightarrow & \overline{k(\mathfrak{p})} & \longrightarrow & k_B \end{array}$$

$\overline{A_{\mathfrak{p}}}$ is local henselian with residue field $\overline{k(\mathfrak{p})}$. So it is real closed local and it is also a formally etale A-algebra . By the functoriality of the factorisation there is an unique morphism $\overline{A_{\mathfrak{p}}} \longrightarrow B$ making everything commute , and it is separably closed . $A \longrightarrow \overline{A_{\mathfrak{p}}} \longrightarrow B$ is thus the wanted factorisation . We can now explicit the bijective correspondance mentioned in the proposition :

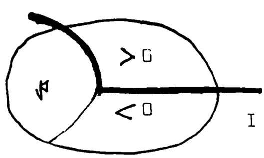
- To a "localisation" $f : A \longrightarrow B$ corresponds the couple formed by $\mathfrak{p} = f^{-1}(\mathfrak{m}_B)$ and the order on $k(\mathfrak{p})$ induced by the order on k_B .

- To a couple $(\mathfrak{p}, \leq)$ corresponds the "localisation" $A \longrightarrow B$ obtained by taking the factorisation formally etale-separably closed of the morphism $A \longrightarrow \overline{k(\mathfrak{p})}$ where $\overline{k(\mathfrak{p})}$ is the real closure of $k(\mathfrak{p})$ with respect to the given order .

Here is an example of such a "localisation" : Consider the ring $\mathbb{R}[X]$, and the real prime ideal (X) together with the unique order on the residue field of $\mathbb{R}[X]_{(X)}$ which is $\mathbb{R}$. The corresponding "localisation" is obtained by taking the factorisation formally etale-separably closed of $\mathbb{R}[X]_{(X)} \longrightarrow \mathbb{R}$ since $\mathbb{R}$ is real closed . This is just the henselisation of $\mathbb{R}[X]_{(X)}$ and it is known that it is the ring of germs of Nash (or algebraic) functions at 0 , i.e. germs of real analytic functions f satisfying a relation $P(x,f(x)) = 0$ with P a non constant polynomial (see [6]) .

II) The etale real spectrum of a ring

a) We have just seen that the "localisations" of a ring A correspond to real prime ideals of A together with an order on the residue field . Suppose we are given a real prime ideal $\mathfrak{p}$ and an order on $k(\mathfrak{p})$; we have then a partition of A in three parts : the elements of A which become 0 in $k(\mathfrak{p})$ (i.e. $\mathfrak{p}$) , those which become strictly positive and those which become strictly negative



Proposition 1 and Definition 1 : Giving a real prime ideal $\mathfrak{p}$ of A together with an order on $k(\mathfrak{p})$ is equivalent to giving a subset I of A satisfying :

1) $1 \in I$

2) $-x^2 \in I$.

3) $x \in I \wedge y \in I \Rightarrow x+y \in I$

4) $xy \in I \iff (x \in I \wedge -y \in I) \vee$ /...

.../ $(-x \in I \wedge y \in I)$.

The condition 4 may be reformulated as the conjunction of the following three conditions :

$4_1'$) $x \in I \wedge -x \in I \Rightarrow xy \in I$

$4_2'$) $x \in I \wedge -y \in I \Rightarrow xy \in I$

$4_3'$) $xy \in I \Rightarrow x \in I \vee y \in I$.

I is the set of elements which become ≤ 0 in $k(\mathfrak{p})$. $\mathfrak{p}$ is $I \cap -I$.

A subset of A satisfying conditions 1 to 4 will be called a prime negideal of A .

Proof : It is obvious that the set of elements which become ≤ 0 in $k(\mathfrak{p})$ satisfy conditions 1 to 4 . Suppose now we are given a prime negideal $I \subset A$. Let $\mathfrak{p}$ be $I \cap -I$. Condition 3 gives $x \in \mathfrak{p} \wedge y \in \mathfrak{p} \Rightarrow x+y \in \mathfrak{p}$, condition $4_1'$ gives $x \in \mathfrak{p} \Rightarrow xy \in \mathfrak{p}$, condition 1 gives $1 \notin \mathfrak{p}$, condition $4_3'$ gives $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \vee y \in \mathfrak{p}$. Suppose we have $x_1^2+x_2^2+\ldots+x_n^2 \in \mathfrak{p}$. By conditions 2 and 3 we have $x_1^2 \in I$, which by condition 4 implies $x_1 \in \mathfrak{p}$. So $\mathfrak{p}$ is a real prime ideal of A ; we shall say that $\mathfrak{p}$ is associated to I .

It is easy to check that there is an order on $A/\mathfrak{p}$ the positive elements of which are the images of $x \in -I$. This is a total order since conditions 2 and 4 imply $x \in I \vee -x \in I$, and it induces an order on the field of fractions $k(\mathfrak{p})$. ■

Remark : THe right thing to consider in the context of "localisation" (from an intuitionistic or topos-theoretic point of view) would be not the prime neg-ideal but its complementary (just as the right thing to consider is not the prime ideal but its complementary) .

b) The topology of the real etale spectrum .

Definition 2 : The real etale spectrum of a ring A (denoted by $\mathrm{Spec}_{Ret}(A)$) is the set of its prime negideals with the topology given by the basis of open sets :

$$D'_{a_1,\ldots,a_n} = \{ I \text{ prime negideal of } A \mid a_1 \notin I \wedge \ldots \wedge a_n \notin I \} .$$

The name of real etale spectrum is justified by the fact that it is related to "localisations" of A which are real formally etale local A-algebras . $D'_{a_1,\ldots,a_n}$ is the set of prime negideals such that in the corresponding ordered field $a_1,\ldots,a_n$ become strictly positive . Spec_{Ret} is obviously a contravariant functor from the category of rings to the category of topological spaces .

Let us describe the real etale spectrum of $\mathbb{R}[X]$. We have already seen that the real primes of $\mathbb{R}[X]$ are the real maximal ideals corresponding to the points of the real line and the ideal 0 which is the generic point of the real line . The residue field at real maximal ideals is R and there is no choice for the order on it : each real point determines a prime negideal .

The residue field at 0 is $\mathbb{R}[X]$. To order it we have to introduce X somewhere in the real line ; the following cases may occur :

$$X = -\infty$$

$$\left.\begin{array}{l} X = a_- \\ X = a_+ \end{array}\right\} \text{ for } a \text{ in } R$$

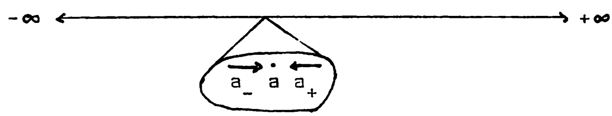
$$X = +\infty \quad .$$

For instance the set of strictly positive elements for the order determined by $X = a_-$ is

$$\{ F \in \mathbb{R}[X] \mid \exists \varepsilon > 0 \ \forall x \in]a-\varepsilon, a[\ \ F(x) > 0 \} \quad .$$

So the generic point of the real line has exploded in many points , each one corresponding to a "half branch" of the line centered at a real point (or at infinity) .

We can make the following picture of the real etale spectrum :



The open sets are generated by the intervals $[a_+, b_-]$ (which is $D^!_{X-a,b-X}$) or $[-\infty, b_-]$ or $[a_+, +\infty]$. So the real etale spectrum of $\mathbb{R}[X]$ contains the real line with its usual topology (this is a general fact for algebraic subsets of $\mathbb{R}^n$) .

It is easy to see that $\mathrm{Spec}_{Ret}(\mathbb{R}[X])$ is compact . This is a general property :

Proposition 2 : The real etale spectrum of a ring is compact .

Proof : Consider the language L which is the language of rings augmented by a unary relational symbol $\mathfrak{J}$. Consider all the sequents $\Phi \vdash \mathfrak{J}(s)$ (where Φ is a conjunction of atomic formulas of L and s is a term) which are consequences of the theory T of rings-with-a-prime-negideal (obtained by adding the four conditions of proposition 1 to the theory of rings) . Let us call C this set of sequents . In C we have for instance the conditions 2 , 3 , $4'_1$, $4'_2$ but not 1 nor $4'_3$. Consider now a ring A and a subset X of A , and let J(X) be the closure of X for C (i.e. the smallest J A containing X such that all the sequents of C are valid when $\mathfrak{J}$ is interpreted by J .

Lemma 1 : J(X) is the intersection of all prime negideals of A containing X .

Proof of the lemma : First it is clear that this intersection contains J(X) . Suppose now that every prime negideal of A containing X contains a . Let f : A $\longrightarrow$ B be an A-algebra and I a prime negideal of B containing f(X) . Then I contains f(a) since $f^{-1}(I)$ must contain a . Consider now the realisation of L given by A and X (for the interpretation of $\mathfrak{J}$) . The preceding remark means that the positive diagram of this realisation (i.e. the set of closed atomic formulas of L with parameters in A which are true in this realisation) and the theory T of rings-with-a-prime-negideal imply $\mathfrak{J}(a)$. By compacity we get a conjunction of atomic formulas of L $\Phi(y_1,\dots,y_n,x)$ such that $\Phi(y_1,\dots,y_n,x) \vdash \mathcal{J}(x)$ is a consequence of T and there are $b_1,\dots,b_n$ in A such that the realisation (A,X) of L satisfies $\Phi(b_1,\dots,b_n,a)$. By the construction of J(X) we have then $a \in J(X)$. ◀

Lemma 2 : If $\mathrm{Spec}_{Ret}(A) = \bigcup_{x\in X} D'_x$, there is a finite subset Y of X such that $\mathrm{Spec}_{Ret}(A) = \bigcup_{x\in Y} D'_x$.

Proof of the lemma : $\mathrm{Spec}_{Ret}(A) = \bigcup_{x\in X} D'_x$ means that there is no prime negideal of A containing X , which is equivalent to $J(X) = A$, or $1 \in J(X)$. From the construction of J(X) it is clear that there is then a finite subset Y of X such that $1 \in J(X)$. ◀

We are now in position to show that $\mathrm{Spec}_{Ret}(A)$ is compact . Suppose we are given an open covering of $\mathrm{Spec}_{Ret}(A)$; we may always suppose this covering is composed of open sets of the basis :

$$\mathrm{Spec}_{Ret}(A) = \bigcup_{i\in I} D'_{a_{i,0},\dots,a_{i,n_i-1}} \quad .$$

This is equivalent to :

$$\forall j \in \prod_{i\in I} n_i \quad \mathrm{Spec}_{Ret}(A) = \bigcup_{i\in I} D'_{a_{i,j(i)}}$$

(where $n_i = \{0,1,\dots,n_i-1\}$) . By lemma 2 for every j there is a finite $I_j \subset I$ such that

$$\mathrm{Spec}_{Ret}(A) = \bigcup_{i\in I_j} D'_{a_{i,j(i)}} \quad .$$

The elements of $\prod_{i\in I} n_i$ which coincide with j on I_j form an open neighbourhood V_j of j in $\prod_{i\in I} n_i$ which is compact . so there are $j_1,\dots,j_m$ such that $\prod_{i\in I} n_i = \bigcup_{k=1}^{m} V_{j_k}$. Let J be the union of the I_{j_k} .

J is finite and we have :

$$\mathrm{Spec}_{Ret}(A) = \bigcup_{i\in J} D'_{a_{i,0},\dots,a_{i,n_i-1}} .$$

Remark : This proof is raher unsatisfactory . It would surely be nicer to have an actual description of the intersection of the prime negideals containing a given subset of a ring (as there is such a description in the case of prime ideals) .

Proposition 3 : Every $D'_{a_1,\dots,a_n}$ is compact .

Proof : Since the image of a compact set by a continuous map is compact , this will result immediately of the following lemma :

Lemma 3 : If $a\in A$, let $A[\sqrt{a}]$ be $(A[X]/_{X^2-a})[(2a)^{-1}]$. It is an etale A-algebra . $D'_{a_1,\dots,a_n}$ is the image of $\mathrm{Spec}_{Ret}(A[\sqrt{a}_1,\dots,\sqrt{a}_n])$ in $\mathrm{Spec}_{Ret}(A)$.

Proof of the lemma : Since $A[\sqrt{a}_1,\dots,\sqrt{a}_n]$ is an etale A-algebra any "localisation" of this ring is a "localisation" of A . Let I be a prime negideal of A . The "localisation" of A at I factors through $A[\sqrt{a}_1,\dots,\sqrt{a}_n]$ iff $a_1,\dots,a_n$ become strictly positive in the ordered field associated to I , that is iff $I\in D'_{a_1,\dots,a_n}$. ∎

III) The real etale spectrum of a field as a ringed space .

Let k be a field . $\mathrm{Spec}_{\mathrm{Ret}}(k)$ is just the set of orders which can be put on k . It is non empty iff k is real . There is an obvious sheaf to put on this topological space which we shall denote by $\mathcal{S}\mathrm{pec}_{\mathrm{Ret}}(k)$:

- If $\mathcal{O}$ is an order on k , the stalk of $\mathcal{S}\mathrm{pec}_{\mathrm{Ret}}(k)$ over $\mathcal{O}$ is $\bar{k}^{\mathcal{O}}$, the real closure of k with respect to $\mathcal{O}$.

- Let α be an element of $\bar{k}^{\mathcal{O}}$. α is algebraic over k ; let P be its irreducible polynomial . All roots of P are simple so we can build a Sturm sequence $P_0 = P$, $P_1 , \ldots , P_m$ for P (see [7]) . Let $a_1, \ldots, a_m$ be the leading coefficients of the P_i's and U be the set of orders on k giving to $a_1, \ldots, a_m$ the same signs that $\mathcal{O}$. U is an open set of $\mathrm{Spec}_{\mathrm{Ret}}(k)$. For every $\mathcal{O}'$ in U , P has a constant number of real roots (roots in $\bar{k}^{\mathcal{O}'}$) which is given by Sturm's theorem . α is , say , the n^{th} root of P in $\bar{k}^{\mathcal{O}}$ for the order on $\bar{k}^{\mathcal{O}}$. For $\mathcal{O}' \in U$ let $\alpha^{\mathcal{O}'}$ be the n^{th} root of P in $\bar{k}^{\mathcal{O}'}$. The sets $\{ \alpha^{\mathcal{O}'} \mid \mathcal{O}' \in V \}$ for V open set contained in U form a basis of open neighbourhoods of α for the topology on the sheaf $\mathcal{S}\mathrm{pec}_{\mathrm{Ret}}(k)$.

$\mathcal{S}\mathrm{pec}_{\mathrm{Ret}}(k)$ has the following universal property :

<u>Proposition 1</u> : Let X be a topological space and $\mathcal{F}$ a sheaf of rings over A such that each

stalk is a real closed field . For every morphism $g : k \longrightarrow \Gamma\mathfrak{F}$ there is an unique couple (φ, f) where $\varphi : X \longrightarrow \mathrm{Spec}_{Ret}(k)$ is continuous and $f : \varphi^{*}\mathcal{S}pec_{Ret}(k) \longrightarrow \mathfrak{F}$ is a morphism of sheaves such that

$$\begin{array}{ccc} k & \longrightarrow & \Gamma\mathcal{S}pec_{Ret}(k) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & \mathfrak{F} \end{array}$$

commutes .

<u>Proof</u> : g induces $g_x : k \longrightarrow \mathfrak{F}_x$ for $x \in X$. The order on $\mathfrak{F}_x$ induces an order $\varphi(x)$ on k . φ is continuous since for $a \in k$ $\varphi^{-1}(D'_a)$ is the open subset of X over which the global section g(a) of $\mathfrak{F}$ is strictly positive , i.e. the value of the formula $\exists x\ \exists t\ (\ xt = 1 \wedge x^2 = g(a)\ )$. Since $\mathfrak{F}_x$ is real closed there is an unique morphism $f_x : \overline{k}^{\varphi(x)} \longrightarrow \mathfrak{F}_x$ and glueing together the f_x's yields a morphism of sheaves $f : \varphi^{*}(\mathcal{S}pec_{Ret}(k)) \longrightarrow \mathfrak{F}$. ∎

We shall come back later on the existence and the eventual universal property of a similar sheaf on the real etale spectrum of a ring .

IV) Real étale topology

We are going to define a Grothendieck topology the real étale topology on the dual of the category of rings.

Let us define first a pretopology:

the family $(A \xrightarrow{f_i} A_i)_{i \in I}$ is in Cov(A) iff for all i in I $f_i : A \longrightarrow A_i$ is an étale A-algebra and

$\bigcup_{i \in I} \mathrm{Im}(\mathrm{Spec}_{Rét}(f_i)) = \mathrm{Spec}_{Rét}(A)$.

- $\{A \xrightarrow{1_A} A\}$ is in Cov(A)

- local character is trivial

- stability under change of basis:

Let $(A \xrightarrow{f_i} A_i)$ be a covering family and $(B \xrightarrow{g_i} B \otimes_A A_i)$ the family obtained by push-out under $f : A \longrightarrow B$.

We must prove that it is a covering family: let J be a prime negideal in B. There exist an i and a prime negideal J_i in A_i such that $f_i^{-1}(J_i) = f^{-1}(J)$.

We get three real closed fields $k(J), k(f^{-1}(J))$, $k(J_i)$ which are respectively the residue fields of the "localisations" $B_J, A_{f^{-1}(J)}, A_{i\,J_i}$; since real closed fields admit amalgamation property (elimination of quantifiers and proposition page 63 ch.13 in Sacks saturated model theory) there is a real closed field K which makes the diagramm

$$\begin{array}{ccccc} & & k(J_i) & & \\ & \nearrow & & \searrow & \\ k(f^{-1}(J)) & & & & K \\ & \searrow & & \nearrow & \\ & & k(J) & & \end{array}$$

commute.

So we get a morphism h from $B\otimes A_i$ to K. It is clear that $J_i'=h^{-1}\{x \mid x\leq 0\}$ is a prime negideal in $B\otimes_A A_i$ such that $g_i^{-1}(J_i')=J$.

Now covering families for the real étale topology are families of arrows of same source containing a sub-family in Cov(A).

Proposition 1: the real étale topology contains the realZariski topology (and the Zariski topology).

proof: Let $(a_1,\ldots,a_n)$ be a covering family for real Zariski topology. Let I be a prime negideal of A such that the images of the a_i's are all in $\mathfrak{m}_{A_I}$ maximal ideal of A_I. The inverse image of $\mathfrak{m}_{A_I}$ in A is then a real prime ideal containing all the a_i: contradiction.

Remark:

-real étale topology does not contain étale topology $\mathbb{C}\longrightarrow 0$ is covering for real étale topology, not for real étale topology.

-étale topology does not contain real étale topology: $\mathbb{R}\longrightarrow\mathbb{C}$ is covering for étale topology, not for real étale topology.

Before going further we are going to state the following result concerning real closed local rings:

Proposition 2: Let Φ_{nm} be the formula with free variables$(x_1,\ldots,x_n,y_1,\ldots,y_m)$:

$$\exists x\ x^n+x_1x^{n-1}+\ldots+x_n=0\ /\ldots$$

$$\ldots/\wedge\ \exists t\ t.(nx^{n-1}+(n-1)x_1x^{n-2}+\ldots+x_{n-1}).(x^m+y_1x^{m-1}+\ldots+y_m)=1$$

There exist polynomial expressions in $(x_1,\ldots,x_n, y_1,\ldots,y_m)$ $E_{11},\ldots,E_{1n_1},\ldots,E_{k1},\ldots,E_{kn_k}$ such that

$\phi_{nm} \leftrightarrow \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n_i} E_{ij} > 0$ (1) is true in the theory of real closed local rings. ($A>0$ stands as an abreviation for "the exist a simple square root of A": $\exists x \ x^2=A \wedge \exists t \ t.2A=1$).

proof: $\{(x_1,\ldots,x_n,y_1,\ldots y_m) \mid \phi_{nm}(x_1,\ldots,x_n,y_1,\ldots,y_m)\}$ is an open subset of $\mathbb{R}^{n+m}$. Hence we can find the required E_{ij} such that (1) is verified in real closed fields from the theorem in appendix 1.

Now a real closed local ring being henselian over its residue field

$A \models \phi_{nm}(a_1,\ldots,a_n,b_1,\ldots,b_m)$ iff

$k_A \models \phi_{nm}(\bar{a}_1,\ldots,\bar{a}_n,\bar{b}_1,\ldots,\bar{b}_m)$

Also since we have only strict inaqualities

$A \models \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n_i} E_{ij}(a_1,\ldots,a_n,b_1,\ldots,b_m) > 0$ iff

$k_A \models \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n_i} E_{ij}(\bar{a}_1,\ldots,\bar{a}_n,\bar{b}_1,\ldots,\bar{b}_m) > 0$.

So (1) is also verified in real closed local rings.

<u>Remark</u>: The proposition 2 tells us that the existence of a simple root in a real closed ring can be expressed by a disjunction of conjonctions of strict inequalities concerning polynomial expressions in the coefficient of the polynomial.But we have no effective way of calculating these expressions as we do for hyperdiscriminants in the case of strictly local henselian rings.

Proposition 3: If f is an étale A-algebra

$\mathrm{Spec}_{Rét}(f)$ is an open map.

proof: Let us prove first that $\mathrm{Im}(\mathrm{Spec}_{Rét}(f))$ is open. Since $\mathrm{Spec}_{Rét}(A(a^{-1}))$ may be identified to the open $D'_a \cup D'_{-a}$ the only case to consider is $A \xrightarrow{f} A[X]/P\,[P'R^{-1}]$ (because of the theorem of local structure of étale A-algebras) with P monic polynomial of degree n and R monic polynomial of degree m.

We know that the formula $\exists x\ P(x)=0 \wedge \exists t.\ P'(x).R(x)=1$ is equivalent to $\bigvee_{i=1}^{k} \bigwedge_{j=1}^{n_i} E'_{ij}>0$ with $E'_{ij}=E_{ij}(a_1,\dots,a_n,b_1,\dots,b_m)$,the a_i's and b_j's being the coefficients of P and R.

Let $U=\bigcup_{i=1}^{k} D'_{E'_{i1},\dots,E'_{in_i}}$ .We are going to prove $U=\mathrm{Im}(\mathrm{Spec}_{Rét}(f))$.

Let I be a prime negideal in U. In the real closed local ring A_I the image of P has a simple root α with $R(\alpha)$ invertible since the required inequalities are verified. Hence $A \longrightarrow A_I$ extends to a morphism from $A[X]/P\,[P'R^{-1}]$ to A_I .The inverse image of the maximal negideal of A_I is a prime negideal J in $A[X]/P\,[P'R^{-1}]$ such that $\mathrm{Spec}_{Rét}(f)(J)=I$.

Conversely let J be a prime negideal in $A[X]/P[P'R^{-1}]$. $A[X]/P\,[P'R^{-1}]_J$ is a real closed local ring and a formally étale A-algebra: it is a "localisation" of A, A_I. P has a simple root α with $R(\alpha)$ invertible in A_I: the image of X. So the coefficients of the images of P and R in A_I verify the required inequalities and I is in U.

Now every $D'_{a_1,\dots,a_n}$ being of the form $\mathrm{Im}(\mathrm{Spec}_{\text{Rét}}(f))$ (II)proposition 3) we are finished.

Proposition 4: Every finite family in Cov(A) is obtained by push-out from a family in $\mathrm{Cov}(A_0)$ where A_0 is a finitely presented ring.

proof:Let $f:A \longrightarrow B$ be étale .By local structure of étale algebras $\exists\, b_1,\dots,b_n$ covering for Zariski topology such that $\forall i \,\exists\, a_i, P_i, R_i$ with $B(b_i^{-1}) \simeq A(a_i^{-1})[X]/P_i\,[P_i'R_i^{-1}]$.
The fact that $(A \longrightarrow B_j)_{j=1,\dots,m}$ is in Cov(A) can be expressed by a geometric formula of the theory of rings with parameters a_i^j and the coefficients of P_i^j and R_i^j.
(For example to the covering family $A \rightrightarrows A(a^{-1}),\ A(b^{-1})$

we associate the formula $a>0 \vee -a>0 \vee b>0 \vee -b>0$,to $A \longrightarrow A[X]/P\,[P'R^{-1}]$ we associate the formula $\bigvee_{i=1}^{k} \bigwedge_{j=1}^{n_i} E'_{ij}>0$ as in proposition 3).

It is clear then that this situation is obtained fr om a similar situation in FPRings .

Proposition 5: Every family in Cov(A) contains a finite family in Cov(A).

proof:clear since $\mathrm{Spec}_{\text{Rét}}(A)$ is compact and for all f étale $\mathrm{Spec}_{\text{Rét}}(f)$ is open .

Definition:The topos of sheaves for the real étale topology on $\mathrm{FPRings}^{op}$ is called the real étale topos Rét.

Proposition 5 gives us immediately the

Theorem 1:The real étale topos is coherent.

Theorem 2: The real étale topos is the classifying topos for real closed local rings.

proof: We are going to prove that A is real closed local iff $Hom_{Rings}(-,A):FPRings^{op} \longrightarrow Sets$ is continuous for the real étale topology.

Consider $(f_i:A \longrightarrow A_i)_{i\in I}$ a covering family in $FPRings^{op}$ and $f:A \longrightarrow B$. The family $g_i:B \longrightarrow B\otimes_A A_i$ obtained by push-out under f is also covering. So the maximal ideal of A, which is real prime comes from a real prime ideal $\mathfrak{p}$ of an A_i. $(B\otimes_A A_i)_{\mathfrak{p}}$ is real locale étale hence isomorphic to A (I proposition 2), this gives $h_i: A_i \longrightarrow A$ such $h_i f_i = f$.

Conversely A is surely a real local ring. We must see that A[i] is separably closed i.e. that axioms like

ReH^2+ImH^2 invertible/...

$.../\vdash \exists a \exists b\ Re(P+iQ)(a+ib)=0 \wedge Im(P+iQ)(a+ib)=0/...$

$.../\wedge Re(P'+iQ')(a+ib)^2+Im(P'+iQ')(a+ib)^2$ invertible

with F=P+iQ monic polynomial and H one of the hyperdiscriminants of F are verified.

$A \longrightarrow A[X,Y]/(Re(F(X+iY)),Im(F(X+iY)))\ [(|F'(X+iY)|^2)^{-1}]$

is étale since $|F'(X+iY)|^2$ is the Jacobian of ReF and ImF with respect to X and Y.

It is now easy to check that if we take $F=(X+iY)^n+(X_1+iY_1)(X+iY)^{n-1}+...+(X_n+iY_n)$ and $A=\mathbb{Z}[X_1,Y_1,...,X_n,Y_n]\ [ReH^2+ImH^2\ ^{-1}]$ the morphism is in this case covering for the real étale topology.

The fact that $Hom_{Rings}(-,A)$ is continuous for the real étale topology gives then the result.

V) The real etale spectrum as the generic "localisation"

a) We are given a ring A . We are now interested in giving a description of the generic "localisation" of A (i.e. the generic real closed local formally etale A-algebra) . We shall denote by $\mathcal{S}pec_{Ret}(A)$ this generic "localisation" and by $\underline{Spec}_{Ret}(A)$ the topos where it lives . $\mathcal{S}pec_{Ret}(A)$ has the following universal property :

Given a real closed local ring B in a topos $\underline{E}$ and a morphism f from (the constant sheaf) A to B , there is an unique (up to iso) couple $(\varphi, \overline{f})$ where φ is a geometric morphism from $\underline{E}$ to $\underline{Spec}_{Ret}(A)$ and $\overline{f}$ is a separably closed morphism from $\varphi^*(\mathcal{S}pec_{Ret}(A))$ to B such that the following diagram commutes :

$$\begin{array}{ccc} A & \longrightarrow & \varphi^*(\mathcal{S}pec_{Ret}(A)) \\ & \searrow^{f} & \downarrow \overline{f} \\ & & B \end{array}$$

Indeed the factorisation formally etale-separably closed of the morphism $f : A \longrightarrow B$ gives $A \xrightarrow{g} C \xrightarrow{\overline{f}} B$ with C real closed local (I b propositions 4 , 5 and 6) . Now the"localisation" $g : A \longrightarrow C$ is classified by a geometric morphism $\varphi : \underline{E} \longrightarrow \underline{Spec}_{Ret}(A)$ such that $\varphi^*(\mathcal{S}pec_{Ret}(A)) = C$.

$\mathcal{S}pec_{Ret}(A)$ is thus a spectrum in the sense of Cole [11] . In the terminology of [2] , it is the

spectrum associated to the localisation triple (Ring,V,$\mathcal{T}$)
where : - Ring is the category of rings ,

- V is the set of etale morphisms between finitely presented rings .
- $\mathcal{T}$ is the real etale topology on $\underline{FPRing}^{op}$, the dual of the category of finitely presented rings .

This triple satisfies the condition for being a localisation triple since the real etale topology is generated by families of etale morphisms . So the general results of [2] give a description of $\mathcal{S}pec_{Ret}(A)$:

Proposition 1 : $\underline{Spec}_{Ret}(A)$ is the topos of sheaves over the dual of the category of etale A-algebras for the real etale topology and $\mathcal{S}pec_{Ret}(A)$ is the sheaf associated to the presheaf given by the inclusion of the category of etale A-algebras in the category of rings .

Proof : This is exactly the description of the spectrum in [2] once we have remarked that :

- by pushing out V under A we get the category of etale A-algebras (cf [4] chapitre V exercice p.55) ,
- by pushing out $\mathcal{T}$ under A we get the real etale topology (cf IV proposition 4) . ∎

Remark : $\underline{Spec}_{Ret}(A)$ is a coherent topos since the real etale topology is generated by finite families .

b) Is the real etale spectrum a sheaf over a topological space ?

If the answer is yes , $\mathcal{S}pec_{Ret}(A)$ is a sheaf over the real etale spectrum of A described in II (this justifies the notation) :

<u>Proposition 2</u> : If $\underline{Spec}_{Ret}(A)$ is spatial it is the topos of sheaves over $Spec_{Ret}(A)$.

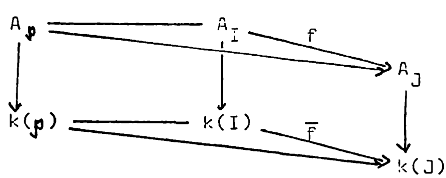
<u>Proof</u> : Suppose $\underline{Spec}_{Ret}(A)$ is the topos of sheaves over a topological space X . Since points of $\underline{Spec}_{Ret}(A)$ correspond to "localisations" of A , X and $Spec_{Ret}(A)$ have the same points . By the construction of $\underline{Spec}_{Ret}(A)$ the family of sets $\{ g : A \longrightarrow B$ "localisation" of A $|$ g factors through $s : A \longrightarrow E \}$ indexed by the etale A-algebras $s : A \longrightarrow E$ is a basis for the topology of X . Since s etale implies $Spec_{Ret}(s)$ open (IV proposition 3) and $D'_{a_1,\dots,a_n}$ is the image of $Spec_{Ret}(A \longrightarrow A[\sqrt{a}_1,\dots,\sqrt{a}_n])$ (II lemma 3) the topology on X coincide with the topology on $Spec_{Ret}(A)$. ■

Now we come to the question itself . We use the following result , proved in Appendix 2 : a coherent topos is spatial iff the category of its points is equivalent to an ordered set . The category of points of $\underline{Spec}_{Ret}(A)$ is equivalent to the category of "localisations" of A , the morphisms being morphisms of A-algebras . So the problem is reduced to the following one : Given any two prime negideals I and J of A , show that there is at most one morphism of A- algebras from the "localisation" A_I to the "localisation" A_J .

Remark that if there is a morphism of A-algebras from A_I to A_J we have necessarily $J \subset I$ since the fact of being strictly positive (i.e. having an invertible square root) is preserved by any morphism .
It looks plausible that the ordered set of prime neg-ideals of A is equivalent to the dual of the category of "localisations" of A with morphisms of A-algebras .
But all we have been able to prove is the following two results (the first one being rather obvious) :

Proposition 3 : If the prime negideals I and J have the same associated real prime ideal , there is a morphism of A-algebras from A_I to A_J iff $I = J$ and in this case there is no other morphism than the identity of A_I .

Proof : Suppose $I \cap -I = J \cap -J = \mathfrak{p}$ and $f : A_I \longrightarrow A_J$ is a morphism of A-algebras . Then f is a morphism of $A_{\mathfrak{p}}$-algebras and since A_I and A_J are local-ind-etale $A_{\mathfrak{p}}$-algebras f is local (see [4]) . Hence we have a commutative diagram :



Since $k(I)$ and $k(J)$ are real closures of $k(\mathfrak{p})$ we have $I = J$ and $\overline{f}$ is the identity (cf [7]) . Now

$$\begin{array}{ccccc} & & A_I & & \\ & \nearrow & \downarrow f & \searrow & \\ A_{\mathfrak{p}} & & & & k(I) \\ & \searrow & & \nearrow & \\ & & A_I & & \end{array}$$

is commutative and

$A_{\mathfrak{p}} \longrightarrow A_I \longrightarrow k(I)$ is the factorisation formally etale-separably closed of $A \longrightarrow k(I)$. This implies $f = Id_{A_I}$.

<u>Proposition 4</u> : Let I be a prime negideal of A and $\mathfrak{p}$ its associated real prime ideal . Suppose $k(\mathfrak{p})$ is dense in $k(I)$ (for the order) . Then for any prime neg-ideal J of A there is at most one morphism of A-algebras from A_I to A_J .

<u>Proof</u> : Let $\mathfrak{q}$ be the real prime ideal associated to J and f a morphism of A-algebras from A_I to A_J . It induces a morphism of A-algebras g from $A_{\mathfrak{p}}$ to $A_{\mathfrak{q}}$:

$$\begin{array}{ccccc} & \nearrow & A_{\mathfrak{p}} & \longrightarrow & A_I \\ A & & \downarrow g & & \downarrow f \\ & \searrow & A_{\mathfrak{q}} & \longrightarrow & A_J \end{array}$$

We know that there is at most one such morphism from A to A .

A_I is a filtered colimit of local-etale $A_{\mathfrak{p}}$-algebras (cf [5] p.10) i.e. of $A_{\mathfrak{p}}$-algebras like $(A_{\mathfrak{p}}[X]/_P)_{\mathfrak{n}}$ where P is monic and $\mathfrak{n}$ is a prime ideal not containing P' ; the image of X in A_I is a simple root of P . So it is sufficient to prove that there is only one way to send simple roots of monic polynomials in $A_{\mathfrak{p}}[X]$ from A_I to A_J .

Let $\overline{P}$ be the image of P in $k(\mathfrak{p})[X]$ and let $\overline{\alpha}$ be the image in k(I) of a simple root α of P in A_I . If $k(\mathfrak{p})$ is dense in k(I) we can surely find $\overline{a}$ and $\overline{b}$ in $k(\mathfrak{p})$ such that $\overline{a} < \overline{\alpha} < \overline{b}$ and that there is no other root of $\overline{P}$ in $[\overline{a},\overline{b}]$. By the theorem of appendix 1 the formula "there is a simple root of P in $]\overline{a},\overline{b}[$ and no other root of P in $[\overline{a},\overline{b}]$" is equivalent in the theory of real closed fields to a disjunction of conjunctions of strict inequalities involving polynomial expressions over $\mathbb{Z}$ in $\overline{a}$, $\overline{b}$ and the coefficients of $\overline{P}$; we shall denote this last formula by $F(\overline{a},\overline{b},\overline{P})$.

Suppose now that there is an other morphism of A-algebras $f' : A_I \longrightarrow A_J$. We have $f(a) = f'(a) = g(a)$ and $f(b) = f'(b) = g(b)$. Since f and f' both preserve strict inequalities we have in k(J) $\overline{g(a)} < \overline{f(\alpha)} < \overline{g(b)}$ and $\overline{g(a)} < \overline{f'(\alpha)} < \overline{g(b)}$ and also $F(\overline{g(a)},\overline{g(b)},\overline{g(P)})$. So there is only one root of $\overline{g(P)}$ between $\overline{g(a)}$ and $\overline{g(b)}$ and hence $\overline{f(\alpha)} = \overline{f'(\alpha)}$. Since $f(\alpha)$ and $f'(\alpha)$ are two simple roots of P which have the same image in the residue field we must have $f(\alpha) = f'(\alpha)$ which is the wanted result . ■

Remark : There are ordered fields which are not dense in their real closure : Take $\mathbb{R}(X)$ with the order associated to $X = +\infty$. Its real closure contains $\sqrt{X}$ and there is no rational function between $\sqrt{X}$ and $2\sqrt{X}$ (this example is borrowed from a paper by McKenna "New facts about Hilbert's 17^{th} problem") .

We can apply proposition 3 in the case where A is a field k . $\mathcal{S}pec_{Ret}(k)$ is then a sheaf on a topological space , and it is the one described in section III .

c) There is another case where we can answer positively the question of §b :

Theorem 1 : Let Γ be an algebraic curve in $\mathbb{R}^n$ and A the ring of polynomial functions on Γ. Then $\mathcal{S}pec_{Ret}(A)$ is a sheaf over the topological space $Spec_{Ret}(A)$. We have already seen that $Spec_{Ret}(A)$ contains as a subspace Γ withthe topology induced by the usual one on $\mathbb{R}^n$. $\mathcal{S}pec_{Ret}(A)$ restricted to Γ is the sheaf of Nash functions on Γ .

Proof : There are two kinds of real prime ideals in A : 1) those which are generic points of irreducible real components of Γ ,

2) those which are real points of Γ .

Consider now two prime negideals I and J of A with associated real primes $\mathfrak{p}$ and $\mathfrak{q}$. Suppose that there is a morphism of A-algebras $f : A_I \longrightarrow A_J$. We want to show that f is the only one .

If $\mathfrak{p}$ and $\mathfrak{q}$ are of the same kind we must have $\mathfrak{p} = \mathfrak{q}$

and then we can apply proposition 3 . If not $\mathfrak{p}$ is of the second kind and $\mathfrak{q}$ of the first kind . $k(\mathfrak{p})$ is surely dense in its real closure since it is $\mathbb{R}$; so we can apply proposition 4 .

The stalk of $\mathcal{S}pec_{Ret}(A)$ at $\mathfrak{p}$ when $\mathfrak{p}$ is a real maximal ideal of A corresponding to a real point s of Γ is the "localisation" of A at the prime negideal over $\mathfrak{p}$ i.e. the henselisation of the local ring of germs of polynomial functions on Γ at s : this is the ring of germs of Nash functions on Γ at s . This gives the last part of the theorem . ∎

We can now give a description of the real etale spectrum of the real line as a ringed space : The topological space $Spec_{Ret}(\mathbb{R}[X])$ has already been described in II a . We know already that the stalk of $\mathcal{S}pec_{Ret}(\mathbb{R}[X])$ at a real point of the line is the ring of germs of Nash functions at this point . We have to look at the stalk at points of $Spec_{Ret}(\mathbb{R}[X])$ like $-\infty$, a_- , a_+ or $+\infty$. Let us consider for instance 0_+ :

<u>Proposition 5</u> : The stalk of $\mathcal{S}pec_{Ret}(\mathbb{R}[X])$ at 0_+ is the field K of germs of Nash functions at the right of 0 , i.e. the filtered colimit of the rings of Nash functions defined on $]0,\varepsilon[$ for $\varepsilon > 0$.

<u>Proof</u> : K is a field since a Nash function which is not identically zero must be invertible on some $]0,\varepsilon[$.

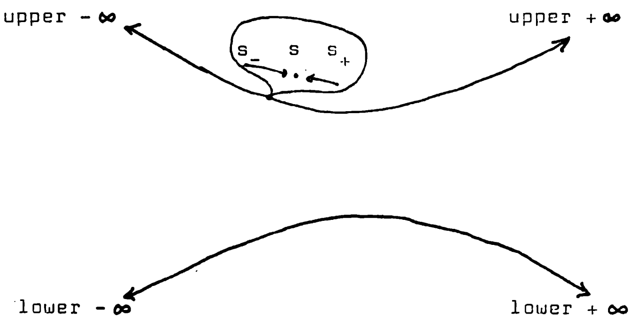
K has an order : the strictly positive elements are germs f which are strictly positive on some $]0,\varepsilon[$. K is then an ordered extension of $\mathbb{R}(X)$ with the order dtermined by $X = 0_+$. By definition of Nash functions K is algebraic over $\mathbb{R}(X)$. It remains to show that it is real closed . Let $P = X^n + f_1 X^{n-1} + \ldots + f_n$ be an irreducible polynomial of $K[X]$. We can construct its Sturm sequence $P_0=P$, $P_1,\ldots,P_m$ and Sturm's theorem gives us the number of roots of P in the real closure of K , say r . Consider now an interval $]0,\varepsilon[$ where the leading coefficients of the P_i's have constant signs . For any x in $]0,\varepsilon[$, $x^n + f_1(x)x^{n-1} + \ldots + f_n(x)$ has r roots in $\mathbb{R}$, and the implicit function theorem yields the fact that , when x describes $]0,\varepsilon[$, these r roots give us r Nash functions on $]0,\varepsilon[$. This proves that K is real closed , and that it is the real closure of $\mathbb{R}(X)$ for the considered order (this ought to be already known) . ■

Remark : We can give another description of K . Let $\mathbb{R}(X)^*$ be the field of fractional power series , i.e. series in $x^{1/p}$ for some positive integer p with a finite number of terms with negative exponents . $\mathbb{R}(X)^*$ is real closed since $\mathbb{C}(X)^*$ is algebraically closed ([12] chapter IV theorem 3.1.) . It contains $\mathbb{R}(X)$, inducing on it the order determined by $X = 0_+$. So K may be identified with the field of fractional power series which are algebraic over $\mathbb{R}(X)$.

Since 0 is in the closure of $\{0_+\}$, we must have a morphism from the ring of germs of Nash functions at 0 to K . This is clear since a germ at 0 determines a germ at the right of 0 .

The ring of sections of $\mathcal{S}pec_{Ret}(\mathbb{R}[X])$ over the open $[a_+, b_-]$ is the ring of Nash functions over $]a,b[$ and in particular the ring of global sections is the ring of Nash functions on $\mathbb{R}$.

We can give a similar description of the real etale spectrum of the ring of polynomial functions on the hyperbola : Let A be $\mathbb{R}[X,Y]/_{X^2-Y^2+1}$. We may first check that the topological space $Spec_{Ret}(A)$ is just the union of two disjoint copies of $Spec_{Ret}(\mathbb{R}[X])$:



It is also true for the sheaves : $\mathcal{S}pec_{Ret}(A)$ is the union of two disjoint copies of $\mathcal{S}pec_{Ret}(\mathbb{R}\ X)$: It is sufficient to seethat the projection of the hyperbola on the x-axis along the y-axis induces isomorphisms on the stalks of the sheaves :

- Let $s = (a,b)$ be a point on the hyperbola. $A_{(X-a,Y-b)}$ is a local-etale $\mathbb{R}[X]_{(X-a)}$-algebra with the same residue field, so they have the same henselisation.
- Consider now a point like s_+ with $s = (a,b)$. The field of fractions of A with the order determined by $(X,Y) = s_+$ is an ordered algebraic extension of $\mathbb{R}[X]$ with the order determined by $X = a_+$, hence the two fields have the same real closure.

So, from the etale real point of view, a hyperbola is just the union of two disjoint lines.

Appendix 1

This appendix is devoted to the proof of the following result :

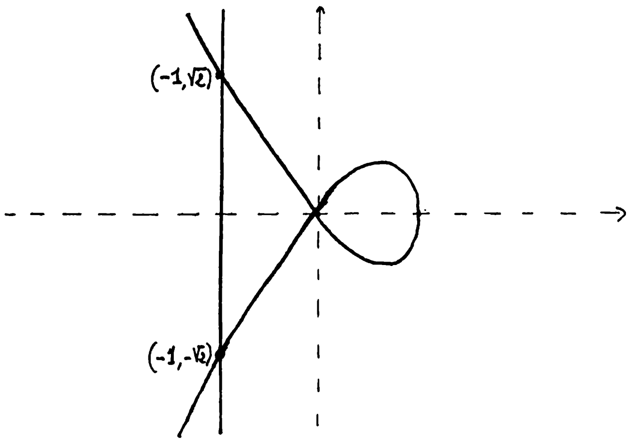
Theorem : Let $A(x_1,\dots,x_n)$ be a formula of the language of ordered fields such that the set of $(a_1,\dots,a_n)$ in $\mathbb{R}^n$ which satisfy $A(a_1,\dots,a_n)$ is open . Then there exist polynomial expressions $E_{1,1},\dots,E_{1,m_1},\dots$ $\dots,E_{p,1},\dots,E_{p,m_p}$ in $x_1,\dots,x_n$ with coefficients in $\mathbb{Z}$ such that $A(x_1,\dots,x_n)$ is equivalent to $\bigvee_{i=1}^{p}(\bigwedge_{j=1}^{m_i} E_{i,j}>0)$ in the theory of real closed fields .

Proof : The theory of real closed fields admits elimination of quantifiers , so we already know that $A(x_1,\dots,x_n)$ is equivalent to a quantifier free formula i.e. a boolean combination of strict inequalities . We have to reduce this boolean combination to a positive one , that is a disjunction of conjunctions of strict inequalities .

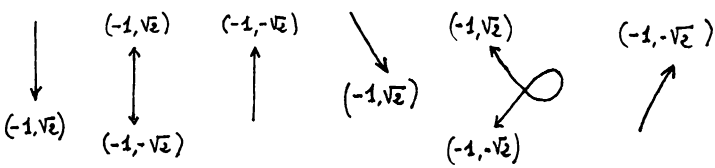
Since the theory of real closed fields is complete it is sufficient to look at interpretations of these formulas in $\mathbb{R}$. Now we have to prove essentially that a semi algebraic subset of $\mathbb{R}^n$ which is open is a finite union of subsets given by a finite number of strict inequalities .

This is a consequence of the "separation lemma" of G. Efroymson [9] . Here is a (slightly modified) statement of this result :

Let $P_1,\dots,P_m$ be polynomials in $\mathbb{R}\ X_1,\dots,X_n$. The roots of $P_1,\dots,P_m$ divide up $\mathbb{R}^n$ into a disjoint union of connected subsets where each P_i is either constantly zero or either constantly >0 or either constantly <0 (cf Theorem 2.1 in [10]) . This partition of $\mathbb{R}^n$ willbe called the partition associated to $(P_1,\dots,P_m)$. For instance the partition of $\mathbb{R}^2$ associated to $(X^3-X^2+Y^2,X+1)$



is composed of the two points $(-1,\sqrt{2})$ and $(-1,-\sqrt{2})$, the six bits of curves



and the six connected open subsets of the plane which can be seen on the picture .

Separation Lemma : Let $P_1,\dots,P_m$ be polynomials in $\mathbb{R}[X_1,\dots,X_n]$. There exist polynomials $P_{m+1},\dots,P_q$ in $\mathbb{R}[X_1,\dots,X_n]$ such that if S and T are any bits of $\mathbb{R}^n$ which belong to the partition associated to $(P_1,\dots,P_q)$ we have the following equivalence :
$S \subset adh(T)$ iff any sign condition (>0 or <0) on the P_i ($i = 1,\dots,q$) which holds on S holds on T . Moreover if $P_1,\dots,P_m$ are in $\mathbb{Z}[X_1,\dots,X_n]$ $P_{m+1},\dots,P_q$ may also be chosen in $\mathbb{Z}[X_1,\dots,X_n]$.

We give now a proof which is simplified and which yields immediately the last part of this result . This is nevertheless essentially the proof of G. Efroymson .

The proof is by induction on n :

1°) n = 1 . Let $P_{m+1},\dots,P_q$ be all the derivatives at any order of the $P_1,\dots,P_m$. Then the property is satisfied (this is known as Thom's lemma) : Let x and y be two points of $\mathbb{R}$ ($x<y$) such that every sign condition satisfied at x is satisfied at y . It is sufficient to show that P_i ($i = 1,\dots,q$) cannot have a root on $]x,y]$. Suppose P_i has a root on $]x,y]$. If it is simple and the only one on $[x,y]$ then ($P_i(x)>0$ and $P_i(y)\leqslant 0$) or ($P_i(x)<0$ and $P_i(y)\geqslant 0$)

which is a contradiction . If the root is not simple or if there are several roots on $[x,y]$ there is a root of P_i' on $]x,y]$ and we can repeat the argument : at the end we always get a contradiction .

2°) From n-1 to n .

Consider $P_1,\dots,P_m$ as polynomials in X_n with coefficients in $\mathbb{R}[X_1,\dots,X_{n-1}]$. We may always suppose that the coefficient of the term of highest degree in X_n is a constant in $\mathbb{R}$: If it is not the case we may change the x_n axis by putting

$$X_1 = X'_1 + a_1X'_n$$
$$\dots\dots\dots\dots\dots\dots$$
$$X_{n-1} = X'_{n-1} + a_{n-1}X'_n$$
$$X_n = X'_n$$

and it is always possible to find $a_1,\dots,a_{n-1}$ in $\mathbb{Q}$ such that the condition is satisfied by the polynomials in $X'_1,\dots,X'_n$.

Now let $P_{m+1},\dots,P_r$ be all the derivatives at any order of the $P_1,\dots,P_m$ withrespect to X_n . In the derivatives also the coefficient of the term of highest degree in X_n is a constant . Let $H_1,\dots,H_s$ be all the hyperresultants of all couples (P_i,P_j) $1 \leqslant i \leqslant r$ $1 \leqslant j \leqslant r$ $i \neq j$ **with** respect to the variable X_n . We must say what are these hyperresultants :

If P is a monic polynomial in one variable of degree d , the d hyperresultants of (P,Q) are the elementary symetric functions of the $Q(\alpha_1),\dots,Q(\alpha_d)$ where $\alpha_1,\dots,\alpha_d$ are the virtual roots of P . The hyperresultants are polynomials over $\mathbb{Z}$ in the coefficients of P and Q . The d^{th} hyperresultant is the ordinary resultant of P and Q and when P'=Q these hyperresultants are the hyperdiscriminants of P (cf I a) . The hyperresultants of (P,Q) have the following property (P and Q are supposed in $\mathbb{R}[X]$) : Let $i \leq d$ be the smallest integer such that for all $j > i$ the j^{th} hyperresultant is zero . Then there are d-i roots of P in $\mathbb{C}$ which are roots of Q .

$H_1,\dots,H_s$ are polynomials in $X_1,\dots,X_{n-1}$. From the induction hypothesis we get a finite set of polynomials $P_{r+1},\dots,P_q$ in $\mathbb{R}[X_1,\dots,X_{n-1}]$ containing $H_1,\dots,H_s$ and such that the partition of $\mathbb{R}^{n-1}$ associated to $P_{r+1},\dots,P_q$ has the separation property of the theorem . We are going to prove that the partition of $\mathbb{R}^n$ associated to $P_1,\dots,P_q$ has also this property .

The key to this fact is to remark that over any bit U of the partition of $\mathbb{R}^{n-1}$ associated to $P_{r+1},\dots,P_q$ the real roots of $P_1,\dots,P_r$ are given by continuous functions $\varphi : U \longrightarrow \mathbb{R}$ and that these functions do not intersect over U (use the property of hyperresultants mentioned above) .

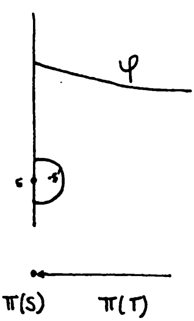
Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ be the projection along the x_n axis . Let S and T be bits of the partition of $\mathbb{R}^n$ associated to $P_1,\dots,P_q$. $\pi(S)$ and $\pi(T)$ are then bits of the partition of $\mathbb{R}^{n-1}$ associated to $P_{r+1},\dots,P_q$.

Suppose $S \not\subset adh(T)$. Several cases may occur :

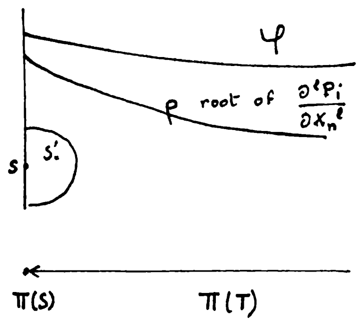
a) If $\pi(S) \not\subset adh(\pi(T))$ there is a sign condition on some P_i $r < i \leq q$ which holds on S and not on T .

b) If $\pi(S) = \pi(T)$ and $S \not\subset adh((T))$ we are reduced to the case $n = 1$ by looking at one fiber for π ; there is a sign condition on some P_i $1 \leq i \leq r$ which holds on S and not on T .

c) We are left with the case $\pi(S) \neq \pi(T)$, $\pi(S) \subset adh(\pi(T))$ and $S \not\subset adh(T)$. In $\pi^{-1}(\pi(T))$ T is bounded (if it is) by two roots φ and ψ ($\varphi < \psi$) - or it is a root φ - . We may choose s in S , $s \notin adh(T)$ t in T and a path Γ in $\mathbb{R}^{n-1}$ going from $\pi(t)$ to $\pi(s)$ inside $\pi(T)$ excepted at $\pi(s)$. When we go from $\pi(t)$ to $\pi(s)$ along Γ , φ and ψ have limits $\varphi(\pi(s))$ and $\psi(\pi(s))$. Since $s \notin adh(T)$ s_n (the n^{th} coordinate of s) must be outside $[\varphi(\pi(s)),\psi(\pi(s))]$, say $s_n < \varphi(\pi(s))$; we are led to a similar inequality when T is reduced to a root φ . Let P_i be a polynomial of minimal degree among those of $P_1,\ldots,P_r$ which have φ as root (with respect to X_n) . We want to show that either $P_i(s) \neq 0$ and for all s' sufficiently near to s over $\pi(T)$ there is no root of $P_i(s'_1,\ldots,s'_{n-1},X_n)$ in $[s'_n,\varphi(\pi(s'))[$



or there is some derivative $\frac{\partial^l P_i}{\partial x_n^l}$ which is not zero at s and such that for every s' sufficiently near to s over $\pi(T)$ there is a simple root of $\frac{\partial^l P_i}{\partial x_n^l}$ in $]s'_n, \varphi(\pi(s'))[$ and no other root on $[s'_n, \varphi(\pi(s'))]$.



In both cases there will surely be a sign condition satisfied at s and not on T . Now the proof of the result mentioned above is an easy consequence of the following fact : If ρ is a root of P_j over $\pi(T)$ and $x \in \pi^{-1}(\pi(s))$ is also a root of P_j with $x_n < \rho(\pi(s))$ there is a root of $\frac{\partial P_j}{\partial x_n}$ over $\pi(T)$, σ , such that $\sigma < \rho$ and $x_n < \sigma(\pi(s))$. This is so because the maximum of $|P_j(s_1, \ldots, s_{n-1}, x_n)|$ on $[x_n, \rho(\pi(s))]$ gives a root of $\frac{\partial P_j}{\partial x_n}$ where the sign of $\frac{\partial P_j}{\partial x_n}$ changes and so this root must be the limit of a root σ of $\frac{\partial P_j}{\partial x_n}$ over $\pi(T)$. ∎

We can now go back to the proof of our theorem . Let $U \subset \mathbb{R}^n$ he the set of $(a_1,\ldots,a_n)$ satisfying a quantifier free formula which is a boolean combination of things like $P = 0$ or $P > 0$ for P belonging to a finite set $P_1,\ldots,P_m$ of polynomials of $\mathbb{Z}[X_1,\ldots,X_n]$. The separation lemma gives $P_{m+1},\ldots,P_q \in \mathbb{Z}[X_1,\ldots,X_n]$ such that the partition of $\mathbb{R}^n$ associated to $P_1,\ldots,P_q$ has the separation property . U is a finite union of bits of this partition : $U = \bigcup_{i=1}^{p} S_i$.

Suppose now U is an open set . Let C_i be the conjunction of sign conditions which hold on S_i . The set of $(a_1,\ldots,a_n)$ satisfying $\bigvee_{i=1}^{p} C_i$ is U for if a bit T of the partition is contained in this set there is an i such that $S_i \subset \mathrm{adh}(T)$ which implies $T \subset U$ since U is open .

This ends the proof of the theorem !

Appendix 2

Theorem : A coherent topos is spatial iff the category of its points is equivalent to an ordered set .

The main idea of the proof and the following proposition 1 were indicated by André Joyal .

Let E be a coherent topos . E is the classifying topos for some finitary geometric (or coherent) theory T . Let R_T be the logical category associated to T (see [B]) . There is an equivalence between the category of models of T and the category of logical functors from R_T to the category of sets . Let S(1) be the lattice of subobjects of the final object in R_T . S(1) as a category is a logical category and the inclusion $I : S(1) \longrightarrow R_T$ is a logical functor . By the construction of R_T , S(1) is equivalent to the lattice of closed finitary geometric (or coherent , or existential positive) formulas of T with the preorder $A \leq B$ when $A \rightarrow B$ is a theorem of T . In the following we shall identify these two lattices .

A logical functor from S(1) to the category of sets factors necessarily through the lattice $\{0,1\}$ and so may be identified with a prime filter on S(1) i.e. a subset F of S(1) satisfying : $0 \notin F \quad 1 \in F$

$A \vee B \in F \longleftrightarrow A \in F \vee B \in F \quad , \quad A \wedge B \in F \longleftrightarrow A \in F \wedge B \in F \quad .$

The prime filter is the set of elements of $S(1)$ which are sent on 1 by the logical functor .

The inclusion $I : S(1) \longrightarrow R_T$ induces then a functor I^* from the category of models of T to the lattice of prime filters on $S(1)$. If M is a model of T , $I^*(M)$ is simply the prime filter of finitary geometric formulas satisfied by M .

Proposition 1 : i) For every prime filter F on $S(1)$ there exists a model M of T such that $I^*(M) = F$.

ii) For every model M of T and every prime filter F on $S(1)$ containing $I^*(M)$ there exist a model N of T and a homomorphism from M to N such that $I^*(N) = F$.

iii) If M and M' are two models of T such that $I^*(M) = I^*(M')$ there exist a third model N of T and homomorphisms $M \longrightarrow N$ and $M' \longrightarrow N$ such that $I^*(N) = I^*(M)$.

Proof : Let $\overline{F}$ be the set of negations of closed finitary geometric formulas which are not in F . We must show that $T + F + \overline{F}$ has a model . If not there are A in F and B with B in $\overline{F}$ such that $A \longrightarrow B$ is a theorem of T , which is absurd .

ii) Let $D(M)$ be the set of closed atomic formulas with parameters in M which are satisfied by M . We have to show that $T + F + \overline{F} + D(M)$ has a model : the reasoning is the same that for i .

iii) We have to show here that $T + F + \overline{F} + D(M) + D(M')$ has a model and we use the same reasoning . ∎

Proposition 2 : Suppose that there is at most one homomorphism between any two models of T . Let $f : M \longrightarrow N$ be a homomorphism with $I^*(M) = I^*(N)$. Then f is an isomorphism .

Proof : Let be any element of a model M of T . Since there is at most one homomorphism between two models of T , $T + D(M)_1 + D(M)_2$ implies $a_1 = a_2$ ($D(M)_1$ and $D(M)_2$ are two distinct copies of $D(M)$ obtained by associating to each element b of M two distinct constants b_1 and b_2) . We can then find a finitary geometric formula $A(x)$ which is satisfied by a and such that $A(x) \wedge A(x') \longrightarrow x = x'$ is a theorem of T (we say that A is univalent .) .

Consider now $f : M \longrightarrow N$ with $I^*(M) = I^*(N)$. If r is a relational symbol of T and if in N we have $r(f(a_1),\dots,f(a_n))$, then in M we have $r(a_1,\dots,a_n)$: We know that there are finitary geometric univalent formulas $A_1,\dots,A_n$ satisfied respectively by $a_1,\dots,a_n$ in M . N satisfies the formula

$$\exists x_1 \ldots \exists x_n \left[r(x_1,\ldots,x_n) \wedge A_1(x_1) \wedge \ldots \wedge A_n(x_n) \right] .$$

This is a finitary geometric formula so it is also satisfied by M , which implies that we have $r(a_1,\ldots,a_n)$ in M .

It remains to show that f is surjective . Let b be an element of N ; it satisfies a finitary geometric univalent formula B . In M the formula $\exists x B(x)$ is also true . Let a be the element of M such that B(a) . Then necessarily f(a) = b . ∎

Proposition 3 : Suppose that there is at most one homomorphism between any two models of T . Then I^* is an equivalence of categories .

Proof : I^* is surjective on objects by proposition 1i . It remains to show that it is full (it is necessarily faithful) . Suppose that we have $F \subset G$ and M and N two models such that $I^*(M) = F$ and $I^*(N) = G$. By proposition 1ii there exist a model N' such that $I^*(N') = G$ and a homomorphism from M to N' . By proposition 1iii there exist a model N" such that $I^*(N") = G$ and homomorphisms from N to N" and from N' to N" . By proposition 2 these homomorphisms are isomorphisms and so we get a homomorphism from M to N . ∎

Here is now the end of the proof of the theorem : By Makkai and Reyes' conceptual completeness theorem (Theroem 7.1.8. p.204 in [8]) since I^* is an equivalence the classifying topos for T , i.e. E , is equivalent to the category of sheaves over S(1) for the topology generated by the following covering families :

A is covered by the $(B_i)_{i \in I}$ (I finite and $B_i \leq B$) when $A = \bigvee_{i \in I} B_i$. This topos is equivalent to the topos of sheaves over the following topological space :

- The points are the prime filters on S(1) i.e. in the case we consider isomorphism classes of models of T .

- A basis of open sets is given by the sets $\hat{A} = \{ F \mid A \in F \}$ (or $\{ M \mid M \text{ satisfies } A \}$) for A element of S(1) (i.e. closed finitary geometric formula of T) .

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Subtoposes of the ring classifier

*E.J. Dubuc and G.E. Reyes**

This paper is a corrected, simplified and extended version of a preprint with the same title written by the second author. It is divided in four sections. In §1 we give a (new) proof of the following theorem of M. Coste, M.F. Coste and A. Kock [1]: the generic model of an ε-stable geometric theory of rings is of line type and $(-)^D$ commutes with colimits. In §2 we show that the Weil topos Sets^W (described in [2]) is the classifying topos of the theory consisting of (all $L_{\infty\omega}$) geometric sequents true in all Weil algebras, and that this theory has a complete axiomatization given by the following two axioms:

$$0 = 1 \Rightarrow \downarrow$$

$$\uparrow \;\Rightarrow \bigvee_{\substack{r\in\mathbb{R} \\ n\in\mathbb{N}}} (x-r)^n = 0$$

We shall say that this theory is the $L_{\infty\omega}$ geometric theory of Weil algebras. In §3 we show that the theory of Archemedian real closed local $\mathbb{R}$-algebras such that every element is either a unit or nilpotent is the $L_{\omega\omega}$ geometric theory of Weil algebras. We don't know yet the $L_{\omega\omega}$ theory of Weil algebras. We suspect it to be simply the theory of real closed local $\mathbb{R}$-algebras. In §4 we extend the results of §1 to algebraic theories in general.

We are grateful to M. Makkai who discovered a mistake in the preprint refered to above (cf. §2). We also acknowledge conversations with A. Joyal, A. Kock and R. Paré.

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§1 Generic rings of line type and ε-stability

We shall be concerned with subtoposes of the ring classifier and, more generally, with subtoposes of the k-algebra classifier (for k a commutative ring with unit). Throught this paper, given any k-algebra A, we shall superline-it, writting $\bar{A}$, to indicate the corresponding object in the dual category.

Let $k \to I$ be a k-algebra of finite (linear) dimension n over k. For any k-algebra $k \to A$, we let $A[I] = A \otimes_k I$. Then $A \to A[I]$ is an algebraic functor (of degree n) (cf. §4) and it thus has a left adjoint $A \to A^*$ such that $k[X_1 \ldots X_s] \to k[X_1 \ldots X_{ns}]$ (cf. [5]). It follows then that if A is finitely presented, so is A*. Since clearly A[I] is also finitely presented, we have:

Proposition

The object $\bar{I}$ is exponentiable in the dual C of the category of finitely presented k-algebras. Furthermore, $p = (-)^{\bar{I}}$ has a left adjoint q given by $q(\bar{A}) = \overline{A[I]}$.

Remark

A direct construction of this exponential is possible. For example, if $A = k[(X_\alpha)_\alpha]$ divided by $(F_\beta)_\beta$ and $I = k[\varepsilon]$ (the "dual numbers"), then $\bar{A}^{\bar{I}} = \overline{A^*}$, and $A^* = k[(X_\alpha)_\alpha (Y_\alpha)_\alpha]$ divided by $(F_\beta, \sum_\alpha Y_\alpha \frac{\partial F_\beta}{\partial X_\alpha})_\beta$.

By definition, $[\bar{A}, p(\overline{k[t]})] = [\overline{A[I]}, \overline{k[t]}] = [k[t], A[I]] = A[I]$. That is, if $\mathcal{U} = [k[t],-]$ is the fortgetful functor $C^{op} \xrightarrow{\mathcal{U}}$ Sets, and if we let $\mathcal{U}[I](A) = A[I]$, we have $\mathcal{U}[I] = [p(k[t]),-]$.

Assume now that C is a site on the dual of the category of finitely presented k-algebras. Then, following [1] we say that C is <u>I-stable</u> if p is continuous. That is, there is a geometric endomorphism (p^*, p_*), $p^* \dashv p_*$ = composition with p, and a commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{p} & C \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon'} \\ \tilde{C} & \xrightarrow{p^*} & \tilde{C} \end{array} \qquad \begin{array}{l} \varepsilon p\, \overline{k[t]} = p^*\, \varepsilon\, \overline{k[t]} \\ \\ g[I] = p^* g \end{array}$$

Where $g = \# U = \varepsilon\, \overline{k[t]}$ is the generic model, $g[I] =_{def} \varepsilon p\, \overline{k[t]} = \#(U[I])$, and $\#$ indicates the associate sheaf functor.

<u>Theorem</u>

Let C be a site in the dual of the category of finitely presented k-algebras. Then C is I-stable if and only if the exponentiation $(-)^{\varepsilon\bar{I}}$ is the inverse image of an (essential) geometric endomorphism of $\tilde{C}$ such that $g^{\varepsilon\bar{I}} = g[I]$.

<u>Proof</u>

First notice that q is always continuous, and that given any sheaf F in $\tilde{C}$ and k-algebra A in C, $F^{\varepsilon\bar{I}}(\bar{A}) = F^{\bar{I}}(\bar{A}) = F(\bar{A}\times\bar{I}) = F(qA)$. That is, $(-)^{\varepsilon\bar{I}} = q_*$ (q_* defined by composition with q).

Then, if C is I-stable, that is, if p is continuous, since $q_* \dashv p_*$, it follows that $p^* = q_* = (-)^{\varepsilon\bar{I}}$. On the other hand, if $(-)^{\varepsilon\bar{I}}$ is the inverse image of a geometric endomorphism $(-)^{\varepsilon\bar{I}} = r^*$ such that $g[I] = r^* g$, we have $\varepsilon p\, \overline{k[t]} = r^* \varepsilon\, \overline{k[t]}$, which implies $\varepsilon p = r^* \varepsilon$. Thus εp is continuous and thus p is (continuous) since ε reflects coverings.

Let $\mathbb{T}$ be a geometric theory of k-algebras, We say that $\mathbb{T}$ is w-stable if the site associated to $\mathbb{T}$ (on the dual of the category of finitely presented k-algebras) is I-stable, for every Weil k-algebra I (cf. §3). In other words, if $g[I] = \varepsilon p\ \overline{k[t]}$ is a model (in the classifying topos) of $\mathbb{T}$, for every Weil k-algebra I.

Corollary

Let $\mathbb{T}$ be a geometric theory of k-algebras. Then, $\mathbb{T}$ is w-stable if and only if for any Weil k-algebra I, $(-)^{\varepsilon \bar{I}}$ is the (essential) geometric endomorphism of the classifying topos $E(\mathbb{T}) = \tilde{C}$ of $\mathbb{T}$ which classifies $g[I]$. In particular, if $\mathbb{T}$ is w-stable, g is of line type and 1-small objects (in the sense of [4]) are internally projective.

Remark

Given a point s of $\tilde{C}$, let s[I] be the point defined by s[I]*h = s*hp, where h is the Yoneda embbeding. If $\tilde{C}$ has enough points, then $\tilde{C}$ is I-stable if and only if s[I] is a point of $\tilde{C}$ whenever s is a point of $\tilde{C}$. This is clear since the family of points of $\tilde{C}$, by assumption, reflects coverings (use definition of s[I] above). This notion had been used in the preprint mentioned in the introduction. A. Joyal suggested to work with the continuity of p directly.

If T has enough models (i.e., $E(T) = \tilde{C}$ has enough points), then $\mathbb{T}$ is I-stable if and only if $A[I] = A \otimes_k I$ is a model of T whenever A is a model of $\mathbb{T}$. This was the original definition of A. Kock [3]. Theories with enough models are, for example, those which are either coherent (i.e. finitary) or $L_{\omega_1 \omega}$ geometric with countably many axioms (cf. [6] Chapter 6). A different type are those who consist of (all) geometric sequents true in a given class of k-algebras.

§2 What does the Weil topos classify in the language of ℝ-algebras?

Following [2], we call Weil Topos the category $\mathrm{Sets}^{\mathcal{W}}$ of set valued functors on the category $\mathcal{W}$ of Weil algebras. Recall that a Weil algebra is a finite dimensional ℝ-algebra of the form $\mathbb{R} \oplus M$ such that every element of M is nilpotent. Finite colimits of Weil algebra are Weil algebras, as well as any quotient ($\neq 0$) and any sub-algebra. It follows then that $\mathcal{W}$ has also finite limits (since given $X = \mathbb{R} \oplus M$ and $Y = \mathbb{R} \oplus N$, then $X \otimes_{\mathbb{R}} Y = \mathbb{R} \oplus M \oplus N$ is a product in $\mathcal{W}$). The real numbers ℝ is a terminal and initial object for $\mathcal{W}$. We shall need the following lemma of general nature:

Lemma 1

Given any diagram $\mathbb{\Gamma} \xrightarrow{X} E$ in a topos E and a cone $X_\alpha \longrightarrow U$ in E, if:

a) $X_\alpha \to U$ is an epimorphic family

b) For every non empty fiber product

$X_\alpha \leftarrow X_\alpha \times_U X_\beta \to X_\beta$ there exist $\alpha \leftarrow \gamma \to \beta$ in $\mathbb{\Gamma}$ such that $X_\gamma = X_\alpha \times_U X_\beta$ (together with the projections).

Then, $X_\alpha \to U$ is a colimiting cone for X.

Proof

Check that any cone $X_\alpha \to F$ is a compatible family (with respect to $X_\alpha \times_U X_\beta$). Then, the claim follows since in a topos all epimorphic families are effective.

__Theorem__

Let C be a site on the dual of the category of finitely presented $\mathbb{R}$-algebras. Consider the following 3 conditions:

0) For any Weil algebra $X \in \mathcal{W}$, the representable functor $C \xrightarrow{[-,X]} \text{Sets}$ is continuous

1) The empty family co-covers the null ring in C

2) The family $(\mathbb{R}[t] \xrightarrow{\alpha} X)_{\alpha \in X \in \mathcal{W}}$ co-covers in C.

Then, 0), 1) and 2) hold in C if and only if $\tilde{C}$ is equivalent to the Weil Topos in such a way that to the generic model $g = \varepsilon\, \mathbb{R}[t] \in \tilde{C}$ corresponds the forgetful functor $U \in \text{Sets}^{\mathcal{W}}$.

__Proof__

Assume 0), 1) and 2). By 0) we obtain a continuous functor $C \xrightarrow{q} \text{Sets}^{\mathcal{W}}$ which induces a geometric morphism (q^*, q_*) making the following diagram commutative:

$$\begin{array}{ccc} \text{Sets}^{\mathcal{W}} & \xleftarrow{q^*} & \tilde{C} \\ & \nwarrow^{q} & \uparrow \varepsilon \\ & & C \end{array} \qquad \begin{array}{l} q\,\overline{\mathbb{R}[t]} = U \\ q^* g = U \end{array}$$

On the other hand, the inclusion $\mathcal{W}^{op} \xhookrightarrow{i} C$ is (vacuosly) continuous and hence induces a geometric morphism (i^*, i_*) making the following diagram commutative:

$$\begin{array}{ccc} \text{Sets}^{\mathcal{W}} & \xrightarrow{i^*} & \tilde{C} \\ \uparrow h & & \uparrow \varepsilon \\ \mathcal{W}^{op} & \xrightarrow{i} & C \end{array} \qquad i^* h \bar{X} = \varepsilon \bar{X},\ X \in \mathcal{W}.$$

To prove the theorem we have to check that there is an isomorphism of $\mathbb{R}$-algebra objects $i^*\mathcal{U} \approx \mathcal{G}$ (that is, $i^*q\ \overline{\mathbb{R}[t]} \approx \varepsilon\ \overline{\mathbb{R}[t]}$ for the co-$\mathbb{R}$-algebra object $\mathbb{R}[t] \in C$). Let $(\mathbb{R}[t] \xrightarrow{\alpha} X)_\alpha = (h\bar{X} \xrightarrow{\theta} \mathcal{U})_\theta$, $\alpha(t) = \theta(id_X) \in X \in \mathcal{W}$, be the canonical diagram of $\mathcal{U}$. The family $i^*hX \xrightarrow{i^*(\theta)} i^*\mathcal{U}$ is a colimiting cone, and there is a cone $\varepsilon\bar{X} \xrightarrow{\varepsilon(\alpha)} \mathcal{G}$. This induces a morphism of $\mathbb{R}$-algebra objects $i^*\mathcal{U} \xrightarrow{\phi} \mathcal{G}$ such that $\phi i^*(\theta) = \varepsilon(\alpha)$. On the other hand, it follows immediatly from condition 1) and 2) that the cone $\varepsilon\bar{X} \xrightarrow{\varepsilon(\alpha)} \mathcal{G}$ satisfies the hypothesis of Lemma 1. This shows that ϕ is an isomorphism. The converse is immediate. Suppose the equivalence to be given by a pair of functors q^* and i^* as before. Since $[-,X] = ev_X q = ev_X q^*\varepsilon$ (where ev_X is the evaluation in X functor of the Weil topos), condition 0) holds. Conditions 1) and 2) follow since there are enough points of the form $[-,X] = ev_X q^*\varepsilon$, with $X \in \mathcal{W}$.

Corollary

The forgetful functor $\mathcal{U} \in Sets^{\mathcal{W}}$ is the generic model of a (geometric) theory $\mathbb{T}$ in the language of $\mathbb{R}$-algebras if and only if $\mathbb{T}$ has a complete axiomatization given by the following two axioms (in $L_{\infty\omega}$):

1) $0 = 1 \Rightarrow \downarrow$

2) $\uparrow \Rightarrow \bigvee_{\substack{r\in\mathbb{R}\\ n\in\mathbb{N}}} (x-r)^n = 0$

Proof

Immediate if theories are viewed as sites (see e.g. [6]). All there is to verify is that $\mathbb{R}[X]/_{(x-r)^n}$ is a Weil algebra.

All such theories are thus the same, that we will denote by $\mathbb{T}_W$. Remark that $\mathbb{T}_W$ is the theory of (all) the geometric sequents true in all Weil algebras. We shall call $\mathbb{T}_W$ the <u>geometric theory of Weil algebras</u>.

<u>Proposition</u>

Condition 2) in the theorem is equivalent to the following condition:

3) i) $\mathcal{C}$ has enough points

ii) Every (left exact) continuous functor $\mathcal{C} \xrightarrow{p}$ Sets (i.e., every point) is a filtered colimit of representable $[-,X]$, with $X \in \mathcal{W}$.

<u>Proof</u>

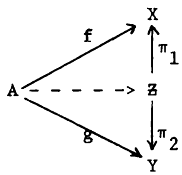
Consider, for each (fixed) $A \in \mathcal{C}$ the following families:

(1) $p(X) \xrightarrow{p(\alpha)} p(A)$, all $A \xrightarrow{\alpha} X$, $X \in \mathcal{W}$.

(2) $[A,X] \xrightarrow{\theta A} p(A)$, all $[-,X] \xrightarrow{\theta} p$, $X \in \mathcal{W}$.

It is an immediate consequence of Yoneda's lemma that (1) is a surjective family if and only if (2) is a surjective family. Condition 3) ii) means that for every point p, the family (2) is surjective. It follows then, by 3) ii) that the family $A \xrightarrow{\alpha} X$, $X \in \mathcal{W}$, co-covers A in $\mathcal{C}$. Taking $A = \mathbb{R}[t]$ this shows that 3) implies 2). Assume 2). Then, from the theorem it follows that $\mathcal{C}$ has enough points of the form $\mathcal{C} \xrightarrow{[-,X]}$ Sets with $X \in \mathcal{W}$. Since for each $A \in \mathcal{C}$ each one of these points is continuous with respect to the family $A \xrightarrow{\alpha} X$, $X \in \mathcal{W}$, it follows that $A \xrightarrow{\alpha} X$, $X \in \mathcal{W}$ co-covers A in $\mathcal{C}$. Thus for any p as in 3) ii), the family $[-,X] \xrightarrow{\theta} p$, $X \in \mathcal{W}$, is an epimorphic family in $\text{Sets}^{\mathcal{C}}$. Given $[-,X] \xrightarrow{\theta} p$ and $[-,Y] \xrightarrow{\eta} p$, the fiber product

$[-,X] \times_p [-,Y]$ is represented by $[-,\mathfrak{Z}]$ where $\mathfrak{Z} \subset X \times Y$ is defined by:
$\mathfrak{Z} = \{(x,y) \mid \theta\, \mathbb{R}[t](x) = \eta\, \mathbb{R}[t](y)\}$. It is not difficult to check that $\mathfrak{Z} \in \mathcal{W}$ and that it satisfies the required universal property:

$\theta\mathfrak{Z}(\pi_1) = \eta\mathfrak{Z}(\pi_2)$ and 

$\forall f,g$ such that $\theta A(F) = \eta A(g)$

Thus the hypothesis in lemma 1 are satisfied. This concludes then the proof.

We remark that in condition 3) ii) we can assume the $[-,X] \xrightarrow{\theta} p$ to be sub-functors. The image of θ is represented by $[-,\mathfrak{Z}]$, where $\mathfrak{Z}$ is X divided the congruence $x \sim 0 \iff \theta\, \mathbb{R}[t](x) = 0$. Since by condition 1) $0 \neq 1$ in $p(\mathbb{R}[t])$, 1 is not congruent to 0, and thus $0 \neq \mathfrak{Z} \in \mathcal{W}$. In fact, the converse is also true. If $[-,X] \rightarrowtail p$, $X \in \mathcal{W}$ is a colimit diagram, then $p(\mathbb{0}) = \emptyset$ and since there are enough p's (by 3) i)), the empty family covers $\mathbb{0}$.

<u>Corollary</u>

The fortgetful functor $\mathcal{U} \in \mathrm{Sets}^{\mathcal{W}}$ is the generic model of a (geometric) theory $\mathbb{T}$ in the language of $\mathbb{R}$-algebras if and only if $\mathbb{T}$ satisfies the following conditions:

o) All Weil algebras are models of $\mathbb{T}$

3) T has enough models and every model of $\mathbb{T}$ is a filtered colimit of its Weil sub-algebras.

It follows then that $\mathbb{T} = \mathbb{T}_W$.

Proof

Immediate if theories are viewed as sites (see e.g. [6]).

Remark

M. Makkai has noticed that no $L_{\omega_1\omega}$ geometric theory in the language of $\mathbb{R}$-algebras can have $Sets^{\mathcal{W}}$ as its classifying topos. (Otherwise $(\mathbb{R}[t] \xrightarrow{\alpha} X)_{\alpha\in X\in\mathcal{W}}$ would have a countable sub-cover).

We shall exhibit our $L_{\omega_1\omega}$ geometric theory $\mathbb{T}_{\mathbb{R}}$ satisfying o) and such that every model is a filtered colimit of its Weil sub-algebras. It follows then that $\mathbb{T}_{\mathbb{R}}$ does not have enough models, and that the models of $\mathbb{T}_{\mathbb{R}}$ are exactly the same that the models of $\mathbb{T}_W$.

Let $\mathbb{T}$ be the $L_{\omega_1\omega}$ geometric theory whose generic model R satisfies the following axioms

1) R is a local ring

2) R is real closed or, equivalently, R[i] is separably closed (cf. [9] for an explicit coherent axiomatization of this notion).

3) R is Archemedian, that is, $\forall x \; \forall y \; (x > 0 \rightarrow \bigvee_{n>0} nx > y)$, where $x > y$ stands for $\exists z \; (z \text{ invertible} \wedge x-y = z^2)$.

4) Every element of R is either invertible or nilpotent, that is, $\forall x \; (x \text{ invertible} \vee \bigvee_{n>0} x^n = 0)$.

We define $\mathbb{T}_k = \mathbb{T} \cup \Delta(k)$ = the theory of k-algebras satisfying 1)-4) in the language of rings with one constant for each element of **k**.

Then clearly every Weil algebra is a model of $\mathbb{T}_{\mathbb{R}}$, and furthermore, every model of $\mathbb{T}_{\mathbb{R}}$ is a filtered colimit of its Weil sub-algebras. For this last claim, see Lemma in §3.

§3 The $L_{\omega_1\omega}$ geometric theory of Weil algebras

In this section, we prove that the theory $\mathbb{T}_{\mathbb{R}}$ introduced in §2 is a complete axiomatization of the $L_{\omega_1\omega}$ geometric sequents true in all Weil $\mathbb{R}$-algebras.

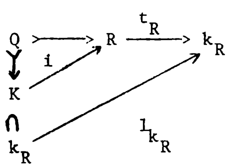
Before turning to the proof, we shall prove a simple (and well known) Cohen's type theorem on the existence of coefficient fields (cf. [9, Cor 2, page 280] for Henselian local rings of characteristic 0 (i.e. $\mathbb{Q}$-algebras).

Proposition

Let R be a Henselian local ring of characteristic 0. Thus the canonical map $R \xrightarrow{t_R} k_R$ has a section, i.e., there is a ring monomorphism $i\colon\ k_R \to R$ such taht $t_R \circ i = 1_{k_R}$. In particular $R \simeq k_R \oplus m_R$.

Proof

By Zorn's lemma, we can find a subfield $K \subset k_R$ with a map $i\colon\ K \to R$ such that the following diagram commutes



and which is maximal in the sense that i admits no proper extensions to $K' \supset K (K' \subset k_R)$. We claim that $K = k_R$. If not, there is $\alpha \in k_R \setminus K$. If α is transcendental over K, we extend i to $K(\alpha) \subset k_R$ by sending α into any $\beta \in t_R^{-1}(\alpha)$. This is possible, since β is a unit. If α is algebraic over K, let $p(t) \in K[t]$ be the irreducible (monic) polynomial of α over K. Since

$ch(K) = 0$, $p'(\alpha) \neq 0$ and hence α can be lifted to a root β of $p(t)$ in R (which is a K-algebra). We extend i to $K(\alpha)$ by sending α into β. In any case, we have contradicted the maximality of (K,i) and this shows that $K = k_R$.

We have proved that the exact sequence $0 \to m_R \rightarrowtail R \twoheadrightarrow k_R \to 0$ splits and this implies that $R \simeq k_R \oplus m_R$.

Theorem

If $E[\mathbb{T}_{\mathbb{R}}] \not\models \sigma$, where σ is a sequent $\phi(\bar{x}) \Rightarrow \Psi(\bar{x})$ with $L_{\omega_1\omega}$ geometric formulas ϕ,Ψ of the language of $\mathbb{T}_{\mathbb{R}}$, then there is a Weil $\mathbb{R}$-algebra X such that σ is false in X.

Proof

By using the formal system introduced in [6, Chapter 6], we reformulate the hypothesis as follows: $\mathbb{T}_{\mathbb{R}} \nvdash \sigma$.

We may clearly assume that ϕ is coherent and $\Psi = \bigvee \{\Psi_n : n > 0\}$ with Ψ_n coherent (by noticing that a $L_{\omega_1\omega}$ geometric formula is equivalent to a countable disjunction of coherent ones).

Let $k \subset \mathbb{R}$ be any countable ring with 1 such that all the interpretations of the countable many constants of ϕ, $\{\Psi_n : n > 0\}$ belong to k.

A fortiori, $\mathbb{T}_k \nvdash \sigma$ and by the completeness theorem for countable $L_{\omega_1\omega}$ geometric theories of [6, Chapter 6], there is a model R of $\mathbb{T}_k$ such that σ is false in R.

For a commutative ring k with 1, we define a Weil k-algebra be a finite dimensional k-algebra of the form $k \oplus m$ such that every element of m is nilpotent.

Lemma

Any model R of the theory $\mathbb{T}$ (defined in §2) is the filtered $\varinjlim$ of its Weil k_R-algebras.

Proof

Since ch(R) = 0, $R \simeq k_R \oplus m_R$ by the Proposition. For each $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle \in m_R^n$ we let

$$k_R \langle \varepsilon_1, \ldots, \varepsilon_n \rangle = \bigoplus_{\alpha_1, \ldots, \alpha_n \in \mathbb{N}} \varepsilon_1^{\alpha_1} \ldots \varepsilon_n^{\alpha_n} i(k_R) \subseteq R$$

(where i is the section of t_R given by the Proposition). Since each ε_i is nilpotent, this + has only a finite number of terms and so $k_R \langle \varepsilon_1 \ldots \varepsilon_n \rangle$ is a Weil k_R-algebra. Obviously $R = \varinjlim_{\langle \varepsilon_1 \ldots \varepsilon_n \rangle \in m_R^n} k_R \langle \varepsilon_1 \ldots \varepsilon_n \rangle$ and this system is filtered.

Coming back to the proof of our theorem, we notice that were σ true in each Weil k_R-such algebra of R, σ would be true in their filtered $\varinjlim$, i.e., in R. Therefore, there is a Weil k_R-algebra X_o and a sequent $\bar{a}$ of elements of X_o such that $X_o \models \phi[\bar{a}]$ and $X_o \not\models \Psi_n[\bar{a}]$ for all n > 0.

To continue the proof, we need the following straightforward reduction of truth in a finite dimensional k-algebra (e.g. a Weil k-algebra) to truth in k.

<u>Lemma</u>

(cf. [3] for a particular, but representative case).

Assume that $k \to I$ is a k-algebra of dimension n. For any finitary sentence σ of the language L_I of the theory of I-algebras, there is another finitary sentence σ_I of the language L_k as the theory of k-algebras such that for every k-algebra $k \to K$.

$$I \underset{k}{\otimes} K \models \sigma \text{ iff } K \models \sigma_I$$

<u>Proof</u>

By hypothesis on I we have a commutative diagram of k-modules

(*) $$\begin{array}{ccc} & & I \\ & \nearrow & \wr\| \\ k & & \\ & \searrow & k^n \end{array}$$

and we can describe the multiplication table of I by means of the basis $e_1,\ldots,e_n$ obtained (via the isomorphism) from the canonical basis of k^n:

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ijk}\, e_k \quad \text{with} \quad \gamma_{ijk} \text{ in } k.$$

For each term t of L_I, we define (by recursion) a sequence $(t_i)_{i\leq n}$ of terms of L_k as follows:

1) if $t = a \in I$, $\quad t_i = a_i$ in L_k (where $a = \sum_{i=1}^{n} a_i e_i$)

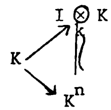
2) if $t = x$, $\quad t_i = x_i$

3) if $t = t' + t''$, $\quad t_i = t'_i + t''_i$

4) if $t = t' \cdot t''$, $\quad t_i = \sum_{jk=1} \gamma_{jki}\, t'_j \cdot t'_k$

We now define $\phi_I(x_1,\ldots,x_n,y_1,\ldots,y_n,\ldots,z_1,\ldots,z_n)$ for each <u>formula</u> $\phi(x,y,\ldots,z)$ of L_I by recursion in the obvious way, e.g., if $\phi \equiv t' = t''$, then $\phi_I \equiv \bigwedge_{i=1}^{n} t_i' = t_i''$; if $\phi \equiv \exists\, x\, \theta$, then $\phi_I \equiv \exists x_o \ldots \exists x_n \theta_I$, etc.

Via the base extension $k \to K$, (*) is transformed into the new diagram



and we prove, by induction on ϕ,

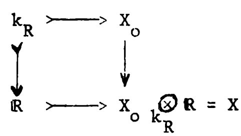
$$K^n \models \phi[(a_1,\ldots,a_n),(b_1,\ldots,b_n),\ldots,(c_1,\ldots,c_n)] \text{ iff}$$

$$K \models \phi_I[a_1,\ldots,a_n,b_1,\ldots,b_n,\ldots,c_1,\ldots,c_n].$$

(K^n is given the obvious K-algebra structure via the isomorphism).

For ϕ atomic this is essentially the statement that the multiplication table of $I \otimes_k K$ is the same as that of I (in terms of the corresponding basis).

To finish the proof of our theorem, we consider the following diagram (noticing that k_R may be embedded in $\mathbb{R}$, since it is Archemedian)



By Tarski's theorem on the elimination of quantifiers in the theory of real closed fields or the fact that this theory is model complete (cf. [7]), $k_R \rightarrowtail \mathbb{R}$ is an elementary extension. The previous lemma allows us to conclude that $X_o \to X$ is again an elementary extension (of Weil algebras) and, in particular,

$$X \models \phi[\bar{a}] \text{ and } X \not\models \Psi_n[\bar{a}], \text{ for all } n > 0.$$

§4 The (general) notion of stability

The notion of ε-stability depends ultimately on a purely algebraic construction, that is, the multiplication table of the ring of dual numbers. In this section we clarify the mechanism that stablish this dependence, and we thus extend the results of section §1 from the theory of ℝ-algebras to algebraic theories in general.

Let k be any commutative ring with unit and let $\mathbb{T} = \{\bar{A}_o, \bar{A}_1, \ldots, \bar{A}_n, \ldots\}$, $A_n = k[X_1, \ldots, X_n]$ be the algebraic theory of k-algebras. A finite n dimensional k-algebra $\mathbb{T} \xrightarrow{I} \text{Sets}$ determines, via its multiplication table, a generic I. That is, a R-algebra object p in $\mathbb{T}$. Thus, p is a co-k-algebra, structure in the polynomial k-algebra $A_n = k[X_o X_1 \ldots X_n]$, or, equivalently, a product preserving functor $\mathbb{T} \xrightarrow{p} \mathbb{T}$. This is done as follows: Let $e_1 e_2, .., e_w$ be a (linear) base of I. The products of the e_i's determine a multiplication table $e_i e_j \overset{(1)}{=} \sum_{k=1}^{n} \gamma_{ijk} e_k$, $\gamma_{ijk} \in k$ which means that the e_i's can be considered as symbols (or indeterminates) satisfying the relation (1). Given any s-ary polynomial F, we can compute then n ns-ary polynomials f_k such that $F(\ldots, \sum_{i=1}^{n} X_{ij} e_i, \ldots) = \sum_{k=1}^{n} F_k(\ldots, X_{ij}, \ldots) e_k$. We define then p by: $(\bar{A}_s \xrightarrow{f} \bar{A}_1) \longmapsto (\bar{A}_{ns} \xrightarrow{(F_1 \ldots F_n)} \bar{A}_n)$.

We call I-construction the (functorial) process which to any k-algebra object in a cartesian category E, $\mathbb{T} \xrightarrow{X} E$, it assignes the R-algebra object $\mathbb{T} \xrightarrow{X[p]} E$ defined as the composite $\mathbb{T} \xrightarrow{p} \mathbb{T} \xrightarrow{X} E$. One verifies immediately that, for $k = A_o$ = free algebra in no generators, $A_o^n = A_o[p] \approx I$ by means of an isomorphisms that transports the canonical base of A_o^n into the (given) base $e_1 e_2, .., e_n$ of I. It follows that if $\mathbb{T} \xrightarrow{A} \text{Sets}$ is any k-algebra, then $A^n = A[p] = A \times R[p] \approx A \otimes I$, which means that the tensor product is built up from A in the same way that I is built up from k. We see that, in particular, the algebraic functor induced by p preserves finite presentability, and thus it restricts into an endofunctor of the category of finitely presented k-algebras. What follows is motivated by the preceeding discution.

Let $\mathcal{C} \xrightarrow{p} \mathcal{C}$ be a left exact endofunctor of a cartesian category $\mathcal{C}$. Given any left exact functor $\mathcal{C} \xrightarrow{X} \mathcal{E}$ into a cartesian category $\mathcal{E}$, let X[p] be the composite $\mathcal{C} \xrightarrow{p} \mathcal{C} \xrightarrow{X} \mathcal{E}$.

<u>Definition</u>

We say that a site structure on $\mathcal{C}$ is <u>p-stable</u> if any one of the following three (equivalent) conditions are satisfied:

i) $\mathcal{C} \xrightarrow{p} \mathcal{C}$ is continuous

ii) Given any fiber $\mathcal{C} \xrightarrow{X} \mathcal{E}$ in a topos $\mathcal{E}$, $X[p_o]$ is also a fiber

iii) $\varepsilon[p]$ is a fiber (for ε = the generic fiber) (by fiber we mean a left exact functor that sends coverings into epimorphic families).

Recall that condition ii) means that there is a geometric morphism $\mathcal{E} \to \tilde{\mathcal{C}}$ which classifies X[P]:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & \mathcal{C} \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle X} \\ \tilde{\mathcal{C}} & \xrightarrow{p^*} & \mathcal{E} \end{array} \qquad \begin{array}{l} p^* \dashv p_* \\ p^*\varepsilon = X[p_o] \end{array}$$

Recall also that when $\mathcal{E} = \tilde{\mathcal{C}}$ and $X = \varepsilon$, then p_* is given by composing with p(which sends sheaves into sheaves).

Observation

If p has a continuous left adjoint $C \xrightarrow{q} C$, $q \dashv p$, then the functor p^* is given by composing with q, and the geometric morphism that classifies $\varepsilon[p]$ is essential.

Proof

Let q_* = composing with q: $\tilde{C} \to \tilde{C}$. Since $q \dashv p$, then $q_* \dashv p_*$. By definition $p^* \dashv p_*$. Thus $q^* \dashv q_* = p^*$ and p is therefore essential.

Let $\mathbb{T} = \{\bar{A}_o \bar{A}_1 \ldots \bar{A}_n \ldots\}$ be any algebraic theory and $\mathbb{T} \xrightarrow{p} \mathbb{T}$ a product preserving functor. There is then a colimit preserving extension of p (that we denote also by p) to the category A of algebras: $\mathbb{T}^{op} \to A \xrightarrow{p} A$ which has as right adjoint $A \xrightarrow{q} A$ the algebraic functor $A \longrightarrow A[p]$ induced by p.

After dualizing we have then:

Proposition

Let $\mathbb{T}$ be any algebraic theory and $\mathbb{T} \xrightarrow{p} \mathbb{T}$ a product preserving functor. Consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{p} & \mathbb{T} \\ \downarrow & & \downarrow \\ C & \xrightarrow{p} & C \end{array}$$

where C is the category dual of that of finitely presented algebras, and $C \xrightarrow{p} C$ is the left exact extension of $\mathbb{T} \xrightarrow{p} \mathbb{T}$. Then:

a) If the algebraic functor induced by p preserves finite presentability, then p has a left adjoint q defined by $q(\bar{A}) = \overline{A[p]}$.

b) If q is given by $q(\bar{A}) = \bar{A} \times \bar{I}$ for some (fixed) $I \in C$, then $I = A_o[p]$ and $p = (-)^{\bar{I}}$.

Proof

Only the last statement needs some proof. By assumption, $\bar{A}_o \times \bar{I} = \overline{A_o[p]}$. Since $\overline{A_o}$ is the terminal object, the first equality follows. For the second equality, one verifies immediately that any left adjoint of an algebraic functor is an extension of the (inducing) product preserving functor between the theories.

In particular, we have proved the following:

Corollary

Let $\mathbb{T}$ be any algebraic theory, $\mathbb{T} \xrightarrow{p} \mathbb{T}$ a product preserving functor, and $C \xrightarrow{p} C$ the left exact extension to the dual of the category of finitely presented algebras. Given a p-stable site structure on C (cf. definition above), consider the following two conditions on the algebraic functor q induced by p.

a) It preserves finite presentability

b) It is given by cartesian product with a (fixed) algebra $I \in C$, which is necessarály equal to $A_o[p]$ (notice that b) => a)).

Then:

If a), the geometric morphism p that classifies $\varepsilon[p]$ is essential. If b), the inverse image $p^* = q_*$ is the exponentiation with $\varepsilon\bar{I}$, $p^* = (-)^{\varepsilon\bar{I}}$. (To verify that last statement, notice that $(-)^{\varepsilon\bar{I}}$ is always given by $(\bar{I} \times (-))_* = q_*$).

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FORMALLY REAL LOCAL RINGS,

AND INFINITESIMAL STABILITY.

Anders Kock

We propose here a topos-theoretic substitute for the theory of formally-real field, and real-closed field. By 'substitute' we mean that the notion is not just a lifting of the corresponding classical notion, but at the same time a generalisation which takes into account the mathematical applications of the specific topos-theoretic features of the notion. Thus in [1], it was argued that the good topos theoretic substitute for the notion of field is the notion of local ring object. Rousseau, in [6], has argued that topos-theoretic results often mathematically are identical to classical results which depend smoothly on a parameter.

We study here properties which depend smoothly on parameters in the sense that they are infinitesimally stable: they are not changed by infinitesimal changes in the parameters. More precisely, we study ring-theoretic properties φ so that if φ holds for a given object A, then φ also holds for the ring object $A[\epsilon]$ of dual numbers over A. It was precisely the ring-of-dual-numbers that motivated [1]. Clearly, the notion of field is not infinitesimally stable, whereas the notion of local ring is.

1. Two basic ring constructions

If A is a commutative ring object in a category $\underline{E}$ with finite products, then there are several ways of making $A \times A$ into a commutative ring object. We are interested in the following two classical ways (in both cases, the additive structure, or even A-module structure, is coordinatewise):

Ring of dual numbers: $A[\epsilon] = A \times A$, with multiplication

$$(a,b)\cdot(c,d) = (a\cdot c, a\cdot d + b\cdot c).$$

Multiplicative unit 1 is (1,0). The element (0,1) is denoted ε; $\varepsilon^2 = 0$.

Gauss-numbers: $A[i] = A \times A$, with multiplication

$$(a,b)\cdot(c,d) = (a\cdot c - b\cdot d, a\cdot d + b\cdot c).$$

Multiplicative unit 1 is (1,0). The element (0,1) is denoted i; $i^2 = -1$.

Assume now that $\underline{E}$ is a category where coherent logic has a good semantics (say, $\underline{E}$ a topos or a pretopos). Consider any finitary coherent formula $\varphi(z_1,\ldots,z_n)$ (in the sense of [3], see e.g. [5] §5) about n-tuples of elements from rings. Then clearly there is a simple way of constructing a coherent formula φ_ε with $2n$ free variables such that for any ring object A,

$$A \models \varphi_\epsilon (x_1,y_1,\ldots,x_n,y_n)$$

(1.1) iff

$$A[\epsilon] \models \varphi((x_1,y_1),\ldots,(x_n,y_n))$$

(there are several examples below). Similarly, an n-ary formula φ gives rise to a 2n-ary formula φ_i such that (1.1) holds when φ_ϵ and $A[\epsilon]$ are replaced by φ_i and $A[i]$, respectively.

Therefore also, if T is a coherent theory of commutative rings, there is a coherent theory T_ϵ such that $A \models T_\epsilon$ iff $A[\epsilon] \models T$. Similarly with T_i: $A \models T_i$ iff $A[i] \models T$.

We say that a theory T is ε-stable or infinitesimally stable if $A \models T$ implies $A[\epsilon] \models T$, or equivalently if $T_\epsilon \subseteq T$. By the well known metatheorem for coherent logic [3] we have in particular:

Proposition 1.1 A coherent theory T of commutative rings is ε-stable if and only if, for every T-model in Set, $A[\epsilon]$ is also a T-model.

An immediate application is

Proposition 1.2 The coherent theory T_L of local rings is ε-stable.

For, if A is a local ring in Set, then so is $A[\epsilon]$. Note also that no coherent field notion is stable; for $A[\epsilon]$ is not always (in fact never) a field.

Remark 1.3 One could similarly talk about i-stable properties and theories, but we do not know of any significant example. The notion of local ring is not i-stable: First, we note that in $A[i] = A\times A$, an element $z = (x,y)$ is invertible if and only if x^2+y^2 is invertible, and then $z^{-1} = (x^2+y^2)^{-1}\bar{z}$ where $\bar{z} = (x,-y)$). Next, let F be a field in Set of characteristic $\neq 2$, and suppose there is an element $j \in F$ with $j^2 = -1$ (for instance $F = \mathbb{C}$). Then certainly F is a local ring, but $F[i] = F\times F$ is not. For, $(1,j)$ and $(1,-j)$ are non-invertible since $1^2+j^2 = 0$, but their sum is $(2,0)$ which is invertible. So $F[i]$ is not local.

2. Some ε-stable theories

Consider for each natural number n the coherent sequent s_n:

$$\forall x_1,\ldots,x_n : \bigvee_{i=1}^{n} (x_i \text{ invertible}) \Rightarrow \sum_{i=1}^{n} x_i^2 \text{ is invertible}$$

We let T_{FR} denote the coherent theory of commutative rings whose axioms are the sequents s_n. We call T_{FR} the theory of formally-real rings.

Proposition 2.1 If K is a field in Set, then $K \models T_{FR}$ if and only if K is formally real in the classical sense: '-1 is not a square sum' (cf. e.g. [2] XI.2).

The proof is straightforward.

Proposition 2.2 The theory T_{FR} of formally-real rings is ε-stable.

Proof By Prop. 1.1, it suffices to consider a formally real ring A in Set and prove that $A[\epsilon]$ is formally real. Let $(x_i, y_i) \in A[\epsilon] = A \times A$ for $i = 1, \ldots, n$. Now $(x,y) \in A[\epsilon]$ is invertible iff x is invertible in A. So one of the (x_i, y_i)'s is invertible iff one of the x_i's is invertible, which implies that $\sum x_i^2$ is invertible (by formal-realness of A). But then also

$$(\sum x_i^2, \sum 2x_i y_i)$$

is invertible in $A[\epsilon]$. The displayed element is the square sum of the (x_i, y_i)'s.

Porposition 2.3 If A is formally real and local, then $A[i]$ is local.

Proof easy, using part of remark 1.3.

Clearly, no coherent theory of algebraically closed field is ε-stable, because the notion of field is not ε-stable. But the notion of algebraically closed local ring is not ε-stable either (algebraically closed ring means: monic polynomials have roots). For, if it were, $\mathbb{C}[\epsilon]$ in Set would be algebraically closed local, which it is not, since ϵ has no square root.

However, Wraith [7] has displayed a coherent theory T_{SC} of 'separably closed local rings'. It has the property that for a ring A in Set, $A \models T_{SC}$ if and only if A is a Henselian local ring with separably closed residue field (or: A is strictly Henselian, in the terminology of [4], chapter VIII). The existence of such a theory has been known for some time, using theorems og Makkai-Reyes, Deligne, and Hakim; cf. [3].

Propositoin 2.4 The theory T_{SC} of separably closed local rings is ϵ-stable.

Proof. Again, by Prop. 1.1, it suffices to consider rings in Set. Let A be a Henselian local ring with separably closed residue field k. Then $A[\epsilon]$ is local, and its residue field is also k. So we just have to prove that $A[\epsilon]$ is Henselian. We use the description of this notion given in [4] VII §3 prop.3. no. 2, so we must prove that for monic polynomials $P(X)$ over $A[\epsilon]$, simple roots in k lift to $A[\epsilon]$. Now we have canonical ring maps

$$A[\epsilon] \xrightarrow[q_2]{} A \xrightarrow[q_1]{} k \quad .$$

If $P(X)$ is a monic polynomial over $A[\epsilon]$, we denote its image under q_2 and $q_1 \circ q_2$ by $\bar{P}(X)$ and $\bar{\bar{P}}(X)$, respectively. Assume $\bar{\bar{P}}$ has a simple root $\in k$. Since A is Henselian, this root may be lifted to a root $b \in A$ of $\bar{P}(X)$, and b is necessarily a simple root (meaning $\bar{P}'(b)$ is invertible). Now

$$P(X) = \bar{P}(X) + \epsilon \cdot Q(X).$$

To lift b means to find a $c \in A$ so that $P(b + \epsilon c) = 0$.

Now

$$P(b + \epsilon c) = \bar{P}(b + \epsilon c) + \epsilon \cdot Q(b + \epsilon c)$$

$$= \bar{P}(b) + \epsilon c\, \bar{P}'(b) + \epsilon \cdot Q(b) + \epsilon\ (\epsilon c\, Q'(b)).$$

The first term vanishes since $\bar{P}(b) = 0$. The last term vanishes since $\epsilon^2 = 0$. Thus to find c means to solve

$$0 = \epsilon\, c\, \bar{P}'(b) + \epsilon\, Q(b)$$

which can be done since $\bar{P}'(b)$ is invertible in A. This proves the proposition.

3. A substitute for the notion of real-closed field

We shall say that a ring object A is a separably-real-closed local ring if A is formally real local, and $A[i]$ is separably closed ($A[i]$ is local by prop. 2.3).

Let, as above, T_L, T_{FR}, and T_{SC} be the (coherent) theories of local, formally real, and separably closed local, rings, respectively. Then the theory of separably-real-closed local ring is

$$T_{SRCL} = T_L \cup T_{FR} \cup (T_{SC})_i,$$

and as such, it is a coherent theory.

Proposition 3:1 The theory T_{SRCL} is ε-stable.

Proof The theories T_L, T_{FR}, and T_{SC} are ε-stable by propositions 1.2, 2.2, and 2.4. The result will now follow from the following general

Lemma If T is an ϵ-stable theory, then so is T_i.

Proof We have

$$(A \models T_i) \Rightarrow (A[i] \models T) \Rightarrow (A[i][\epsilon] \models T)$$

(by ϵ-stability of T)

$$\Rightarrow (A[\epsilon][i] \models T) \Rightarrow (A[\epsilon] \models T_i),$$

since obviously $A[\epsilon][i] = A[i][\epsilon]$.

Besides (or related to) the ϵ-stability of T_{SRCL}, a justification of this theory lies in the followin conjecture: The Dedekind reals in an elementary topos with NNO satisfy T_{SRCL}. A support for the conjecture is

Proposition 3.2 The sheaf R of germs of continuous real-valued functions on a topological space X is a separably-real-closed local ring object.

Proof We have $R[i] = C$ = sheaf of germs of continuous complex-valued functions. To see that $C \models T_{SC}$, it suffices, since T_{SC} is a coherent theory, to see that for each $x \in X$, $C_x \models T_{SC}$. But C_x is well known to be Henselian (and have $\mathbb{C}$ as residue field), see e.g. [4] VII §4.

We note that R is not a real-closed local ring in the sense of $R[i] = C$ being an algebraically closed local ring. For, if it were, then one could solve $x^2 = id$ around the origin of $X = \mathbb{C}$, which cannot be done continuosly (there is homotopy obstruction).

4. Strict order structure.

In this section, A will denote a fixed separably real closed local ring object. Any formally real ring, and in particular A, is an algebra over the rationals $\mathbb{Q}$; for, $n = 1^2 + \ldots + 1^2$ (n times) and is thus invertible.

We equip A with a binary "Strict order relation" $<$ by posing for arbitraty $a: X \longrightarrow A$

$$a > 0 \quad \text{iff} \quad \vdash_X \exists y (y^2 = a \text{ and } y \text{ invertible}).$$

We put $a > b$ if $a - b > 0$.

Proposition 4.1. The following coherent sentences hold:

1) $\forall a$: $a > 0 \Rightarrow a$ invertible
2) $\forall a_1, a_2$: $a_1 > 0$ and $a_2 > 0$ implies $a_1 \cdot a_2 > 0$.
3) $\forall a$: a invertible implies $a > 0 \vee (-a) > 0$
4) $\forall e, f$: $e > 0$ and $f > 0$ implies $e+f > 0$ (and hence $a_1 > a_2$ and $b_1 > b_2$ implies $a_1 + b_1 > a_2 + b_2$).
5) $\forall e, f$: $e+f > 0$ implies $e>0 \vee f>0$. (I am indebted to Peter Johnstone for this observation.)

Proof. Again, by coherence, it suffices to prove these in Set. The first and second are immediate. To prove the third, consider the monic polynomial $X^2 - a$. Since $A[i]$ is separably closed and 2 is invertible, this polynomial has a root, $x+iy$, say, with x and $y \in A$. So

$$(x+iy)^2 = a$$

that is,

$$x^2 - y^2 = a \tag{4.1}$$

and

$$2xy = 0. \tag{4.2}$$

Since a is invertible, we conclude from (4.1) and localness:

$$x \text{ invertible or } y \text{ invertible,}$$

whence from (4.2)

$$y = 0 \quad \text{or} \quad x = 0.$$

If $y = 0$, $x^2 = a$. If $x = 0$, $y^2 = -a$, whence $a > 0$ or $(-a) > 0$, respectively.

To prove 4), assume $x^2 = e$ and $y^2 = f$, with x and y invertible. By A being formal-real, we conclude $e+f$ invertible so by 3)

$$e+f > 0 \quad \text{or} \quad -(e+f) > 0.$$

We just have to exclude the latter possibility. But $-(e+f) > 0$ implies

$$-(x^2 + y^2) = -(e+f) = z^2$$

for some invertible z, whence $x^2+y^2+x^2 = 0$, contradicting formal real-ness.

To prove 5): if $e+f > 0$, then $e+f$ is invertible. Since A is a local ring, either e or f is invertible, say e is. Then by 3) either $e > 0$ (in which case we are done) or $(-e) > 0$, whence $f = (e+f) + (-e) > 0$ by 4).

Corollary 4.2 The relation $>$ is transitive.

Proof $((a>b)$ and $(b>c))$ implies $((a-b) > 0$ and $(b-c) > 0)$, which in turn implies $((a-b) + (b-c) > 0)$, thus $a-c > 0$, thus $a>c$.

Corollary 4.3 if n is a positive natural number, then $n>0$ and $n^{-1} > 0$ in A.

Proof. By prop. 4.1(3), it suffices to exclude $-n > 0$ and $-n^{-1} > 0$, which is easy.

We now leave the world og coherent logic by introducing the predicate '$\leq$'. We put $b \leq 0$ iff $\forall a$: $a>0$ implies $a>b$. Also, put $b \leq c$ if $b-c \leq 0$.

Proposition 4.3 $\forall a,b$: $a\leq0$ and $b\leq0$ implies $a+b \leq 0$.

Proof. Let $c>o$. We must prove $a+b < c$. Now $c = \frac{1}{2}c + \frac{1}{2}c$, and $\frac{1}{2}c > o$ by $c>o$ and Coroll. 4.3.. Thus $(a<\frac{1}{2}c)$ and $(b<\frac{1}{2}c)$, whence $a+b < \frac{1}{2}c+\frac{1}{2}c=c$ (using Prop. 4.1 (4)). This proof is intuitionistically valid, hence valid in $\underline{E}$.

Again, it is clear that this Proposition implies the transitive law for $\leq$. Also, $\forall a: a\leq a$, so that $\leq$ is a preorder. We cannot conclude that it is a partial order.

The next Propositions have evident corollaries obtained by adding elements to both sides of the various (strict or nonstrict) inequality signs. We omit these corollaries.

Proposition 4.4. $\forall a,b : a\geq o$ and $b>o$ implies $a+b > o$.

Proof. Since $\forall d: d<o$ implies $d<a$, we also have

$$\forall d: \quad d < b \quad \text{implies} \quad d < a+b \ .$$

In particular, this holds for $d = \frac{1}{2}b$, thus $o < \frac{1}{2}b < a+b$, whence $a+b > o$ by transitivity of $>$.

Proposition 4.5. For all a and b , we have

1) $a>o$ implies $a\geq o$

2) $a\geq o$ and $b\geq o$ implies $a\cdot b \geq o$

3) $a\geq o$ and $b>o$ implies $a\cdot b \geq o$.

Proof. 1) If $a>o$ and $c<o$, then by transitivity of $<$, $c<a$. Since this holds for any $c<o$, $a\geq o$.

2) The following corrects my erroneous proof in the originally circulated version (June 1977) of the present paper. It depends on the following Proposition, due to Peter Johnstone; this Proposition at the same time refutes a remark to the contrary effect in the June 1977 version.

Proposition 4.6. We have

$$\neg (z > o) \iff z \leq o \ .$$

Proof. By Prop. 4.1. (5) we have

$$\forall z,y : \quad z+y > o \quad \Rightarrow \quad z>o \ \vee \ y>o$$

hence

$$\forall z,y : \quad z+y > o \ \wedge \ \neg(z>o) \quad \Rightarrow \quad y>o \ ;$$

hence

$$\forall z ; \quad \neg(z>o) \quad \Rightarrow \ [\forall y : z+y > o \ \Rightarrow \ y>o] \ .$$

But it is easy to see that the formula in the square bracket is equivalent to $z \leq o$:

$$\forall y : \ z+y > o \quad \Rightarrow \quad y>o$$

$$\iff$$

$$\forall a : \ z+(a-z) > o \ \Rightarrow \ (a-z) > o$$

$$\iff$$

$$\forall a : \ a > o \quad \Rightarrow \quad a > z \ .$$

This proves the implication $\Rightarrow$. For the other one, observe that

$$z \leq o \ \wedge \ (z>o) \ \Rightarrow \ z>z \ ,$$

using the definition of $\leq$. But $z>z$ is false.

Proof of Prop. 4.5.(2) and (3). Assume $a \geq o$ and $b \geq o$. To prove $a \cdot b \geq o$, it suffices to prove $\neg (a \cdot b) < o$. But if $a \cdot b < o$, $a \cdot b$ is invertible, so in particular, b is ; by Proposition 4.1.(3), $b>o$ or $b<o$. By Proposition 4.6., $b<o$ is incompatible with $b \geq o$, so that $b>o$. Similarly $a>o$. Thus $a \cdot b > o$, by Proposition 4.1.(2) , contradicting the assumption $a \cdot b < o$.

Now (3) in the Proposition follows by combining (1) and (2). Again, the present proofs are intuitionistically valid, hence valid in E .

Let us finally remark that we cannot conclude $a \leq b$ and $b \leq a$ implies $a=b$. ($\mathbb{R}[\epsilon]$ in Set furnishes a counterexample.)

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REMARKS ON THE PREVIOUS PAPER

Extracts from two letters from Peter Johnstone to Anders Kock, March 1978.

Concerning your conjecture that the Dedekind reals are always a separably real-closed local ring: As you remark (Proposition 3.2.) this is true in any spatial topos, from the "classical" fact that the stalks of the sheaf of continous real- (or complex-) valued functions on a space are always Henselian. In fact this observation is sufficient to prove your conjecture in any Grothendieck topos, for the simple reason that "the generic unramifiable polynomial over $\mathbb{C}$" lives in a spatial topos. Explicitly, let f be a monic polynomial of degree n over $\mathbb{C}$ in a topos $\mathcal{E}$. Then the coefficients of f define a geometric morphism $\bar{f} : \mathcal{E} \longrightarrow \mathrm{Shv}(\mathbb{C}^n)$, and f is unramifiable precisely if the image of $\bar{f}$ is contained in the open subtopos of points in $\mathbb{C}^n$ where at least one of the hyperdiscriminants is nonzero. But over this space, the sheaf of continuous $\mathbb{C}$-valued functions is separably closed, and so we can cover the space with open subsets on which the generic polynomial has a simple root. Pulling back this cover along $\bar{f}$, we get a localization of $\mathcal{E}$ over which f has a simple root.

Given that the result is true, there clearly ought to be a better proof of it than this. I suspect that in order to get a direct proof we are going to need a formulation of T_{SRCL} which does not involve mention of the extension ring $A[i]$.

I feel that there ought to be something analogous to the hyperdiscriminants which would tell you (in the classical case) when a given polynomial over a formally real field has at least one simple root in a formally real extension; but so far I have not found a way of distinguishing between real and complex roots that can be expressed coherently.

Re-reading what I wrote earlier, it occurs to me that perhaps the theory of separably real-closed local rings needs to be written in the language of ordered rings rather than the language of rings. Define the theory OLR of ordered local rings to consist of the theory of rings plus a unary predicate P satisfying

$$
\begin{aligned}
P(a) &\vdash \exists b \; (ab=1) \\
\exists b \; (ab=1) &\vdash P(a) \vee P(-a) \\
P(a) \wedge P(b) &\vdash P(ab) \wedge P(a+b) \\
P(a+b) &\vdash P(a) \vee P(b) \\
P(0) &\vdash \text{false}
\end{aligned}
$$

Then the underlying ring of an OLR is formally real local; conversely, in Set every formally real local ring admits an ordering (since we can order its residue field, and then pull back). You showed that every SRCL ring admits a unique ordering (the positive elements being invertible squares). But in a topos it is not true even locally that a formally real local ring can be ordered: consider, in the topos of diagrams of the form $\cdot \begin{smallmatrix} \nearrow \cdot \\ \searrow \cdot \end{smallmatrix}$, the formally real field $Q(\sqrt{2}) \begin{smallmatrix} \nearrow \mathbb{R} \\ \searrow \mathbb{R} \end{smallmatrix}$ where the two embeddings are different. Now it seems to me that one has rather better chances, in the theory of ordered fields, of saying that a polynomial has a simple root in an <u>ordered</u> extension field (not, of course, the same thing as a formally real extension field); for example, with a quadratic one can make the assertion that its discriminant is positive.

ON THE SYNTHETIC THEORY OF VECTOR FIELDS

Anders Kock

The present note is an exposition of some of the general "synthetic differential geometry". The style of exposition is that it expresses maps, subobjects, and statements in set theoretic language. As long as one stays inside what Lawvere calls "cartesian logic", which is essentially negation free (but higher order) logic, then the maps, subobjects etc. described can be interpreted in any cartesian closed category with equalizers. So when we for instance say "ring", we mean "ring object in such a category".

Let A be a commutative ring with 1. Let $D \subseteq A$ be the set of elements of square zero. We say that A is of <u>line type</u> if every map $t: D \to A$ is of form

$$(0) \qquad t(d) = b + d \cdot c \quad \forall d \in D$$

for some unique b and $c \in A$. Clearly $b = t(0)$. We denote the c occurring here by $t'(0)$. Similarly, if $f: A \to A$ is arbitrary, and $a \in A$, we define $f'(a)$ to be that unique element in A such that

$$(1) \qquad f(a+d) = f(a) + d \cdot f'(a) \quad \forall d \in D$$

(this element exists uniquely in virtue of A being of line type). We call (1) the Taylor expansion of f at a.

To a map $f: A \to A$ we have thus associated a new map, $f': A \to A$, its derivative. It is easy from (1) to prove

$$(f+g)' = f' + g' \qquad (f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(f \circ g)' = (f' \circ g) \cdot g' \qquad (\text{identity})' \equiv 1$$

$$(\text{constant})' \equiv 0;$$

see [5]. In fact proofs of these laws explicitly using elements with vanishing square were used very early in the history of calculus (Fermat), but were later abandoned, perhaps due to

Proposition 1. No non-trivial rings in the category of sets are of line type.

Proof. If A is non-trivial, then D must contain more than just $0 \in D$ (for, otherwise the c occurring in (0) could not be uniquely determined). So take some $\delta \in D$ with $\delta \neq 0$. Define a function $t\colon D \to A$ by

$$* \qquad \begin{cases} t(\delta) = 1 \\ t(d) = 0 \quad \text{for } d \neq \delta. \end{cases}$$

By the line type axiom, t is of form $t(d) = b + d \cdot c$. Obviously $b = 0$, so $t(d) = d \cdot c \;\; \forall d \in D$. In particular

$$1 = t(\delta) = \delta \cdot c.$$

Multiplying this equation by δ, we obtain $\delta = \delta^2 \cdot c = 0$, (since $\delta \in D$), contradicting the assumption $\delta \neq 0$.

The proof hinges on the construction principle *, which has no place in cartesian logic.

For the rest of this note, A is a fixed ring, assumed to be of line type.

We note that the uniqueness assertion about c in the line type notion can be formulated: for any $c \in A$

$$(c \cdot d = 0 \quad \forall d \in D) \Rightarrow (c = 0).$$

This principle, we refer to as "cancelling universally quantified d's".

Geometrically, D is the intersection of the unit circle around $(0,1) \in A \times A$ and the x-axis $A \times \{0\} \subseteq A \times A$, and is thus a unity of the opposites: "curved" and "straight". In fact, for any object M, a map $t: D \to M$ should be thought of as a tangent vector on M at the point $t(0) \in M$ (Lawvere, [3]). Likewise (ibid.), a vector field X on M is a law which to each $m \in M$ associates a tangent vector $X(m,-): D \to M$. Thus, a vector field on M is a map

$$X: M \times D \to M$$

satisfying

$$X(m,0) = m \qquad \forall m \in M$$

Keeping a $d \in D$ fixed, we get a map

(2) $$X(-,d): M \to M$$

called an infinitesimal transformation belonging to X.

The classical work of Lie on differential equations (see e.g. [2]) makes wide use of these endomaps of M, which have no place in modern rigourous treatments.

It is natural to ask whether $X(-,d)$ is a bijective map, with inverse

$$X(-,-d): M \to M.$$

A condition on M that will guarantee this, and also will allow us to add tangent vectors at the same point, is the condition that M is <u>infinitesimally linear</u> in the following sense. For each natural number n, we let $D(n) \subseteq A^n$ be the subset

$$\{(d_1,\dots,d_n) \in A^n \mid d_i \cdot d_j = 0 \quad \forall i,j\}$$

(in particular $d_i^2 = 0 \ \forall i$). For $i = 1,\dots,n$, we have the "i'th inclusion"

$$incl_i: D \to D(n)$$

given by

$$incl_i(d) = (0,0,\dots,d,\dots,0)$$

(the d in the i'th place).

We say that M is <u>infinitesimally linear</u> [6], [8], if for each n and each n-tuple $t_i: D \to M$ $(i=1,\dots,n)$ of tangent vectors at the same point $m \in M$, there exists a <u>unique</u> $l: D(n) \to M$ with

(3) $$l \circ incl_i = t_i \quad i = 1,\dots,n.$$

In particular, if M is infinitesimally linear, and t_1, t_2 are two tangent vectors at $m \in M$, there is a unique $l\colon D(2) \to M$ with (3) holding $(n=2)$, and we define $(t_1+t_2)\colon D \to M$ to be the map given by

$$(t_1 + t_2)(d) = l(d,d)$$

(note that $d \in D$ implies $(d,d) \in D(2)$).

Likewise, if $t\colon D \to M$ is a tangent vector and $a \in A$ is a scalar, we define $a\cdot t$ to be the map $D \to M$ given by

$$(a\cdot t)(d) = t(a\cdot d)$$

(note that $d \in D$ and $a \in A$ implies $a\cdot d \in D$).

It is then easy to prove ([6],[8],[9]) that the set of tangent vectors at any given point m of M becomes an A-module, with the structures thus defined (one uses $D(3)$ to prove associativity; the higher $D(n)$'s are not used).

To prove

(4) $$X(X(m,d),-d) = m,$$

we shall more generally prove, for $(d_1,d_2) \in D(2)$

(5) $$X(X(m,d_1),d_2) = X(m,d_1 + d_2)$$

(note that $(d_1,d_2) \in D(2) \Rightarrow d_1 + d_2 \in D$, because when squaring $d_1 + d_2$, the double product vanishes by assumption). To prove (5), note that both sides define maps

$$l\colon D(2) \to M$$

with $1 \circ \mathrm{incl}_i = X(m,-)$ $(i=1,2)$, and thus are equal, by the uniqueness assertion in the infinitesimal linearity assumption.

We can add two vector fields X and Y on an infinitesimally linear object M, by letting $(X+Y)(m,-)$ be the sum (as already defined) of the two tangent vectors at m, $X(m,-)$ and $Y(m,-)$. We can also multiply a vectorfield X with a scalar valued function $\varphi: M \to A$, namely by putting

$$(\varphi \cdot X)(m,d) = X(m, \varphi(m) \cdot d).$$

In this way, the set of vector fields on M becomes a module over the ring of functions $M \to A$.

Recall [6] [7] that an A-module M is called Euclidean if each $t: D \to M$ is of form

$$t(d) = t(0) + d \cdot \underline{v}$$

for some unique $\underline{v} \in M$, called the principal part of t.

Proposition 2. If M is a Euclidean A-module which is also infinitesimally linear, then addition of tangent vectors at a given $\underline{m} \in M$ using infinitesimal linearity agrees with the obvious addition "adding principal parts". Similarly for multiplication by scalars.

Proof. Let

$$t_i(d) = \underline{m} + d \cdot \underline{v}_i \qquad i = 1,2$$

be two vectors at $\underline{m} \in M$. Their sum, using infinitesimal linearity is found from $l\colon D(2) \to M$ given by

$$l(d_1,d_2) = \underline{m} + d_1 \cdot \underline{v}_1 + d_2 \cdot \underline{v}_2$$

since $l \circ \mathrm{incl}_i = t_i$. So we have, for all $d \in D$,

$$(t_1+t_2)(d) = l(d,d) = \underline{m} + d \cdot \underline{v}_1 + d \cdot \underline{v}_2$$
$$= \underline{m} + d \cdot (\underline{v}_1 + \underline{v}_2),$$

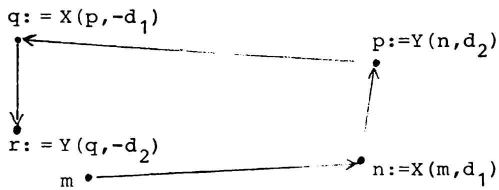
proving that $t_1 + t_2$ has principal part $\underline{v}_1 + \underline{v}_2$.

The last assertion of the Proposition is trivial.

We henceforth assume that A is of line type (hence Euclidean as an A-module), and infinitesimally linear; and M is assumed to be an arbitrary infinitesimally linear object.

We proceed to consider Poisson bracket of two vector fields X and Y on M. For fixed $d_1 \in D$ and $d_2 \in D$, we may consider the commutator of the two bijective endomaps $X(-,d_1)$ and $Y(-,d_2)$ of M. In other words, for fixed m, we consider the "circuit"

(6)



(recall from (4) that $X(-,d_1)^{-1} = X(-,-d_1)$, and similarly for Y). For fixed m, the r obtained depends on $(d_1,d_2) \in D \times D$, so that we have a map

(7)
$$D \times D \to M$$
$$(d_1,d_2) \mapsto r$$

If $d_1 = 0$, we have $n = m$ and $q = p$, so that

$$r = Y(q,-d_2) = Y(p,-d_2) = n = m$$

the third equality sign by (4) and $Y(n,d_2) = p$. Similarly if $d_2 = 0$, we get likewise $r = m$. So the map (7) satisfies the condition for τ in the following requirement on M, [6]:

<u>Requirement</u>. For any map $\tau: D \times D \to M$ with

$$\tau(d,0) = \tau(0,d) = \tau(0,0) \quad \forall d \in M$$

there is a unique map $t: D \to M$ with

$$t(d_1 \cdot d_2) = \tau(d_1,d_2) \quad \forall (d_1,d_2) \in D \times D.$$

We assume henceforth that M satisfies this. Thus the map described in (7) is of form $(d_1,d_2) \to t(d_1 \cdot d_2)$ for some unique $t: D \to M$ with $t(0) = m$. We denote this t $[X,Y](m,-)$. Letting m vary, we obtain in this way a vector field $[X,Y]$ on M. It is characterized by

$$[X,Y](m,d_1 \cdot d_2) = r \quad \forall (d_1,d_2) \in D \times D,$$

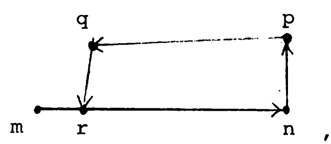
r obtained as in (6).

It is easy to prove that $[X,Y] = 0$ and $[X,Y] = -[Y,X]$. I belive that bilinearity and Jacobi identity for the bracket operation described here can be obtained by reinterpretation of the proofs for similar facts about the Lie algebra object of a monoid in [6].

Easier proofs exist (using Proposition 2) for the case where M is a Euclidean module, essentially by using the notion of "directional derivation along a vector field" which we shall discuss in a moment. However, we do not want to perform "a double-dualization" by identifying a vector field with a differential operator on a ring of functions. Thus, the following Theorem, which is essential in Lie's theory of differential equations, is stated and proved entirely in geometric terms (no differential operators!).

We shall call a vector field X proper if each $X(m,-): D \to M$ is an injective map (thus we make a positive assumption on X instead of the classical negative "$X(m,-)$ is always non-zero".) The theorem deals with two vector fields X,Y (with X proper) such that all circuits are X- trapezia, i.e. have shape

(8)



which, to be precise, we take to mean that for each $m \in M$ and $(d_1,d_2) \in D \times D$ the r constructed in (6) is of form $X(m,\delta)$ for some $\delta \in D$ (necessarily unique since X is proper).

We shall finally assume that A also satisfies the Requirement above. Then

Theorem 3. Let X,Y be vector-fields on M, with X proper. Then the following two conditions are equivalent:

i) all circuits of form (6) are X-trapezia, (8).

ii) $[X,Y] = \rho\cdot X$ for some scalar valued function $\rho\colon M\to A$.

<u>Proof</u>. Assume (i). Let m be fixed, and consider for $(d_1,d_2)\in D\times D$ that unique $\delta = \delta(d_1,d_2)$ such that

(9) $$r = X(m,\delta).$$

Arguing as for the map described in (7), we see that $\delta(d,0) = \delta(0,d)=0$. Therefore, by the Requirement for A, we have $\delta(d_1,d_2) = t(d_1\cdot d_2)$ for some unique $t\colon D\to A$. Since $t(0)=0$, we get, since A is of line type, a unique $b\in A$ such that $t(d) = b\cdot d$ for all $d\in A$, so that

$$\delta(d_1,d_2) = b\cdot d_1\cdot d_2 \qquad \forall\ (d_1,d_2)\in D\times D.$$

Now let m vary, and record the dependence of b on m by writing $b=\rho(m)$. Thus we have, for all $(d_1,d_2)\in D\times D$,

$$\begin{aligned}[X,Y](m,d_1\cdot d_2) &= r = X(m,b\cdot d_1\cdot d_2)\\ &= X(m,\rho(m)\cdot d_1\cdot d_2) = (\rho\cdot X)(m,d_1\cdot d_2).\end{aligned}$$

From the uniqueness in the Requirement then follows

$$[X,Y](m,d) = (\rho\cdot X)(m,d) \quad \text{for all} \quad d\in D$$

(and all m). This proves (ii).

The converse implication is trivial; if $r=(\rho\cdot X)(m,d_1\cdot d_2)$, we have $r=X(m,\rho(m)\cdot d_1\cdot d_2)$ witnessing that r is of form $X(m,\delta)$.

If we call two elements m_1 and m_2 of M <u>X-neighbours</u> provided there exists a $d \in D$ with

$$X(m_1,d) = m_2,$$

then it is easy to see that the conditions of the theorem in turn are equivalent to: for any $d \in D$, the permutation $Y(-,d)$ preserves the relation "being X-neighbours". Lie uses the phrase: "X admits Y". The phrase "Y permutes X" makes a certain sense too in this connection, since by integration (which has no place in the present set up) the X-neighbour-relation passes into the relation"being on the same streamline for the flow generated by X", so that $Y(-,d)$ permutes the streamlines of X (possibly reparametrizing them).

We now discuss directional derivatives. Let X be a vector field on M, and $f: M \to V$ a function with values in a Euclidean module V (in particular, V might be A itself). Consider for fixed $m \in M$ the map $D \to V$ given by

$$d \mapsto f(X(m,d)).$$

By Euclidean-ness of V, this map is of form

$$d \mapsto f(m) + d \cdot \underline{v}$$

for some unique $\underline{v} \in V$, which we denote $X(f)(m)$. Thus $X(f): M \to V$ is the function characterized by

$$f(X(m,d)) = f(m) + d \cdot X(f)(m) \qquad \forall d \in D, \forall m \in M \tag{10}$$

("generalized Taylor formula").

The construction $f \mapsto f'$ previously mentioned is a special case, namely for X the vector field $\hat{+}$ on A given by

$$\hat{+}(a,d) = a + d.$$

It is proved in [7], Prop. 1.2 that $f \mapsto X(f)$ is A-linear, and satisfies appropriate evident generalizations of Leibniz-rule:

$$X(\varphi \cdot f) = X(\varphi) \cdot f + \varphi \cdot X(f)$$

whenever $f: M \to V$ and $\varphi: M \to A$. (The proofs are easy from (10)). We proceed to investigate how $X(f)$ depends on X. We shall prove

<u>Proposition 4</u>. For any vector fields X_1, X_2, Y on M, and any $\varphi: M \to A$, we have

(i) $$(X_1 + X_2)(f) = X_1(f) + X_2(f)$$

(ii) $$(\varphi \cdot X)(f) = \varphi \cdot (X(f))$$

(iii) $$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

for any $f: M \to V$ (V a Euclidean infinitesimally linear module).

<u>Proof</u> (i): Let $L: M \times D(2) \to M$ be defined so that for any $m \in M$, $l = L(m,-,-): D(2) \to M$ has

$$l \circ incl_i = X_i(m,-) \qquad i = 1,2$$

Consider for fixed $m \in M$ the map $h: D(2) \to V$ given by

$$h(d_1,d_2) = f(L(m,d_1,d_2))$$

We then have (for $i = 1,2$) that $h \circ incl_i: D \to V$ is the tangent vector at $f(m)$ with principal part $X_i(f)(m)$; to see this, for $i = 2$, say

$$h(incl_2(d)) = h(0,d) = f(L(m,0,d))$$

$$= f(X_2(m,d))$$

$$= f(m) + d \cdot X_2(f)(m).$$

From the uniqueness assertion in the statement that V is infinitesimally linear, it then follows that

$$h(d_1,d_2) = f(m) + d_1 \cdot X_1(f)(m) + d_2 \cdot X_2(f)(m).$$

We have, for all $d \in D$,

$$f((X_1+X_2)(m,d)) = f(m) + d \cdot (X_1+X_2)(f)(m).$$

On the other hand, for all $d \in D$,

$$f((X_1+X_2)(m,d)) = f(L(m,d,d)) = h(d,d)$$

$$= f(m) + d \cdot X_1(f)(m) + d \cdot X_2(f)(m)$$

Comparing these two expressions for $f((X_1+X_2)(m,d))$ and cancelling the universally quantified d, we get (i), as desired. The proof of (ii) is easier, and omitted. Let us finally prove (iii). For fixed m, d_1, d_2, we consider the circuit (6) and the elements n, p, q, r described there. We consider $f(r) - f(m)$. First

$$f(r) = f(q) - d_2 \cdot Y(f)(q)$$

$$= f(p) - d_1 \cdot X(f)(p) - d_2 \cdot Y(f)(q)$$

using generalized Taylor (10) twice. Again using generalized Taylor (10) twice, (noting $m = X(n, -d_1)$ and $n = Y(p, -d_2)$ by (4)),

$$f(m) = f(n) - d_1 \cdot X(f)(n)$$

$$= f(p) - d_2 \cdot Y(f)(p) - d_1 \cdot X(f)(n).$$

Subtracting these two equations, we get

(11) $$f(r) - f(m) = d_1 \cdot \{X(f)(n) - X(f)(p)\}$$

$$+ d_2 \cdot \{Y(f)(p) - Y(f)(q)\}$$

$$= -d_1 \cdot d_2 \cdot Y(X(f))(p) + d_1 \cdot d_2 \cdot X(Y(f))(p)$$

using generalized Taylor (10) for the function $X(f)$ and for the function $Y(f)$. Now we have

$$d_2 \cdot g(n) = d_2 \cdot g(p)$$

and

$$d_1 \cdot g(n) = d_1 \cdot g(m),$$

since

$$d_2 \cdot g(p) = d_2 \cdot g(Y(n, d_2))$$
$$= d_2 \cdot (g(n) + d_2 \cdot Y(g)(n))$$
$$= d_2 \cdot g(n),$$

the last term vanishing because $d_2^2 = 0$. Similarly for the other equation. Since the terms on the right hand side of (11) occur with both a d_1-factor and a d_2-factor we may apply this principle for the functions $Y(X(f))$ and $X(Y(f))$ to replace the argument p by, first n, and then m. Thus

$$(12) \qquad f(r) - f(m) = d_1 \cdot d_2 \cdot (X(Y(f))(m) - Y(X(f))(m)).$$

On the other hand

$$[X,Y](m, d_1 \cdot d_2) = r$$

so that

$$(13) \qquad f(r) - f(m) = d_1 \cdot d_2 \cdot [X,Y](f)(m).$$

Comparing (12) and (13), we see that for all $(d_1, d_2) \in D \times D$,

$$d_1 \cdot d_2 \cdot (X(Y(f))(m) - Y(X(f))(m)) = d_1 \cdot d_2 \cdot [X,Y](f)(m),$$

and cancelling the universally quantified d_i's, we get (iii).

A final useful classical result about Lie brackets of vector fields on M

$$(14) \qquad [X, f \cdot Y] = f \cdot [X,Y] + X(f) \cdot Y,$$

(where f is a scalar valued function) is easy to prove if M is a Euclidean module and infinitesimally linear. I do not know how to prove it without the module structure on M.

A function $f: M \to V$ (M and V infinitesimally linear, M satisfying the Requirement, V being a Euclidean module) is called an <u>integral</u> of the vector field X on M if $X(f) \equiv 0$. This is equivalent to saying that for any

$$t: D \to M$$

which is a vector of the field X, i.e. $X(t(0),-) = t$, the function f is constant on t,

$$f \circ t \equiv f(t(0)).$$

Then the level set $f^{-1}(f(m))$ contains the tangent vector $X(m,-)$ (meaning that $X(m,-): D \to M$ factors through the level set).

An integral $f: M \to V$ of X is called <u>universal</u> if for any other integral $g: M \to W$ of X,

$$g = \omega \circ f$$

for some $\omega: V \to W$ (not necessarily linear). This definition should really be made a <u>local</u> one, but we are not going very far in this direction anyway. It is reasonable to think of the level sets of a universal integral of X as being precisely the streamlines of X (viewed as unparametrized 1-manifolds). Here we shall use "level set of universal integral" as <u>definition</u> of "streamline". We then have

Proposition 5. If the proper vector field X admits the vector field Y, in the sense of the conditions of Theorem 3, then for each $d \in D$, the infinitesimal transformation $Y(-,d): M \to M$ permutes the streamlines of X.

Proof. We have by assumption

$$[X,Y] = \rho \cdot X$$

for some $\rho: M \to A$. Assume $f: M \to V$ is a universal integral. We claim $Y(f)$ is an integral also. For

$$\begin{aligned} 0 \equiv \rho \cdot X(f) &= (\rho \cdot X)(f) \\ &= [X,Y](f) = X(Y(f)) - Y(X(f)) \\ &= X(Y(f)) - Y(0) \\ &= X(Y(f)), \end{aligned}$$

using Proposition 4 (ii) and (iii). By universality of f we get $\omega: V \to V$ with

$$Y(f) = \omega \circ f.$$

Now we claim that $Y(-,d)$ takes the level set $f^{-1}(c)$ into $f^{-1}(c+d \cdot \omega(c))$. For, let $f(m) = c$. Then

$$\begin{aligned} f(Y(m,d)) &= f(m) + d \cdot Y(f)(m) \\ &= f(m) + d \cdot \omega(f(m)) \\ &= f(m) + d \cdot \omega(c). \end{aligned}$$

Since $Y(-,d)$ is bijective, we actually get that it takes the level set $f^{-1}(c)$ onto $f^{-1}(c+d\cdot\omega(c))$.

This proves the Proposition. Of course, we have no way presently of proving existence of universal integrals.

The use of Theorem 3 for differential equations [2] is that for the case $M =$ the plane $A \times A$, if Y permutes X in the sense of Theorem 3 or Proposition 5, then the function which to $m \in M$ associates the reciprocal of the determinant of (the principal parts of) the two vectors $X(m), Y(m)$ in A^2 is an integrating factor for the differential equation, $X(f) = 0$, meaning that

$$\frac{1}{\det(X,Y)} \cdot X$$

is a source-free vector field, and thus an integral for it, and thus for X, can be found by curve integration (the orthogonal field is a gradient field: its potential function will work).

Lie states [1] that he found these theorems "by synthetic considerations" but found it difficult to write down the proofs synthetically, whence his articles present mainly analytic proofs in coordinates. I believe that the above proofs may be closely related to the synthetic theories of Lie.

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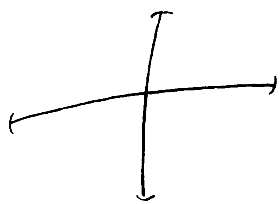
CONNECTIONS IN FORMAL DIFFERENTIAL GEOMETRY

A. Kock and G.E. Reyes*

This paper is a contribution to formal or synthetic differential geometry (see [4], [5], [6], [7], [9]). We recall that the basic idea (suggested by Lawvere [7]) is to work in a category with a ring object A ("the line") and an object D ("the generic tangent vector") by means of which one may interpret directly geometric entities on suitable objects M ("manifolds") in the category M, by performing simple operations of the category on A, D, M. The tangent bundle of M becomes M^D, etc. In this paper we study connections, parallel translations on "vector bundles", covariant differentiation, and related ideas in this synthetic context. Thus, in § 1, the notion of connection on the tangent bundle of an object M is defined as a data which completes each infinitesimal configuration in M

(0.1) 

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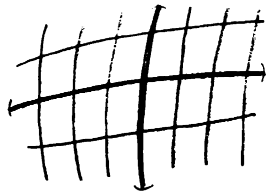
*) Partially supported by a grant of The National Research Council of Canada

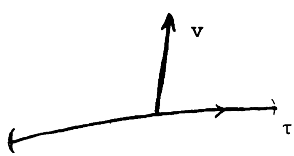
into a configuration

(0.2)

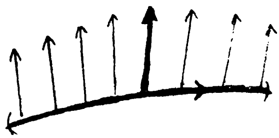


Such completion data in our context is simply a splitting ∇ of the restriction map $M^{D\times D} \longrightarrow M^{DvD}$ where $D\times D$ and DvD are certain objects derived out of D. From this idea (and the pictures (0.1) and (0.2) associated to them) it is immediate that the completion data ∇ provides the infinitesimal germ of parallel transport (the picture (0.2) being full of small parallellograms). Alternatively, under suitable assumptions on M, $M^D \simeq M\times V$ (at least "locally" in a sense to be explained in §3) where V is a vector space (= A-module object in the category). Under this identification the data of a connection becomes a splitting of a certain map $(M\times V)^D = M^D\times V^D \longrightarrow M^D\times V$, so pictorially is a data which completes

(0.3)



into

(0.4) 

This reflects the distinction between the "active" and "passive" tangent vector in a parallel transport situation; thus in (0.3), τ is the active aspect, the one that transports.
The v in (0.3) is the passive vector (the one that is transported in (0.4)). The passive tangent vectors may be replaced by vectors is an arbitrary vector bundle E over M, cf. §2. (A few more pictures appear in §6).

For some of the technical work with this data, we derive from it an equivalent data, namely a connection map C in the sense of Dombrowski [2] and Patterson [8]. In particular we use this form of the data to define covariant differentiation, and, under natural assumptions on M and ∇, to prove Koszul's laws for it. We also discuss torsion and curvature. Since infinitesimals occur explicitely (in the form of the object D), the geometric interpretation of the curvature tensor of E. Cartan [1] is obtained rigourously.

A remark on method: since all categorical operations we perform rest entirely on use of pull-backs and cartesian closedness (exponential objects), all equational arguments and constructions can be performed as if we were in the category of sets (which we are not: in the category of sets, no models for synthetic differential geometry exist, essentially because in the category of sets non-differentiable mappings exist). Thus, we work with elements, just as in [4], [5], [6], [9].

We would like to thank G.C. Wraith for several discussions on this subject. Furthermore, the second author would like to thank J.-M. Terrier for his obstinate efforts to teach him some differential geometry.

§1. The geometric notion of connection in the tangent bundle.

As in [6], we work in a category $\underline{E}$ with finite inverse limits, with a ring object A. We let $D(n) \rightarrowtail A^n$ be defined by

$$D(n) = \{(a_1,\ldots,a_n) \in A^n \mid a_i \cdot a_j = 0 \quad \forall i,j = 1,\ldots,n\},$$

(using set theoretical notation). We shall in particular be interested in D(1) and D(2), which we also denote D and DvD, respectively. The 0 of A, $\mathbb{1} \xrightarrow{0} A$ (where $\mathbb{1}$ is the terminal object) factors through D, $0: \mathbb{1} \longrightarrow D$. Similarly, for D(n). We have furthermore n inclusion maps

$$i_r: D \longrightarrow D(n)$$

given by

$$d \longmapsto (0,0,\ldots,d,\ldots,0)$$

with d placed in the r'th position. We shall assume from now on that D(n) is exponentiable for any n.

An object M is <u>infinitesimally linear</u> (cf. [6]) if for each n

$$M^{D(n)} \begin{array}{c} \xrightarrow{M^{i_1}} \\ \vdots \\ \xrightarrow[M^{i_n}]{} \end{array} M^D$$

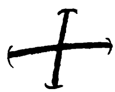
makes $M^0: M^{D(n)} \longrightarrow M$ into an n-fold product of $M^0: M^D \longrightarrow M$, in $\underline{E}/M$. (We denote $M^0: M^D \longrightarrow M$ by π; geometrically, it associates to a tangent vector the point of M where it is attached).

As in [4], we say that A is of <u>line type</u> if the map $\alpha: A\times A \longrightarrow A^D$ defined by $\langle a_0, a_1\rangle \longmapsto [d \mapsto [a_0+a_1 d]]$ is invertible. Throughout this paper, we assume that A is of line type, and is infinitesimally linear. Also, M denotes henceforth a fixed but arbitrary infinitesimally linear object in $\underline{E}$ (M is to be thought of as a manifold).

Since M is infinitesimally linear, we have in particular

$$M^{DvD} = M^{D(2)} \cong M^D \times_M M^D$$

so that an "element" in M^{DvD} looks like

(1.1)  ("a cross"),

a pair of tangent vectors t_1, t_2 attached at the same point of M (which is the justification for the notation DvD).

Clearly, we have

$$D \vee D \subseteq D \times D \subseteq A \times A.$$

We denote the inclusion $DvD \subseteq D \times D$ by j. We thus have a restriction map

$$M^{D\times D} \xrightarrow{M^j} M^{DvD}.$$

Definition 1.1. A connection on M is a splitting $M^{DvD} \xrightarrow{\nabla} M^{D\times D}$ of the mapping M^j.

(We shall later add an equational condition on such ∇, defining the notion of affine connection). We gave some of the geometric heuristics of this notion in the introduction. We elaborate a little on it. The (almost) vertical lines in the infinitesimal grid (0.2) are to be viewed as ∇-parallel translates of the given vertical line in (0.1) along the given horizontal line (0.1). ("Line" here means "curve parametrized by the infinitesimal segment D of the global line A" = "tangent vector to M").

Thus, in the picture (0.1) and (0.2), the given horizontal "line" is the active tangent (the one that transports), and the given vertical "line" is the passive vector (the one that is transported).

To arrive at finite parallel transport from this infinitesimal parallel transport given by ∇, of course means integrating a certain differential equation.

To be specific, we want to define the notion of when a curve of tangents to M is ∇-parallel. A curve in any object N is, by definition, a map $A \longrightarrow N$ (since A is "the line"). Since M^D is the tangent bundle of M (cf. [5], [6], [7], [9]), a curve-of-tangents on M is a map

$$(1.3) \qquad A \xrightarrow{k} M^D.$$

Now any curve $h:A \to N$ on any object N determines a curve-of-tangents on N, "the speed curve of h" which we shall also denote h', namely the composite

$$A \xrightarrow{\hat{+}} A^D \xrightarrow{h^D} N^D,$$

where $\hat{+}(a) = [d \longmapsto a+d]$. Take in particular $N = M^D$, $h = k$ as in (1.3); then we get the speed curve of k

$$(1.4) \qquad A \xrightarrow{k'} (M^D)^D \xrightarrow[\cong]{\varphi} M^{D\times D}.$$

(The isomorphism $\varphi^{-1}:M^{D\times D} \longrightarrow (M^D)^D$ here sends $f:D\times D \longrightarrow M$ into $[d_1 \longmapsto [d_2 \mapsto f(d_1,d_2)]]$). On the other hand, if we denote by k_1

the composite of (1.3) with $\pi : M^D \to M$ (the "footpoint curve of k") we get, using k as well as the speed curve k_1' of k_1 a map

$$A \xrightarrow{<k_1',k>} M^D \times_M M^D \cong M^{D \vee D} \tag{1.5}$$

<u>Definition 1.2</u>. The curve-of-tangents k is parallel according to ∇ if "∇-parallel transport of k-vectors along k_1' yield k-vectors", or more precisely, if (1.5) composed with $\nabla : M^{D\vee D} \longrightarrow M^{D\times D}$ yields (1.4), that is, if

$$\nabla \circ <k_1',k> = \varphi \circ k' .$$

In particular

<u>Definition 1.3</u>. A curve $h : A \longrightarrow M$ is <u>geodesic</u> with respect to ∇ if its speed curve h' is a ∇-parallel curve-of-tangents, that is, if

$$\nabla \circ <h',h'> = \varphi \circ h'' .$$

A proposition concerning the "parameter invariance" of the notions of the two last definitions will be proved in Proposition 2.9 under the assumption that ∇ is an <u>affine</u> connection (Definition 2.7).

The notion of ∇-parallel curve of tangents, and ∇-geodesic, can be relativized for curves

$$k\colon U \longrightarrow M^D$$
$$h\colon U \longrightarrow M$$

defined on subobjects U of A. In fact, if $U' \subseteqq U$ is so that "adding elements from D to elements from U' yields elements of U", i.e., if there exists a factorization

$$\begin{array}{ccc} U' \times D & \overset{+}{\cdots\cdots\to} & U \\ \downarrow & & \downarrow \\ A \times A & \xrightarrow[+]{} & A \end{array}$$

we derive

$$(1.4') \qquad U' \xrightarrow{\hat{+}} U^D \xrightarrow{k^D} (M^D)^D \simeq M^{D \times D}$$

and

$$(1.5') \qquad U' \xrightarrow{\langle k_1', k\rangle} M^D \times_M M^D \cong M^{D \vee D} \xrightarrow{\nabla} M^{D \times D}.$$

The curve k is then (generalizing Definition 1.2) said to be ∇-parallel-according to ∇ on U' if $(1.4')$ and $(1.5')$ agree. Note in particular that we may take $U = D$, $U' = \{0\}$ and talk about when a curve $k\colon D \longrightarrow M^D$ (a very short curve!) is ∇-parallel at 0.

Similar relativization can be made for the notion of geodesic. There we need $U' \subseteqq U$ stable under addition of <u>two</u> elements from D (because of occurrence of the double derivative in Definition 1.3).

A vector field on M is a cross-section X of the map $\pi\colon M^D \to M$. Given two vector fields X and Y, one can consider the diagram

$$(1.6) \qquad \begin{array}{ccccc} M & & \xrightarrow{\quad X \quad} & & M^D \\ {\scriptstyle \langle X,Y \rangle}\Big\downarrow & & & & \Big\downarrow {\scriptstyle Y^D} \\ M^{D \vee D} & \xrightarrow[\nabla]{} & M^{D \times D} & \xrightarrow[\varphi]{\cong} & (M^D)^D \end{array}$$

The difference of the two ways round in this diagram (taken fibrewise in the tangent bundle of M^D, $(M^D)^D \longrightarrow M^D$) will lead to the notion of covariant differentiation, $\nabla_X Y$, of the vector field Y along the vector field X, cf. Definition 2.6 below. The geometric heuristics can in our context be objectivized as follows: given $m \in M$. Then $X(m): D \to M$ is a (small piece of) a curve ("integral curve of X through m") in M, and $Y \circ X(m): D \longrightarrow M^D$ is a (small) curve k of Y-vectors along it. On the other hand, the counterclockwise composite in (1.6) is (modulo the isomorphism φ) a map $D \times D \to M$, the grid obtained by parallel transporting Y-vectors along the tangent vector $X(m)$.

So the difference of the two ways round in (1.6) measures how much the curve of Y-vectors along the integral curves of X differs from being parallel according to ∇. (Note that we do not really integrate the vector field X, since we only need to know the integral curves on small bits of length D anyway).

The difference of the two ways round in (1.6) is a map $M \longrightarrow (M^D)^D$ or $M \longrightarrow M^{D \times D}$, so is not yet a vector field on M, but rather a "grid field". To extract a vector field $\nabla_X Y$ from it, we need some preparations of technical nature. For these technical preparations it is convenient for notational reasons, to generalize slightly in the sense that the"passive tangent vector" is replaced by a vector in an arbitrary vector bundle $E \longrightarrow M$ (= A-module object in $\underline{E}/M$).

§2. Tangent bundles of vector bundles.

In the following, $p:E \to M$ is a fixed map in $\underline{E}$ with both the objects E and M infinitesimally linear. We shall also assume that $E \longrightarrow M$ is equipped with a vector bundle structure (that is, structure of A-module object in $\underline{E}/M$). However, for our first proposition, this structure is not needed:

Proposition 2.1. The object $p:E \longrightarrow M$ in $\underline{E}/M$ is infinitesimally linear.

Proof. Let us denote by $(\ )_M$ the functor $\underline{E} \longrightarrow \underline{E}/M$ "crossing with M". It preserves those exponentials that exist, and inverse limits, so that $(A)_M$ is a ring object of line type in $\underline{E}/M$. Of course, the statements we make about formal differential geometry in $\underline{E}/M$ refers to this ring object. Now, given

$$t_i : (D)_M = D \times M \longrightarrow E \qquad (i = 1,2)$$

in $\underline{E}/M$ (so that $p \circ t_1 = p \circ t_2$ = projection to second factor) with $t_1(0,m) = t_2(0,m)$ (= $t(m)$, say) for all m, we should prove unique existence of an

$$\ell : (D \vee D)_M \longrightarrow E \qquad \text{in } \underline{E}/M$$

restricting to t_1 and t_2 on the two axes of $(D \vee D)_M$. For each $m \in M$, we get by infinitesimal linearity of E a map $\ell_m : D \vee D \longrightarrow E$ restricting to $t_1(-,m)$ and $t_2(-,m)$. The ℓ_m's together define a map $(D \vee D) \times M \longrightarrow E$. We must prove that it is a map in $\underline{E}/M$, meaning that we should prove commutativity of

$$\begin{array}{ccc} (D \vee D) \times M & \xrightarrow{\ell} & E \\ & \underset{\text{proj}}{\searrow} & \downarrow p \\ & & M. \end{array}$$

This means that we should prove that $p \circ \ell_m : D \vee D \longrightarrow M$ has constant value (= m). On the two axed of $D \vee D$, this is certainly so, since ℓ_m restricts to t_1 and t_2 and $p(t_i(d,m)) = m$ for $i = 1,2$, since t_1 and t_2 are maps in $\underline{E}/M$. Now, knowing that $p \circ \ell_m$ restricts to the map "constant m" on the axes of $D \vee D$ implies, by the uniqueness assertion in infinitesimal-linearity assumption on M that $p \circ \ell_m$ must be constant m on the whole of $D \vee D$, as was to be proved. The uniqueness of ℓ follows just from the infinitesimal linearity of E.

We remind the reader (cf. [6] and [9]) that if N is an infinitesimally linear object, then $\pi : N^D \longrightarrow N$ has a natural vector bundle structure: given two tangent vectors t_1 and t_2 at $m \in M$ (so $t_i : D \longrightarrow M$ with $t_i(0) = m$, $i = 1,2$), we first find the unique $\ell : D \vee D \longrightarrow M$ which restricts to t_1 and t_2 on the axes of $D \vee D$; and then we define $t_1 + t_2$ by $(t_1 + t_2)(d) = \ell(d,d)$. We call this the tangential addition. In the present paper it is denoted by $\oplus$:

$$(t_1 \oplus t_2)(d) = \ell(d,d) \qquad d \in D .$$

The associated "multiplication by scalars from A" is denoted $\odot$ and given by

$$(a \odot t)(d) = t(a \cdot d) \qquad d \in D$$

(note $d \in D \Rightarrow a \cdot d \in D$).

We now utilize the vector bundle structure on $p:E \longrightarrow M$ to derive a natural diagram associated to it. We largely use notation from [3]. The diagram in question is

$$E \times_M E \xrightarrow{H} E^D \xrightarrow{K} M^D \times_M E \tag{2.1}$$

where H and K are given (in set theoretic notation) as follows:

$$H: \langle \underline{u}, \underline{v} \rangle \longmapsto [d \mapsto \underline{u} + d \cdot \underline{v}]$$

where $\underline{u}$ and $\underline{v}$ are in the same fibre of $E \longrightarrow M$, and $\underline{u} + d \cdot \underline{v}$ refers to the A-module structure of that fibre. Next,

$$K: f \longmapsto \langle p \circ f, f(0) \rangle$$

where $f: D \longrightarrow E$. (Recall that $f(0)$ is also denoted $\pi(f)$).

In the case where $E \longrightarrow M$ is $M^D \longrightarrow M$, it is easy to see that K equals the restriction map $M^{D \times D} \longrightarrow M^{D \vee D}$, modulo the canonical isomorphisms $\varphi: M^{D \times D} \cong (M^D)^D$ and $M^{D \vee D} \simeq M^D \times_M M^D$. Therefore we can generalize Definition 1.1 into

<u>Definition 2.2</u>. A connection on the vector bundle $E \longrightarrow M$ is a splitting ∇ of the map K in (2.1).

Pictorially, this can be represented as data for completing (0.3) into (0.4).

Let us note that each of the three objects occurring in (2.1) carry <u>two</u> vector bundle structures (over different bases), which in set theoretic notation may be tabulated as follows

(2.2):

structure map	fibrewise addition	notation
$proj_1: E\times_M E \longrightarrow E$	$((\underline{u},\underline{v}_1),(\underline{u},\underline{v}_2)) \longrightarrow (\underline{u},\underline{v}_1+\underline{v}_2)$	$\oplus$
$poproj_2: E\times_M E \longrightarrow M$ (= $poproj_1$)	$((\underline{u}_1,\underline{v}_1),(\underline{u}_2,\underline{v}_2)) \longrightarrow (\underline{u}_1+\underline{u}_2,\underline{v}_1+\underline{v}_2)$	$+$
$\pi: E^D \longrightarrow E$	tangential addition	$\oplus$
$p^D: E^D \longrightarrow M^D$	$(f,g) \longrightarrow [d \mapsto f(d) + g(d)]$	$+$
$proj_1: M^D\times_M E \longrightarrow M^D$	$((t,\underline{v}_1),(t,\underline{v}_2)) \longrightarrow (t,\underline{v}_1+\underline{v}_2)$	$+$
$proj_2: M^D\times_M E \longrightarrow E$	$((t_1,\underline{v}),(t_2,\underline{v})) \longrightarrow (t_1\oplus t_2,\underline{v})$	$\oplus$

In each case, it is understood that the entries live in the same fibre for the relevant structure map, so that the indicated operations can be performed. For instance, in the last case, $t_1(0)=t_2(0)=p(\underline{v})$.

In the following proposition are implicit the following easily seen commutation relations:

$$proj_1 \circ K = p^D \qquad \text{and} \qquad proj_2 \circ K = \pi$$

with K as in (2.1).

<u>Proposition 2.3</u>. The map K is linear with respect to the structures denoted $\oplus$, as well as with respect to the structures denoted $+$.

Proof. Linearity of K with respect to $\oplus$ follows because $K = \langle p^D, \pi \rangle$ and $p^D: E^D \longrightarrow M^D$ is linear with respect to tangential addition, by functorality of the vector-bundle construction $(-)^D$. To see the second assertion, let f,g be given tangent vectors to E in the same fibre of $E^D \longrightarrow M^D$, meaning $p \circ f = p \circ g$. ($= t$, say; $t: D \longrightarrow M$). Then

$$\begin{aligned} K(f+g) &= \langle p \circ (f+g), (f+g)(0) \rangle \\ &= \langle t, f(0)+g(0) \rangle \\ K(f)+K(g) &= \langle p \circ f, f(0) \rangle + \langle p \circ g, f(0) \rangle \\ &= \langle t, f(0) \rangle + \langle t, g(0) \rangle . \end{aligned}$$

which are the same.

In the following Proposition is implicit the following easily seen commutativly: $\pi \cdot H = \mathrm{proj}_1$.

Proposition 2.4. The map H is linear with respect to the $\oplus$-structure, as well as with respect to the $+$-structure.

Proof. Given $(\underline{u}, \underline{v}_1)$ and $(\underline{u}, \underline{v}_2)$. Then

$$H((\underline{u}, \underline{v}_1) \oplus (\underline{u}, \underline{v}_2)) = H(\underline{u}, \underline{v}_1 + \underline{v}_2) = [d \longmapsto \underline{u} + d \cdot (\underline{v}_1 + \underline{v}_2)].$$

On the other hand, to compute the $\oplus$-sum of

$$H(\underline{u}, \underline{v}_1) = [d \longmapsto \underline{u} + d \cdot \underline{v}_1]$$

and

$$H(\underline{u}, \underline{v}_2) = [d \longmapsto \underline{u} + d \cdot \underline{v}_2],$$

we need to find an $\ell: D \vee D \longrightarrow E$ which restricts to these two mappings, and then look at $d \longmapsto \ell(d,d)$. Now ℓ is easily given explicitely: it is clear that

$$\ell(d_1,d_2) := \underline{u} + d_1\cdot\underline{v}_1 + d_2\cdot\underline{v}_2$$

will work. Setting $d_1 = d_2 = d$ gives $\underline{u} + d\cdot\underline{v}_1 + d\cdot\underline{v}_2$ again.

The proof of linearity of H with respect to the +-structure is trivial.

Recall [5], [9] that if A is a ring object in a category $\underline{E}'$, and N is an A-module object, then we say that N is <u>Euclidean</u> if the map

$$\alpha: N\times N \longrightarrow N^D$$

given by

$$(\underline{u},\underline{v}) \longmapsto [d \mapsto \underline{u} + d\cdot\underline{v}]$$

is invertible (with $D = \{a \in A \mid a^2 = 0\}$).

These notions in particular apply to the A-module object $E \longrightarrow M$ in $\underline{E}/M$. It is clear that the H in (2.1) is closely related to α for this object.

We have

<u>Proposition 2.5</u>. If $p: E \longrightarrow M$ is Euclidean in $\underline{E}/M$ ("E is fibrewise Euclidean"), the sequence (2.1) is left exact with respect to the addition structures given by $\oplus$.

<u>Proof</u>. If Z and Y are objects in $\underline{E}/M$ such that the exponential object Y^X exists in $\underline{E}/M$, we shall denote it $X \pitchfork_M Y$. Now it is easy to see that the object in $\underline{E}/M$

$$D_M \pitchfork_M (E \longrightarrow M)$$

is the left hand vertical arrow in a pull-back diagram in $\underline{E}$

$$\begin{array}{ccc} Q & \longrightarrow & E^D \\ \downarrow & \text{p.b} & \downarrow p^D \\ M & \underset{\Delta}{\longrightarrow} & M^D \end{array}$$

Δ being the exponential adjoint of proj: $M \times D \to M$; so Δ is the zero for the vector bundle structure on $M^D \longrightarrow M$, i.e. the structure $\oplus$. Thus, Q is the kernel of p^D (for the structure $\oplus$). On the other hand, the Euclidean-ness of $E \to M$ says $E \times_M E \cong D_M \pitchfork_M (E \to M)$ via H, so that, under this identification, $E \times_M E$ (more precisely, H) is kernel for p^D, or equivalently for K, with respect to the structures $\oplus$.

We now consider a fibrewise Euclidean vector bundle object $p: E \to M$ (with E and M infinitesimally linear, as always). Since (2.1) is left exact with respect to the $\oplus$-structures, it follows that if we have a connection

$$\nabla : M^D \times_M E \longrightarrow E^D$$

on E, the difference (with respect to the structure $\oplus$; we denote the corresponding subtraction by $\ominus$)

(2.3) $$\mathrm{id}_{E^d} \ominus \nabla \circ K$$

factors through the kernel H of K, whence we get a map

$$C_1 : E^D \longrightarrow E \times_M E,$$

and therefore also a map $\mathrm{proj}_2 \circ C_1 : E^D \longrightarrow E$, which we denote C,

$$C : E^D \longrightarrow E,$$

from which the original ∇ can be reconstructed. Certain of the calculations we shall make are more readily expressed in terms of C than in terms of ∇. However, the geometric meaning of C is less direct than that of ∇. The analogue of C in "classical" differential geometry we learned from Dombrowski [3] and Patterson [8] . Also, the following notions, expressed in terms of C we learned from [8].

Definition 2.6. Let $X: M \to M^D$ be a tangent vector field and $Y: M \longrightarrow E$ an E-vector field. Then $\nabla_X Y: M \longrightarrow E$ is defined to be the following composite

$$M \xrightarrow{X} M^D \xrightarrow{Y^D} E^D \xrightarrow{C} E,$$

called the covariant derivative of Y along X (with respect to ∇; C is derived as above from ∇).

This definition can easily be extended to "partially defined vector fields", i.e., given

$$X: N \longrightarrow M^D \quad , \quad Y: N \longrightarrow E$$

with $\pi o X = p o Y$ (= h, say), we can define $\nabla_X Y: N \longrightarrow E$ with $p o \nabla_X Y = h$.

In the case where $E \longrightarrow M$ is $M^D \longrightarrow M$, the difference considered in a preliminary way in (1.6) can be obtained from the difference (2.3), so that Definition 2.6 above provides a way of getting a vector field $\nabla_X Y$ from "deviation of Y being parallel along the integral curves of X", as alluded to earlier.

Now, if we have a connection ∇ on $E \to M$, that is, a splitting of K in (2.1), it follows that K is split epic. Also, because ∇ is a splitting of K, it is easily seen that ∇ preserves the fibrations of E^D and $M^D \times_M E$ over E as well as their fibrations over M^D, whence it makes sense to ask whether ∇ preserves the linear structures given in the table (2.2).

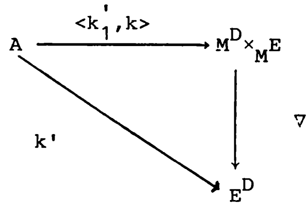
Definition 2.7. We say ∇ is an affine connection on $p: E \to M$ if it preserves both the linear structures $+$ and $\oplus$.

Remark. For the case where $E = M^D$, it is geometrically reasonable to ask that ∇ preserves the structure $+$ (linear structure on passive tangent vectors). It can be viewed as an infinitesimal version of the statement: parallel transport along a path from m_1 to m_2 defines a linear map from the tangent-space at m_1 to the tangent-space at m_2.

Concerning the linearity of ∇ w.r. to the structure $\oplus$ ("linearity with respect to the active tangent vectors"), we can best understand its geometric significance by using it to prove parameter-invariance of the notion of ∇-parallel-curve of tangent vectors, Proposition 2.9 below. We may as well do this for the more general case of an (affine) connection in the fibrewise Euclidean vector bundle $p: E \to M$. We need first generalize Definition 1.2.

Let $k: A \to E$ be a curve in E, and denote by k_1 the composite $p \circ k$.

Definition 2.8. We say k is a parallel-curve with respect to ∇ if the following diagram commutes



Proposition 2.9. Let $k: A \longrightarrow E$ be a parallel curve w.r.t. to the affine connection ∇, and let $f: A' \rightarrow A$ be arbitrary. Then the curve $h = k \circ f: A' \rightarrow E$ is parallel w.r.t. to ∇.

Proof. We have $h_1 = k_1 \circ f$. We must prove

$$\nabla \circ < h_1', h_1 > = h'$$

We compute on the left hand side

$$\begin{aligned}
\nabla \circ < h_1', h > &= \nabla \circ < (k_1 \circ f)', k \circ f > \\
&\overset{*}{=} \nabla \circ < (k_1' \circ f) \odot f', k \circ f > \\
&\overset{**}{=} f' \odot (\nabla \circ < k_1' \circ f, k \circ f >) \\
&= f' \odot (\nabla \circ < k_1', k > \circ f) \\
&= f' \odot (k' \circ f) \\
&\overset{*}{=} (k \circ f)' = h',
\end{aligned}$$

as desired. At the two equality signs marked * we have used an evident and easily proved chain rule analogous to that of [4], and at ** we have used the $\oplus$-linearity of ∇ ($\odot$ denotes that multiplication by scalars that goes together with the addition $\oplus$).

Proposition 2.10. Let ∇ be an affine connection. Then $C: E^D \longrightarrow E$ satisfies both possible linearity laws.

Proof. Since $C_1 = id_{E^D} \ominus (\nabla \circ K)$, it is clear from $\oplus$-linearity of K and ∇ that C_1 is $\oplus$-linear: $E^D \longrightarrow E\times_M E$. Then since the $\oplus$-structure on $E\times_M E$ is just the structure of the second factor, it is clear that $proj_2 \cdot C_1$ sends $\oplus$-structure of E^D to the (unique) structure $+$ of E (everything fibrewise over M).

To see the other linearity condition, let $f,g \in E^D$, with $p\circ f = p\circ g$ (= $t: D \longrightarrow M$, say), so that f and g can be added according to the structure $+$ on E^D. We must prove

$$C(f+g) = C(f) + C(g).$$

It suffices to prove

$$H(C_1(f+g)) = H(C_1(f) + C_1(g))$$

Since H is linear w.r. to the $+$-structure, and $H\circ C_1 = id_{E^D} \ominus \nabla\circ K$, we are required to prove

$$(f+g) \ominus \nabla(K(f+g)) = (f \ominus \nabla Kf) + (g \ominus \nabla Kg).$$

Since K is linear with respect to $+$, we may rewrite the left hand side, so that our problem now is whether

$$(f+g) \ominus (\nabla Kf + \nabla Kg) = (f \ominus \nabla Kf) + (g \ominus \nabla Kg). \tag{2.4}$$

The result then follows from a distributivity law between the two structures $+$ and $\oplus$ on E^D, which is expressed in the following

Lemma 2.11. Suppose $f_1, g_1, f_2, g_2: D \rightarrow E$ are tangent vectors to E, and that

(a) $\qquad p\circ f_i = p\circ g_i \quad (= t_i,$ say, $t_i: D \rightarrow M) \qquad$ for $i = 1,2$

and that

(b) $$f_1(0) = f_2(0) \quad \text{and} \quad g_1(0) = g_2(0).$$

Then all additions occurring in the following equation can be performed, and the equation holds:

$$(f_1 + g_1) \oplus (f_2 + g_2) = (f_1 \oplus f_2) + (g_1 \oplus g_2).$$

Similarly if $\oplus$ is replaced by $\ominus$.

Proof. Tangential addition is natural with respect to maps between infinitesimally linear objects (cf. [9]). Therefore,

$$(E\times_M E)^D \xrightarrow{(+)^D} E^D$$

is a fibrewise linear map with respect to the $\oplus$ structure. (Note that $E\times_M E$ is infinitesimally linear since E and M are). But the vector bundle $\pi: (E\times_M E)^D \longrightarrow E\times_M E$ can be identified with

$$E^D \times_{M^D} E^D \longrightarrow E\times_M E$$

with addition in the domain being given by

$$(f_1,f_2),(g_1,g_2) \longmapsto (f_1 \oplus f_2, g_1 \oplus g_2)$$

where $f_1,\ldots,g_2$ satisfy (a) and (b). The lemma now easily follows.

To apply the lemma in proving (2.4), we just have to verify the conditions (a) and (b) which here say

(a) $$\begin{aligned} p\circ f &= p\circ g \\ p\circ \nabla K f &= p\circ \nabla K g \end{aligned}$$

(b) $$f(0) = (\nabla K f)(0)$$ (and similarly for g).

It is clear that (b) holds; and (a) follows because all four expressions are equal to $t: D \longrightarrow M$. This proves Proposition 2.10.

§3. Koszul's law.

In this section, $p: E \to M$ denotes a fibrewise Euclidean vector bundle (cf. Proposition 2.5), and $\nabla: M^D \times_M E \longrightarrow E^D$ denotes an affine connection on it.

We defined on basis of this the covariant differentiation structure $\nabla_X Y$. We shall prove the following identities

$$\nabla_{X_1 \oplus X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y \tag{3.1}$$

$$\nabla_{f \odot X} Y = f \cdot \nabla_X Y \tag{3.2}$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \tag{3.3}$$

and, under a further assumption of local trivializability of $E \longrightarrow B$ (see below), the "Koszul law"

$$\nabla_X (f \cdot Y) = f \cdot \nabla_X Y + X(f) \cdot Y. \tag{3.4}$$

In this latter, f denotes a map $M \to A$, and $X(f)$ denotes the derivation of f in the direction of the vector field X, [5], that is, the composite

$$M \xrightarrow{\ X\ } M^D \xrightarrow{\ f^D\ } A^D \simeq A \times A \xrightarrow{\ proj_2\ } A. \tag{3.5}$$

Proposition 3.1. The equations (3.1),(3.2),(3.3) hold for an arbitrary connection ∇.

Proof. Let C denote the Dombrowski-Patterson connection map associated with ∇. To prove (3.1), recall that by definition of covariant differentiation, the left hand side of (3.1) denotes the composite map

$$(3.4) \qquad M \xrightarrow{X_1 \oplus X_2} M^D \xrightarrow{Y^D} E^D \xrightarrow{C} E.$$

Now Y^D is linear with respect to the $\oplus$ structure, by functorality of the tangent-bundle construction $(\ )^D$. Also C has a linearity property w.r.t. to $\oplus$, by Proposition 2.10. Thus (3.4) can be written $(C \circ Y^D \circ X_1) + (C \circ Y^D \circ X_2)$, which is just the right hand side of (3.1). The same argument proves (3.2). To see (3.3), note that the left hand side of (3.3) denotes

$$M \xrightarrow{X} M^D \xrightarrow{(Y+Z)^D} E^D \xrightarrow{C} E,$$

but for the $+$ structure on E^D, clearly $(Y+Z)^D = Y^D + Z^D$. The result is now clear from the other linearity property of C.

The proof of (3.4) (in the cases where we can prove it), depends not surprisingly on a Leibniz rule for differentiation. Let F be a Euclidean A-module. To any

$$g: D \longrightarrow F$$

we can (as in [5]) associate $g'(0): \mathfrak{A} \longrightarrow F$. If also $f: D \longrightarrow A$ is given, then one proves easily (much as in [4]) the

Leibniz rule: The following two maps $D \longrightarrow F$ agree:

$$d \longmapsto f(d) \cdot g(d)$$

$$d \longmapsto f(0) \cdot g(0) + d \cdot (f'(0) \cdot g(0) + f(0) \cdot g'(0))$$

or equivalently

$$(f\cdot g)'(0) = f'(0)\cdot g(0) + f(0)\cdot g'(0).$$

Remark that in the ( )'-notation, the derivation $X(f)$ of f in direction of the field X introduced in (3.5) can be written

$$X(f)(m) = (f\circ X(m))'(0).$$

We now prove a special case of (3.4), namely for trival bundles (product bundles).

Proposition 3.2. Assume $E \longrightarrow M$ is of form $\mathrm{proj}_1 : M\times F \longrightarrow M$ with F a Euclidean A-module. Then (3.4) holds.

Proof. The section $Y: M \longrightarrow E = M\times F$ can be written $<\mathrm{id}, Y_2>$, where $Y_2: M \longrightarrow F$ ("the principal part of Y"). Furthermore

$$E^D = (M\times F)^D \cong M^D\times F^D \cong M^D\times F\times F.$$

Under this identification, the linear structure $+$ on E^D is simply given by the linear structure on $F\times F$, whereas the linear structure $\oplus$ on E^D is given by the linear structure of M^D over M and the linear structure on the last factor F:

$$(t_1,\underline{u},\underline{v}_1) \oplus (t_2,\underline{u},\underline{v}_2) = (t_1 \oplus t_2,\underline{u},\underline{v}_1 + \underline{v}_2).$$

Under the identification, $H: E\times_M E \longrightarrow E^D$ can be described $(m,\underline{u},\underline{v}) \longmapsto (0_m,\underline{u},\underline{v})$ where 0_m denotes the zero vector for the $\oplus$ structure in the fibre over m. We have further that $Y^D\circ X: M \longrightarrow E^D$ can be described as

$$m \longmapsto <X(m), Y_2(m), (Y_2\circ X(m))'(0)>$$

whence by definition of $\nabla_X Y$ in terms of C

$$(\nabla_X Y)(m) = C(X(m), Y_2(m), (Y_2 \circ X(m))'(0)).$$

Note that for $m \in M$, we have $X(m): D \longrightarrow M$, so that the succession of symbols makes sense.

Since

$$(f \cdot Y_2) \circ X(m) = (f \circ X(m)) \cdot (Y_2 \circ X(m))$$

we get from the Leibniz rule that

$$((f \cdot Y_2) \circ X(m))'(0) = (f \circ X(m))'(0) \cdot Y_2(m) + f(m) \cdot (Y_2 \circ X(m))'(0)$$

(note $X(m)(0) = m$, so $(Y_2 \circ X(m))(0) = Y_2(m)$).

Thus

$$(3.6) \quad \nabla_X(f \cdot Y) = C(X(m), f(m) \cdot Y_2(m), (f \circ X(m))'(0) \cdot Y_2(m) + f(m) \cdot (Y_2 \circ X(m))'(0)).$$

Let us denote by O_m the zero vector over M in M^D. It is the map $D \longrightarrow M$ given by $d \mapsto m$. Then using the $\oplus$-linearity of C, we may rewrite (3.6) as C applied to the expression

$$(O_m, f(m) \cdot Y_2(m), (f \circ X(m))'(0) \cdot Y_2(m))$$
$$\oplus$$
$$(X(m), f(m) \cdot Y_2(m), f(m) \cdot (Y_2 \circ X(m))'(0))$$

$$= (O_m, f(m) \cdot Y_2(m), O) + (O_m, O, (f \circ X(m))'(0) \cdot Y_2(m))$$
$$\oplus$$
$$f(m) \cdot (X(m), Y_2(m), (Y_2 \circ X(m))'(0)).$$

Note that the first of our three terms in the zero vector for the $\oplus$ structure. Now applying C and using its linearity with respect to both the $+$ and the $\oplus$ structure (Proposition 2.10) yields

$$C(O_m,O,(f\circ X(m))'(0)\cdot Y_2(m)) \;+\; f(m)\cdot C(X(m).Y_2(m),(Y_2\circ X(m))'(0))$$

$$= \; C(H(\,m,O,(f\circ X(m))'(0)\cdot Y_2(m)))+ f(m)\cdot C(X(m),Y_2(m),(Y_2\circ X(m))'(0))$$

$$= \; (f\circ X(m))'(0)\cdot Y_2(m) \;+\; f(m)\cdot C(X(m),Y_2(m),(Y_2\circ X(m))'(0)),$$

using

(3.7) $$C(H(m,o,\underline{v})) = proj_2(C_1(H(m,o,\underline{v})) = proj_2(m,o,\underline{v}) = (m,\underline{v})$$

which we denote just $\underline{v}$, m being understood. Thus we get

$$X(f)(m)\cdot Y_2(m) + f(m)\cdot C(X(m),Y_2(m),(Y_2\circ X(m))'(0)$$

$$= X(f)(m)\cdot Y_2(m) \;+\; f(m)\cdot(\nabla_X Y)(m),$$

which proves the Proposition.

We can prove the Koszul law (3.4) for bundles $E \to M$ which only locally are trivial. We call a vector bundle $p\colon E \to M$ locally trivial if there exists an epic étale* map $\mu: M' \to M$ and a pull-back diagram of form

(3.8) $$\begin{array}{ccc} E' = (M'\times F) & \xrightarrow{\;\varepsilon\;} & E \\ \downarrow & & \downarrow p \\ M' & \xrightarrow[\mu]{} & M \end{array}$$

with F a Euclidean A-module and ε fibrewise linear. Even without using μ epic, it is easy to see that a connection ∇ on $E \to M$ gives rise to a connection ∇' on $E' \to M'$ which is affine if ∇ is (to define $\nabla'\colon E'\times_M E' \to (E')^D$, one needs that $(E')^D$ sits in a pull-back diagram

---

*) For this notion, see [6].

$$\begin{array}{ccc} (E')^D & \longrightarrow & E^D \\ \downarrow & & \downarrow \\ E' & \underset{\varepsilon}{\longrightarrow} & E \end{array}$$

which is a consequence of ε being étale (which in turn follows from μ being étale and (3.8) being a pull-back). Also, C belonging to ∇ pulls back to C' belonging to ∇', vector fields $X: M \to M^D$ and $Y: M \longrightarrow E$ pull back to vector fields $X': M' \to M'^D$ (using μ étale) and $Y': M' \longrightarrow E'$.

Finally, if $f: M \to A$, we denote by f' the composite

$$M' \underset{\mu}{\longrightarrow} M \underset{f}{\longrightarrow} A.$$

Using the assumption that $E' \to M'$ is a trivial bundle $M'\times F \to M$ and that Koszul's law holds for trivial bundles (Proposition 3.2), we get

$$\nabla'_{X'}(f'\cdot Y') = f'\cdot\nabla'_{X'}Y' + X'(f')\cdot Y'. \tag{3.9}$$

However, it is easy to prove that pulling back commutes with the operations defined in terms of the connection, so that (3.9) implies

$$(\nabla_X(f\cdot Y))' = (f\cdot\nabla_X Y + X(f)\cdot Y)' \tag{3.10}$$

which expresses an equality of two vector fields $M' \to E'$ that arise by pulling back two vector fields $M \longrightarrow E$ along $\mu: M' \longrightarrow M$. Under the assumption that μ is epic, we therefore conclude equality of the two vector fields $M \to E$, that is, of (3.10) without the primes. But this is (3.4). We have thus proved

<u>Proposition 3.3</u>. Koszul's law (3.4) (and also the laws (3.1)-(3.3)) hold for any affine connection ∇ on a locally trivial vector bundle $E \longrightarrow M$.

§4. Further structure associated to a connection.

From Proposition 2.10 it follows that an affine connection $\nabla: M^D \times_M E \longrightarrow E^D$ in our sense gives rise to a map $C: E^D \longrightarrow E$ which satisfies the formal analogues of the conditions (1) - (2) in Patterson's characterization Theorem ([8], Theorem 1). The condition (3) in loc.cit. is in our context, for the case of a product bundle, the equation (3.7);by the technique of étale descent used in the proof of Proposition 3.3, we can generalize it to any locally trivial vector bundle E, and prove

$$C \circ \nu = id_E, \tag{4.1}$$

where $\nu: E \to E^D$ is the exponential adjoint of fibrewise multiplication by scalars from D

$$E \times D \longrightarrow E.$$

So connections in our sense give rise to (the formal analogue of) connection maps in Patterson's sense. The notions, and equations proved for them, can now be mimicked in our setting. For those equations and relations that essentially use coordinate calculations, we can mimick these also, under the assumption that the objects in question locally (in the sense of étale maps) can be covered with coordinates: this is the technique of étale descent, as used in §3.

It should be recalled that in the context in which Patterson works, he can prove that a connection map C in his sense is equivalent to a connection in the sense of Koszul, which is by definition an operation that work on vector fields, not on individual vectors, as Patterson and we do. Patterson employs a partition-of-unity argument to come from a Koszul connection to a C. The passage the other way works quite generally, and, as Proposition 3.3 shows, it works in our formal context.

We briefly indicate which of the connection-related notions and equations, which Patterson succeeds in expressing in terms of C also can be expressed/proved in our context. We already mentioned covariant differentiation, and the laws for it.

For an affine connection on the bundle $M^D \to M$

$$\nabla: M^{D\vee D} \longrightarrow M^{D\times D}$$

we define (cf. [8], Theorem 3) the torsion θ of ∇ as the map

$$\theta: M^{D\times D} \longrightarrow M^D$$

given as the difference between C (= the Patterson-Dombrowski connection map associated to ∇), and $C\circ S$ (where $S: M^{D\times D} \longrightarrow M^{D\times D}$ is the map induced by the interchange of the two factors of $D\times D$). Out of θ, we can derive a "tensor" T as follows

$$T(X,Y) = \theta\circ Y^D\circ X$$

(X and Y tangent vector fields on M). To express the relationship of this tensor to covariant differentiation, we need that M is an infinitesimally linear object, which is <u>locally good</u> in the sense that there exists an étale epic $N \twoheadrightarrow M$ with N satisfying:

(i) N is parallellizable (meaning $N^D \to N$ is isomorphic to a product bundle $N\times F \to N$, with F a Euclidean module)

(ii) N satisfies Axiom 2 of [9] (this is an axiom that implies a natural Lie algebra structure $[-,-]$ on the set of vector fields on N).

One can prove that under these assumptions, M itself also satisfies Axiom 2 of [9]. It is now possible to prove

Proposition 4.1. If M is locally good in the sense explained, then for any affine connection ∇ on $M^D \to M$, the torsion tensor T introduced above satisfies

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

We shall not give the proof, since it is standard (using small segments of suitable Taylor series [5]), and does not employ or reveal specific geometric features of our method.

In a similar vein, we follow Patterson in introducing the curvature tensor of an affine connection ∇ in a vector bundle $E \to M$ (not necessarily of form $M^D \to M$). It is defined to be the map

$$\kappa\colon E^{D\times D} \longrightarrow E$$

given by

$$C \circ C^D - C \circ C^D \circ S$$

where C is the Patterson connection map associated to ∇, and $S\colon E^{D\times D} \longrightarrow E^{D\times D}$ (as above) the "twist" map.

Out of κ we can form a "tensor" R as follows

$$R(X,Y,Z) = \kappa \circ Z^{D\times D} \circ Y^D \circ X,$$

where X and Y are tangent vector fields $M \longrightarrow M^D$ and Z is a vector field $M \to E$. (Note: we are using the identification $M^{D\times D} \cong (M^D)^D$.) It is now possible to prove

Proposition 4.2. If M is locally good in the sense explained above, then for any affine connection ∇ on $E \to M$, the curvature tensor R introduced above satisfies

$$R(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Again we omit the proof.

However, in this case, the fact that ∇ is defined for individual pairs of vectors makes it possible to give an elementwise geometric interpretation of curvature, which we present in the following §.

§5. Coordinate neighbourhoods.

In this §, we compute in coordinates some of the notions introduced. So we assume M is a étale subobject $U \rightarrowtail A^n$ of A^n (U is "subeuclidean" in the terminology of [5]). We assume that A is of line type and infinitesimally linear. By 2.3 of [6], U is infinitesimally linear. Furthermore since

$$\begin{array}{ccc} U\times A^n & \longrightarrow & A^n\times A^n = (A^n)^D \\ \downarrow & & \downarrow \text{proj}_1 \\ U & \longrightarrow & A^n \end{array}$$

is a pull-back, $U^D \cong U\times A^n$ as a vector bundle over U, so U is parallelizable (in particular, it is an n-dimensioned manifold in the sense of [6]).

A connection on the bundle $U^D \to U$ becomes a map

(5.1) $$U^D\times_U U^D \simeq U\times A^n\times A^n \xrightarrow{\nabla} U\times A^n\times A^n\times A^n \simeq U^{D\times D}$$

which is completely determined by its fourth component, because of the condition that ∇ should be a splitting of K. Note, namely that under the identifications in (5.1), K is given by

$$(\underline{u},\underline{v}_1,\underline{v}_2,\underline{v}_3) \longmapsto (\underline{u},\underline{v}_1,\underline{v}_2).$$

The two additions $\oplus$ and $+$ in $U^{D\times D} \simeq (U^D)^D$ are given by

$$(\underline{u},\underline{v}_1',\underline{v}_2',\underline{v}_3') \oplus (\underline{u},\underline{v}_1'',\underline{v}_2,\underline{v}_3'') = (\underline{u},\underline{v}_1'+\underline{v}_1'',\underline{v}_2,\underline{v}_3'+\underline{v}_3'')$$

and

$$(\underline{u},\underline{v}_1,\underline{v}_2',\underline{v}_3') + (\underline{u},\underline{v}_1,\underline{v}_2'',\underline{v}_3'') = (\underline{u},\underline{v}_1,\underline{v}_2'+\underline{v}_2'',\underline{v}_3'+\underline{v}_3''),$$

respectively. Similar for multiplication by scalars. Let us denote the fourth component of ∇ by $\bar{\nabla}$, so that

$$\nabla(\underline{u},\underline{v}_1,\underline{v}_2) = (\underline{u},\underline{v}_1,\underline{v}_2,\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2)).$$

Saying that ∇ is affine therefore in this case amounts to saying that $\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2)$ depends bilinearily on $\underline{v}_1,\underline{v}_2$, (whence we can describe $\bar{\nabla}(-,-,-)$ by a 3n-indexed family Γ^{ℓ}_{ij} of functions $U \to A$).

To a pair $\underline{v}_1,\underline{v}_2$ of tangent vectors at $\underline{u}$, or equivalently to $D \vee D \to U$ with analytic expression

$$(d_1,d_2) \longmapsto \underline{u} + d_1\underline{v}_1 + d_2\underline{v}_2 \qquad \forall (d_1,d_2) \in D \vee D,$$

∇ associates a map $D \times D \to U$ with analytic expression

$$(d_1,d_2) \longmapsto \underline{u} + d_1\underline{v}_1 + d_2\underline{v}_2 + d_1 d_2 \bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2)$$

$$= (\underline{u} + d_1\underline{v}_1) + d_2 \cdot (\underline{v}_2 + d_1 \cdot \bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2)).$$

Thus,"to each d_1 is associated a tangent vector at $\underline{u} + d_1\underline{v}_1$", namely $\underline{v}_2 + d_1 \cdot \bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2)$. (We are here using the basic identification α of $A^n \times A^n$ with $(A^n)^D$ which identifies $(\underline{x},\underline{y})$ with $d \mapsto \underline{x} + d\cdot\underline{\mathbf{y}}$.) Thus,

$$\underline{v}_2 + d_1 \cdot \bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2)$$

can be called "<u>the result of ∇-parallel transport of v_2 along d_1-units of v_1</u>"; its base point is $\underline{u} + d_1\underline{v}_1$.

Since $H: U^D \times_U U^D \longrightarrow (U^D)^D$ in the coordinatization used can be seen to have the effect

$$(\underline{u},\underline{v},\underline{w}) \longmapsto (\underline{u},\underline{o},\underline{v},\underline{w}),$$

the connection map $C: (U^D)^D \to U^D = U \times A^n$, or rather, this C followed by projection to the second factor, can be identified with

$$(\underline{u},\underline{v}_1,\underline{v}_2,\underline{v}_3) \longmapsto \underline{v}_3 - \bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_2).$$

We can now interpret, in the terminology of parallel transport, the curvature κ of ∇ introduced in §4. This geometric interpretation is given in terms of "infinitesimal parallellograms" (see [1]), but these have now been objectivized into maps $D\times D \to U$. The curvature then measures the difference in transporting a vector parallel along the two ways round in the parallellogram. To wit, given $\underline{u} \in U$ and two tangent vectors $\underline{v}_1$ and $\underline{v}_2$. Let $\underline{v}_3$ be a third tangent vector. We can then, according to the description above transport $\underline{v}_3$ parallel d_1 units along $\underline{v}_1$ and then d_2 units along $\underline{v}_2$; or we can do it in the reverse order. Then we can subtract. Transporting $\underline{v}_3$ d_1 units along $\underline{v}_1$ yields the following tangent vector at $\underline{u}+d_1\cdot\underline{v}_1$

$$d \mapsto (\underline{u}+d_1\cdot\underline{v}_1) + d\cdot(\underline{v}_3+d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)),$$

that is, the tangent vector $\underline{v}_3+d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)$ attached at $\underline{u}+d_1\cdot\underline{v}_1$. This we now transport d_2 units along $\underline{v}_2$ which yields the following tangent vector attached at $\underline{u}+d_1\cdot\underline{v}_1+d_2\cdot\underline{v}_2$:

$$(5.2)\qquad \underline{v}_3+d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)+d_2\cdot\bar{\nabla}(\underline{u}+d_1\,\underline{v}_1,\underline{v}_2,\underline{v}_3+d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)).$$

Similarly, if we first transport $\underline{v}_3$ d_2 units along $\underline{v}_2$ and then d_1 units along $\underline{v}_1$, we arrive at the following tangent vector attached at $\underline{u}+d_1\cdot\underline{v}_1+d_2\cdot\underline{v}_2$

$$(5.3)\qquad \underline{v}_3+d_2\cdot\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3) + d_1\cdot\bar{\nabla}(\underline{u}+d_2\underline{v}_2,\underline{v}_1,\underline{v}_3+d_2\cdot\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3)).$$

We rewrite (5.2) using linearity of $\bar{\nabla}$ in the third variable, and using Taylor series development [5] in the first ($D_{\underline{v}_1}\bar{\nabla}$ denotes directioned derivative in the direction $\underline{v}_1$ viewing $\bar{\nabla}(-,-,-)$ as a function in the first variable only):

$$\begin{aligned}&\underline{v}_3 + d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)\\ &+ d_2\cdot[\bar{\nabla}(\underline{u}+d_1\underline{v}_1,\underline{v}_2,\underline{v}_3) + d_1\cdot\bar{\nabla}(\underline{u}+d_1\underline{v}_1,\underline{v}_2,\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3))]\end{aligned}$$

$$\begin{aligned}= \quad &\underline{v}_3 + d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)\\ &+ d_2\cdot[\bar{\nabla}(u,\underline{v}_2,\underline{v}_3) + d_1 D_{\underline{v}_1}\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3) + d_1\cdot\bar{\nabla}(\underline{u},\underline{v}_2,\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3)].\end{aligned}$$

Similarly, we get that (5.3) equals

$$\begin{aligned}&\underline{v}_3 + d_2\cdot\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3)\\ &+ d_1\cdot[\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3) + d_2\cdot D_{\underline{v}_2}(\underline{u},\underline{v}_1,\underline{v}_3) + d_2\cdot\bar{\nabla}(\underline{u},\underline{v}_1,\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3)].\end{aligned}$$

The difference is

$$\begin{aligned}d_1 d_2\cdot[&D_{\underline{v}_1}\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3) + \bar{\nabla}(\underline{u},\underline{v}_2,\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3))]\\ &- D_{\underline{v}_2}\bar{\nabla}(\underline{u},\underline{v}_1,\underline{v}_3) - \bar{\nabla}(\underline{u},\underline{v}_1,\bar{\nabla}(\underline{u},\underline{v}_2,\underline{v}_3))].\end{aligned}$$

If we compute out the square bracket in coordinates, putting $\underline{v}_1 = \underline{e}_i$ (i'th canonical basis vector) and $\underline{v}_2 = \underline{e}_j$, $\underline{v}_3 = \underline{e}_k$, we get for the ℓ'th coordinate of the expression in the square bracket (using bilinearity of $\nabla(\underline{u},-,-)$:

$$\frac{\partial}{\partial x_i}\Gamma^{\ell}_{jk}(\underline{u}) - \frac{\partial}{\partial x_j}\Gamma^{\ell}_{ik}(\underline{u}) + \sum_{\alpha}\Gamma^{\alpha}_{ik}(\underline{u})\cdot\Gamma^{\ell}_{j\alpha}(\underline{u}) - \Gamma^{\alpha}_{jk}(\underline{u})\cdot\Gamma^{\ell}_{i\alpha}(\underline{u}),$$

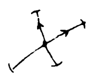
where $\Gamma^k_{ij}(\underline{u})$ = k'th coordinate of $\bar{\nabla}(\underline{u},\underline{e}_i,\underline{e}_j)$. This agrees with the classical analytic expression for the curvature tensor of the connection ∇ with coordinates $\Gamma^k_{ij}(\underline{u})$.

§6. Pictures.

We append a few pictures analogous to those of (0.1) - (0.4). Note that a tangent vector at $m \in M$ in our context is a map $D \to M$, $D \subseteq A$ being a certain definite (but small) piece of the line. This is of course the same idea as defining a tangent vector on M to be an equivalence class of curves passing through m. Because individual maps $D \to M$ are conceptually simpler than <u>equivalence classes of</u> maps $A \to M$, they are also easier to represent by a picture.

Elements in M:

Elements in M^D: 

Elements in $M^D \times_M M^D$: 

Elements in $M^{D \vee D}$: 

(since an element here may contain some more information than an element in $M^D \times_M M^D$, if M is not infinitesimally linear.

Elements in $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$: 

Elements in $M^{D \times D} \cong (M^D)^D$: see (0.2)

Elements in $E \times_M M^D$: see (0.3)

Elements in E^D : see (0.4)

R E F E R E N C E S

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COMPLEX STRUCTURES ON TOPOI

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Introduction:

For a few years work has been done in expressing classical concepts in the language of a topos and in working on these concepts within the language . For example Mulvey proved Swan's theorem by means of a "Kaplansky's theorem" inside the topos of sheaves over the base space . Some other work has been done in that direction by Fourman and by the author . Here we show how we can do some differential geometry on a manifold M , using the internal language of the topos Sh(M) of sheaves over M . Let us consider the topos Sh(M) , in which we distinguish two objects: $\mathbb{R}_C$ and $\mathbb{R}_\infty$, the sheaves of locally constant (resp. differentiable i.e. C^∞) real-valued functions on M . In this context we can express that M is a complex manifold , and construct H , the sheaf of germs of holomorphic functions on M . We can work with vector fields and differential forms , in particular we can construt the differential of a function . Then we look at connections by looking at their covariant derivatives . We construct the riemannian connection on a riemannian manifold , and we give an internal proof of the following theorem: an almost complex manifold is a Kähler manifold iff the riemannian connection is almost complex .

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We now come to the motivation for the present work . In [FR] we show that a differentiable (resp. complex analytic) family of complex structures can be represented as a complex manifold in the topos Sh(M) , where M is the space of parameters . In the first case the complex numbers object is $\mathbb{R}_\infty^2$, in the second case it is H . Kodaira and Spencer proved in [KS] that a complex analytic family which is differentiably locally trivial is analytically locally trivial . We ask if there exists an internal proof of this fact . This was the motivation for the present work . Here we have constructed the object H from $\mathbb{R}_\infty$ and $\mathbb{R}_C$, in the internal language of Sh(M) , provided we have a complex structure tensor J . So we have a relation between $\mathbb{R}_\infty$ and H .

The theory presented below is merely algebraic . We stopped solving the problem mentionned above precisely where classically a Lie equation is solved . We stop at the same place when we try recovering a linear connection from its covariant derivative , i.e. building a splitting of the second tangent bundle into horizontal and vertical vectors . On the other hand it seems that the theory can be done if one replaces $\mathbb{R}_C \hookrightarrow \mathbb{R}_\infty$ by any ring inclusion $A \hookrightarrow B$, at least for the first 4 sections .Of course the theory is uninteresting if there are not enough derivations of B with respect to A . For the last 3 sections it seems enough to have an inclusion $A \hookrightarrow B$ of ordered apartness fields . However we do not present the theory in such an axiomatic context: this work is considered as unfinished . We are interested to know if there is an axiomatisation of $\mathbb{R}_\infty$ over $\mathbb{R}_C$, in terms of , for example,the Lie equations that are solvable .

1. Real numbers in Sh(M):

In the topos Sh(M) of sheaves over the manifold M we have two "objects of reals numbers", $\mathbb{R}_C$ and $\mathbb{R}_M$, the objects of Cauchy, resp. Dedekind, real numbers, given by:

$\mathbb{R}_C(U) = \{f:U \longrightarrow \mathbb{R} \mid f \text{ is locally constant}\}$

$\mathbb{R}_M(U) = \{f:U \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$

Now it seems natural to consider $\mathbb{R}_\infty$, the sheaf of germs of differentiable real-valued functions on M, which "represents" the differentiable structure of M. In [FR] we show that we can consider $\mathbb{R}_\infty$ as an object of real numbers in Sh(M), in the sense that $\mathbb{R}_\infty$ is a suitable object for doing real analysis. We have in Sh(M) the following ring inclusions: $\mathbb{R}_C \subset \mathbb{R}_\infty \subset \mathbb{R}_M$.

2. Differential of a function:

Fourman noticed in [F] that the tangent bundle of M can be represented in Sh(M) by the object of derivations of $\mathbb{R}_\infty$ with respect to $\mathbb{R}_C$, namely if:

$\mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C) = \{X:\mathbb{R}_\infty \longrightarrow \mathbb{R}_\infty \mid X \text{ is } \mathbb{R}_C\text{-linear and } X(fg) = X(f)g + fX(g)\}$

then $\mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)$ is the sheaf of differentiable vector fields on M, i.e. the sheaf of differentiable sections of the tangent bundle of M.

Remark: $\mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)$ is a $\mathbb{R}_\infty$-module

Definition: 1) Let $f \in \mathbb{R}_\infty$, then the differential of f is defined as:

$$f_* : \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C) \xrightarrow{ev_f} \mathbb{R}_\infty$$
$$X \longmapsto f_*(X) = X(f)$$

2) The gradient of f is a differential form : $\nabla f = df \in \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)^*$

We have $\nabla f(X) = df(X) = X(f) \quad \forall X \in \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)$

Proposition: Let U be an open set of M and let $f:U \longrightarrow R$ be differentiable ($f \in \mathbb{R}_\infty(U)$). Then f_* is the sheaf map generated by the classical differential of f. We say that f_* represents the classical differential of f and we use the same notation than in the classical case.

3. Almost complex manifolds:

If M is a complex manifold, then M has local coordinates $z_1 = x_1 + iy_1$, ... , $z_n = x_n + iy_n$. The tangent bundle T(M) of M has locally the following basis : $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$. These vector fields give locally a basis of $\mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)$ over $\mathbb{R}_\infty$. We can define a "complex structure tensor" J by:

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \quad , \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

J gives a $\mathbb{R}_\infty$-linear map $J: \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C) \longrightarrow \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)$, such that $J^2 = -1$.

Definition: A map $J: \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C) \longrightarrow \mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C)$, which is $\mathbb{R}_\infty$-linear and satisfies $J^2 = -1$ is called an (almost) complex structure on $\mathbb{R}_\infty$.

Proposition: M is an almost complex manifold in the classical sense iff there is a complex structure on $\mathbb{R}_\infty$ in Sh(M).

Remarks:

1) $\mathbb{C}_\infty = \mathbb{R}_\infty^2$ is the sheaf of germs of differentiable complex-valued functions on M. There is a map $J':\mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$, $J'(z) = iz$, such that $J'^2 = -1$.

2) $\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2) \cong \mathrm{Der}(\mathbb{R}_\infty, \mathbb{R}_C)^2$

3) $J:\mathrm{Der}(\mathbb{R}_\infty, \mathbb{R}_C) \longrightarrow \mathrm{Der}(\mathbb{R}_\infty, \mathbb{R}_C)$ extends canonically to a $\mathbb{R}_\infty^2$-linear map $J: \mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2) \longrightarrow \mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)$, which satisfies $J^2 = -1$.

4) We can speak of the differential of $f \in \mathbb{R}_\infty^2 = \mathbb{C}_\infty$, as a map

$$f_*: \mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2) \longrightarrow \mathbb{R}_\infty^2 \qquad f_*(X) = X(f).$$

Definition: $f \in \mathbb{R}_\infty^2$ is almost complex iff $J'f_* = f_*J$. (A section $f \in \mathbb{R}_\infty^2(U)$ is almost complex in the sense above iff almost complex in the classical sense).

Now let us consider again a complex manifold M with local coordinates $z_1, \ldots, z_n$. The basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$, of $\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)$ over $\mathbb{R}_\infty^2$ can be replaced by the basis $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$, $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$, where $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$.
We can consider the subspace generated by $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$, and call it $\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)^+$. In the same way $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$ generate a subspace $\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)^-$. ($\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)^+$ is the sheaf of differentiable sections of the complex tangent bundle of M).

But $J\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j}$ and $J\left(\frac{\partial}{\partial \bar{z}_j}\right) = -i\frac{\partial}{\partial \bar{z}_j}$.

So $\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)^+ = \{X \mid J(X) = iX\}$ and $\mathrm{Der}(\mathbb{R}_\infty^2, \mathbb{R}_C^2)^- = \{X \mid J(X) = -iX\}$

We can generalize this to the case of an almost complex manifold.

Notation: In order to simplify the notation we write $\mathrm{Der}(\mathbb{R}_\infty,\mathbb{R}_C) = \mathrm{Der}$, $\mathrm{Der}(\mathbb{R}_\infty{}^2,\mathbb{R}_C{}^2) = \mathrm{Der}^C$, $\mathrm{Der}(\mathbb{R}_\infty{}^2,\mathbb{R}_C{}^2)^+ = \mathrm{Der}^+$, $\mathrm{Der}(\mathbb{R}_\infty{}^2,\mathbb{R}_C{}^2) = \mathrm{Der}^-$.

Proposition: Let J be an almost complex structure on $\mathbb{R}_\infty$. There is a splitting : $\mathrm{Der}^C = \mathrm{Der}^+ \oplus \mathrm{Der}^-$, where $\mathrm{Der}^+ = \{X \mid JX = iX\}$, and $\mathrm{Der}^- = \{X \mid JX = -iX\}$. Moreover $\mathrm{Der}^+ = \mathrm{Der} - iJ\mathrm{Der}$ and $\mathrm{Der}^- = \mathrm{Der} + iJ\mathrm{Der}$

Proof: Let $X \in \mathrm{Der}^C$. Then $X = \frac{1}{2}(X - iJX) + \frac{1}{2}(X + iJX)$, and $J(X - iJX) = JX + iX = i(X - iJX)$, $J(X + iJX) = -i(X + iJX)$.
If $Z = X + iY \in \mathrm{Der}^C$, where $X,Y \in \mathrm{Der}$, then $JZ = iZ$ iff $JX + iJY = iX - Y$ iff $JX = -Y$ iff $Z = X - iJX$.

Proposition: $f \in \mathbb{R}_\infty{}^2$ is almost complex iff $\forall Z \in \mathrm{Der}^-$ $Z(f) = 0$.
Proof: algebraic calculations.

Definition: Let H be the subobject of $\mathbb{R}_\infty{}^2$ of almost complex elements: $H = \{f \in \mathbb{R}_\infty{}^2 \mid \forall X \in \mathrm{Der}^- \; X(f) = 0\}$. (H is the sheaf of germs of almost complex functions on M).

Proposition: If M is a complex manifold, then H is the sheaf of germs of holomorphic functions on M.
Proof: Der^- is locally generated by the $\frac{\partial}{\partial \overline{z}_j}$.

Lie bracket operation:

$$[-,-] : \mathrm{Der}^2 \longrightarrow \mathrm{Der} \qquad \text{or} \qquad [-,-] : (\mathrm{Der}^C)^2 \longrightarrow \mathrm{Der}^C$$
$$[X,Y] = XY - YX$$

Proposition : The following are equivalent:

1) $\forall X,Y \in Der^+ \quad [X,Y] \in Der^+$

2) $\forall X,Y \in Der^- \quad [X,Y] \in Der^-$

3) $\forall X,Y \in Der \quad N(X,Y) = 2([JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]) = 0$

(N is called the torsion of J) .

Proof:

$1) \Leftrightarrow 2)$ comes from: $X \in Der^+$ iff $\overline{X} \in Der^-$, $\overline{[X,Y]} = [\overline{X},\overline{Y}]$, $J(\overline{X}) = \overline{J(X)}$.

$1) \Leftrightarrow 3)$ Let $X,Y \in Der$, let $X' = X - iJX$, $Y' = Y - iJY$.

Then $[X',Y'] = [X,Y] - [JX,JY] - i[JX,Y] - i[X,JY]$

So $[X',Y'] \in Der^+$ iff $J[X',Y'] = i[X',Y']$

iff $J[X,Y] - J[JX,JY] - iJ[JX,Y] - iJ[X,JY]$

$= i[X,Y] - i[JX,JY] + [JX,Y] + [X,JY]$

iff $i\,N(X,Y) = JN(X,Y)$ iff $N(X,Y) = 0$, because $N(X,Y) \in Der$.

Definition: J is said to be integrable iff J satisfies the equivalences of the previous proposition .

Theorem: M is a complex manifold iff J is integrable .

Proof: J is integrable in the sense of our definition iff J is classically integrable , iff M is a complex manifold (by the theorem of Newlander and Nirenberg) .

Proposition: Let M be a complex manifold . Then

$Der(H,\mathbb{R}_C^2) = \{X:H \longrightarrow H \mid X \text{ is } \mathbb{R}_C^2\text{-linear and } X(fg) = X(f)g + fX(g)$

$= \{X \in Der^+ \mid X(H) \subset H\}$, is the sheaf of holomorphic sections of the complex tangent bundle $T^C(M)$ of M .

Proof: Any $\mathbb{R}_C^2$-derivation $X:H \longrightarrow H$ extends to a derivation $X:\mathbb{R}_\infty^2 \longrightarrow \mathbb{R}_\infty^2$, since $Der(H,\mathbb{R}_C^2)$ has locally the basis $\frac{\partial}{\partial z_1}$, ... , $\frac{\partial}{\partial z_n}$

4. Differential forms:

Proposition: The sheaf of differential forms on M is constructed in Sh(M) as the exterior algebra of Der^* : $\Lambda Der^* = \bigoplus_{p\geq 0} \Lambda^p Der^*$.
In the same way the sheaf of complex differential forms is given by ΛDer^{C*} .

Proposition: There is an operator d: $\Lambda^p Der^* \longrightarrow \Lambda^{p+1} Der^*$, given by

$$d(\omega)(X) = X(\omega) \quad \text{if } p = 0$$

$$d(\omega)(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1})$$

$d(\omega)$ is called the exterior derivative of ω . In the same way we define d for complex differential forms . $d^2 = 0$.
Proof: same as classical proof .

Remark: $Der^{C*} = Der^{+*} \oplus Der^{-*}$, where $Der^{+*} = \{\omega | \omega(X) = 0 \mid \forall X \in Der^-\}$.

Definition: The object of complex forms of bidegree (p,q) , where p + q = r , is the subobject of Der^{C*} defined by :

$\Lambda^{p,q}Der^{C*} = \{\omega | \omega(X_1,\ldots,X_r) = 0$ if $p' \neq p$ of the X_i belong to Der^+ and the remaining $(r - p')$ X_i belong to $Der^-\}$.

Proposition: $\Lambda^r Der^{C*} = \bigoplus_{p+q=r} \Lambda^{p,q} Der^{C*}$.

Theorem: The following are equivalent for a complex structure J :

1) J is integrable
2) $d(\Lambda^{p,q}Der^{C*}) \subset \Lambda^{p,q+1}Der^{C*} \oplus \Lambda^{p+1,q}Der^{C*}$

3) $d(\Lambda^{0,1}Der^{C*}) \subset \Lambda^{1,1}Der^{C*} \oplus \Lambda^{0,2}Der^{C*}$

4) $d(\Lambda^{1,0}Der^{C*}) \subset \Lambda^{1,1}Der^{C*} \oplus \Lambda^{2,0}Der^{C*}$

Proof: 1)$\Rightarrow$2) Let $\omega\in\Lambda^{p,q}Der^{C*}$, let $X_1,\dots,X_s\in Der^+$, and $Y_1,\dots,Y_t\in Der^-$, with $s\neq p,p+1$ and $s+t = p+q+1$ (so $t\neq q,q+1$) .

Then
$$d\omega(X_1,\dots,X_s,Y_1,\dots,Y_t) = \sum_{i=1}^{s}(-1)^{i+1}X_i(\omega(X_1,..\hat{X}_i,..X_s,Y_1,..Y_t))$$
$$+\sum_{i=1}^{t}(-1)^{s+i+1}Y_i(\omega(X_1,..,X_s,Y_1,..,\hat{Y}_i,..,Y_t))$$
$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,..\hat{X}_i,..\hat{X}_j,..X_s,Y_1,..Y_t)$$
$$+\sum_{i<j}(-1)^{2s+i+j}\omega([Y_i,Y_j],X_1,..X_s,Y_1,..\hat{Y}_i,..\hat{Y}_j,..Y_t)$$
$$+\sum_{i=1}^{s}\sum_{j=1}^{t}(-1)^{s+i+j}\omega([X_i,Y_j],X_1,..\hat{X}_i,..X_s,Y_1,..\hat{Y}_j,..Y_t)$$

Each of the terms is zero , since $X_i,X_j\in Der^+$ implies $[X_i,X_j]\in Der^+$, $[Y_i,Y_j]\in Der^-$, and $[X_i,Y_j] = Z^+ + Z^-$ with $Z^+\in Der^+$ and $Z^-\in Der^-$.

2)$\Rightarrow$3) , 2)$\Rightarrow$4)

3)$\Rightarrow$1) Let $X,Y\in Der^-$, and $\omega\in\Lambda^{1,0}Der^{C*}$.

Then $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = -\omega([X,Y]) = 0$

So $\omega([X,Y]) = 0 \quad \forall\omega\,\Lambda^{1,0}Der^{C*}$. This means $[X,Y]\in Der^-$.

4)$\Rightarrow$1) in the same way.

<u>Definition</u>: Let $d'\omega$ be the component of $d\omega$ in $\Lambda^{p+1,q}Der^{C*}$, and let $d''\omega$ be the component of $d\omega$ in $\Lambda^{p,q+1}Der^{C*}$. This gives:

$$d': \Lambda^{p,q}Der^{C*} \longrightarrow \Lambda^{p+1,q}Der^{C*}$$
$$d'': \Lambda^{p,q}Der^{C*} \longrightarrow \Lambda^{p,q+1}Der^{C*}$$

<u>Proposition</u>: 1) $d'^2 = 0$, $d''^2 = 0$, $d'd'' + d''d' = 0$

2) If $\omega\in\Lambda^{1,0}Der^{C*} = R_\infty^2$, then $d'(\omega)(X) = \frac{1}{2}(X - iJX)(\omega)$, and $d''(\omega)(X) = \frac{1}{2}(X + iJX)(\omega)$.

3) $H = \{f\in R_\infty^2 \mid d''(f) = 0\}$.

5. Riemannian manifolds:

Definitions: 1) $f \in R_\infty$ is apart from 0 ($f \# 0$) iff f is invertible, iff $|f| > 0$.

2) $X \in Der$ is apart from 0 ($X \# 0$) iff $\forall f \in R_\infty$ $X(f) \# 0$.

Proposition: Let M be a differentiable manifold. M is a riemannian manifold iff there is a morphism of sheaves in Sh(M), $g: Der^2 \longrightarrow R_\infty$, called a metric for M, which is an inner product on Der, i.e.

1) g is bilinear

2) $g(X,Y) = g(Y,X)$ $\forall X,Y \in Der$

3) $\forall X \in Der$ $g(X,X) \geqslant 0$ and $g(X,X) > 0$ iff $X \# 0$

Proof: A riemannian metric extends to an inner product on the vector fields over any open set U of M.

Definition: Let M be an almost complex manifold. A metric g for M is hermitian iff $\forall X,Y \in Der$ $g(JX,JY) = g(X,Y)$.

Proposition: The following are equivalent:

1) g is hermitian.

2) $\forall X,Y \in Der^+$ $g(X,Y) = 0$ and $\forall X,Y \in Der^-$ $g(X,Y) = 0$.

(g extends uniquely to $g: (Der^C)^2 \longrightarrow R_\infty^2$, R_∞^2-bilinear).

Definition: Let g be a hermitian metric on M. We define $\omega: Der^2 \longrightarrow R_\infty$ by $\omega(X,Y) = g(X,JY)$. ω is the Kähler form of M.

Proposition: $\omega \in \Lambda^2 Der^*$

Proof: $\omega(X,Y) = g(X,JY) = g(JX,J^2Y) = -g(JX,Y) = -\omega(Y,X)$

Proposition: M is an almost complex Kähler manifold (in the classical sense) iff $d\omega = 0$ in Sh(M) (i.e. ω is closed).

6. Basis of Der($\mathbb{R}_\infty$,$\mathbb{R}_C$). Dimension:

Proposition: There exists a basis for Der in Sh(M) in the following sense: $\exists X_1,\dots,X_n \in \mathrm{Der}\ \ \forall Y \in \mathrm{Der}\ \ \exists f_1,\dots,f_n \in \mathbb{R}_\infty\ \ Y = \sum_{i=1}^{n} f_i X_i$ and $\forall \lambda_1,\dots,\lambda_n \in \mathbb{R}_\infty\ \ \exists \lambda_i \# 0 \longrightarrow \sum_{i=1}^{n} \lambda_i X_i \# 0$.

Proof: If $x_1,\dots,x_n$ are local coordinates for M, then $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$, is a local basis for Der. $\frac{\partial}{\partial x_i} \# 0$, since $\frac{\partial}{\partial x_i}(x_i) = 1$.

Remark: With the definition of a basis given above we can prove that the vectors of a basis are apart from zero : this fact is essential for the rest of the development.

Proposition: The dimension n of M is given by the number of elements of any basis $X_1,\dots,X_n$ of Der.

Proof. external proof by interpreting.

(There is an internal proof in Sh(M) that two basis have the same number of elements, using elementary linear algebra on an apartness field).

Proposition: Let $g:\mathrm{Der}^2 \longrightarrow \mathbb{R}_\infty$ be a metric on a riemannian manifold M. From a basis $X_1,\dots,X_n$, we can construct an orthonormal basis.

Proof: Gram-Schmidt's orthogonalization process works because $g(X_i,X_i)$ is invertible $\forall i$.

Proposition: Let $g:\mathrm{Der}^2 \longrightarrow \mathbb{R}_\infty$ be a metric on a riemannian manifold M . Then there is an isomorphism :

$$\mathrm{Der} \xrightarrow{\ \phi\ } \mathrm{Der}^*$$
$$X \longmapsto g(X,-)$$

Proof: ϕ injective: suppose $g(X,Y) = 0 \ \forall Y$. In particular $g(X,X) = 0$, so $X = 0$.

ϕ surjective: let $X_1,\dots,X_n$ be an orthonormal basis of Der and let $F \in \mathrm{Der}^*$. Let $f_i = F(X_i)$. Then $F = g(\sum_{i=1}^{n} f_i X_i, -)$.

7. Covariant derivative and connections:

Definition: let M be a differentiable manifold .

1) A derivation law (or connection) is a $\mathbb{R}_\infty$-linear map

$$\nabla : \mathrm{Der} \longrightarrow \mathrm{Hom}_{\mathbb{R}_C}(\mathrm{Der},\mathrm{Der}) \qquad X \longmapsto \nabla_X$$

such that $\nabla_X(fY) = X(f)Y + f\nabla_X Y \quad \forall f \in \mathbb{R}_\infty \quad \forall Y \in \mathrm{Der}$

2) The curvature of ∇ is $K: \mathrm{Der}^2 \longrightarrow \mathrm{Hom}_{\mathbb{R}_\infty}(\mathrm{Der},\mathrm{Der})$

$$(X,Y) \longmapsto K(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

3) The torsion of ∇ is $T: \mathrm{Der}^2 \longrightarrow \mathrm{Der}$ given by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad .$$

Remark: We can extend ∇_X to tensors of type (r,s) by :

If $K:\mathrm{Der}^s \longrightarrow \mathrm{Der}^r$ is $\mathbb{R}_\infty$-multilinear , then

$$\nabla_X K(X_1,\dots,X_s) = \nabla_X(K(X_1,\dots,X_s)) - \sum_{i=1}^{s} K(X_1,\dots,\nabla_X X_i,\dots,X_s) ,$$

where , if $K(X_1,\dots X_s) = (Y_1,\dots,Y_r)$ then $\nabla_X(K(X_1,\dots,X_s)) = (\nabla_X Y_1,\dots,\nabla_X Y_r)$.

<u>Theorem</u>: Let M be a riemannian manifold with metric $g:\mathrm{Der}^2 \longrightarrow \mathbb{R}_\infty$. Then there exists a unique connection ∇ such that $\forall X \in \mathrm{Der}$ $\nabla_X g = 0$ and ∇ has no torsion i.e. $T = 0$. ∇ is called the riemannian connection .

Proof: $\nabla_X Y$ is given by :

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) + g([Z,X],Y) + g(X,[Z,Y]) .$$

Then $g(\nabla_X Y,Z) + g(\nabla_X Z,Y) = Xg(Y,Z)$ $\forall X,Y,Z$, i.e. $\nabla g = 0$.

$T = 0$, since $g(\nabla_X Y,Z) - g(\nabla_Y X,Z) = g([X,Y],Z)$ $\forall X,Y,Z$.

Conversely let $\nabla g = 0$ and $T = 0$. From $\nabla g = 0$ we get:

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z) \quad , \quad Yg(X,Z) = g(\nabla_Y X,Z) + g(\nabla_Y Z,X)$$

$$-Zg(X,Y) = -g(\nabla_Z X,Y) - g(X,\nabla_Z Y)$$

So $Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) = g([X,Z],Y) + g([Y,Z],X) + G(\nabla_X Y + \nabla_Y X,Z)$

$$= g([X,Z],Y) + g([Y,Z],X) + 2g(\nabla_X Y,Z) + g([Y,X],Z) .$$

<u>Definition</u>: let M be an almost complex manifold with complex structure J . A connection ∇ is almost complex iff $\nabla J = 0$.

<u>Theorem</u>: Let M be an almost complex **hermitian** manifold . Then the riemannian connection is almost complex iff $N = 0$ and $d\omega = 0$ i.e. M is a complex Kähler manifold (recall that $\omega(X,Y) = g(Z,JY)$) .

Proof: The proof follows from the following lemmas .

<u>Lemma 1</u>: Let M be an almost complex manifold with complex structure J . Then $\forall X,Y,Z \in \mathrm{Der}$ we have:

$$4g((\nabla_X J)Y,Z) = 2d\omega(X,JY,JZ) - 2d\omega(X,Y,Z) + g(N(Y,Z),JX) ,$$ where

$N(X,Y) = 2([JX,JY] - [X,Y] - J[JX,Y] - J[X,JY])$ is the torsion of J .

Proof: $g((\nabla_X J)Y,Z) = g(\nabla_X JY,Z) - g(J(\nabla_X Y),Z) = g(\nabla_X JY,Z) + g(\nabla_X Y,JZ)$.

$$d\omega(X,Y,Z) = X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X)$$

$$= Xg(Y,JZ) - Yg(X,JZ) + Zg(X,JY) - g([X,Y],JZ) + g([X,Z],JY) - g([Y,Z],JX)$$

$$d\omega(X,JY,JZ) = -Xg(JY,Z) + JYg(X,Z) - JZg(X,Y) + g([X,JY],Z) - g([X,JZ],Y) - g([JY,JZ],JX)$$

$$g(N(Y,Z),JX) = 2(g([JY,JZ],JX) - g([Y,Z],JX) - g([Y,JZ],X) - g([JY,Z],X))$$

So $g(N(Y,Z),JX) + 2d\omega(X,JY,JZ) - 2d\omega(X,Y,Z)$

$$= 2JYg(X,Z) + 2Yg(X,JZ) - 2JZg(X,Y) - 2Zg(X,JY) + 2g([X,Y],JZ) - 2g([X,Z],JY) + 2g([X,JY],Z) - 2g([X,JZ],Y) - 2g([Y,JZ],X) - 2g([JY,Z],X)$$

$$= 4(g(\nabla_X JY,Z) + g(\nabla_X Y,JZ)) = 4g((\nabla_X J)Y,Z)$$

<u>Lemma 2</u>: Let M be an almost complex hermitian manifold . If the riemannian connection has no torsion then $d\omega = \frac{1}{3}\,\mathrm{Alt}\,\nabla\omega$, where $\nabla\omega(X,Y,Z) = (\nabla_X\omega)(Y,Z)$ and $6\,\mathrm{Alt}\,\omega(X,Y,Z)$ is the sum of the $\omega(\sigma X,\sigma Y,\sigma Z)$ for all permutations σ of X,Y,Z .

Proof: $\nabla_X\omega(Y,Z) - \nabla_X\omega(Z,Y) = X\omega(Y,Z) - X\omega(Z,Y) - \omega(\nabla_X Y,Z) - \omega(Y,\nabla_X Z) + \omega(\nabla_X Z,Y) + \omega(Z,\nabla_X Y)$

$$= 2X\omega(Y,Z) - 2\omega(\nabla_X Y,Z) - 2\omega(Y,\nabla_X Z)$$

$$\nabla_Y\omega(Z,X) - \nabla_Y\omega(X,Z) = 2Y\omega(Z,X) - 2\omega(\nabla_Y Z,X) - 2\omega(Z,\nabla_Y X)$$

$$\nabla_Z\omega(X,Y) - \nabla_Z\omega(Y,X) = 2Z\omega(X,Y) - 2\omega(\nabla_Z X,Y) - 2\omega(X,\nabla_Z Y)$$

So $6\,\mathrm{Alt}\,\omega(X,Y,Z) = 2X\omega(Y,Z) + 2Y\omega(Z,X) + 2Z\omega(X,Y) - 2\omega([X,Y],Z) + 2\omega([X,Z],Y) - 2\omega([Y,Z],X) = 2d\omega(X,Y,Z)$

<u>Proof of the theorem</u>:

Suppose that ∇ is almost complex i.e. $\nabla J = 0$. We show $\nabla\omega = 0$.

$$\nabla_X\omega(Y,Z) = X\omega(Y,Z) - \omega(\nabla_X Y,Z) - \omega(Y,\nabla_X Z)$$

$$= Xg(Y,JZ) - g(\nabla_X Y,JZ) - g(Y,\nabla_X(JZ)) = \nabla_X g(Y,JZ) = 0 .$$

By lemma 2 $d\omega = 0$. By lemma 1 $g(N(Y,Z),JX) = 0 \ \forall X,Y,Z$. So $N = 0$.
Conversely if $d\omega = 0$ and $N = 0$ then $g((\nabla_X J)Y,Z) = 0 \ \forall X,Y,Z$ **by lemmma 1**, so $\nabla J = 0$.

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COMPLEXIFICATION OF VARIABLE FIELDS

G.C.Wraith

This article is written in appreciation of J. Kennison's visit to Sussex University in 1977. Problems concerning sheaf representations of rings led him to enquire about extensions of fields in toposes. He is responsible for most of the ideas presented here, and has meanwhile carried the theme further.

We shall restrict ourselves in this article to investigating the generic adjunction of $\sqrt{-1}$ to a field in a topos. To construct explicitly this generic adjunction we shall use two basic tools, namely torsors and glueing. It is hoped that the modest scope of this article will eliminate complication without sacrifice of the essential ideas.

By a field we mean, of course, a commutative ring with unity satisfying the geometric Axioms:

i) $\neg\ (o=1)$

ii) $\forall x \quad x=o \vee \exists y. \quad xy=1.$

By the generic adjunction of $\sqrt{-1}$ to a field K in a Topos ε we mean the generic model of the ε-theory T whose models are fields containing K and satisfying

$$\forall x \quad \exists y \quad (y^2=-1) \wedge \bigvee_{a_1, a_2 \in K} (x=a_1+ya_2)$$

We shall simplify matters by supposing that 2 is invertible in K.

Now in Sets the problem is relatively simple: either (i) K already contains a $\sqrt{-1}$ in which case the trivial extension solves our problem, or else (ii) it does not, in which case we want the extension

$$K \subseteq K(i)$$

where K(i) is $K \times K$ as K-vector space, with the Gaussian rule for complex multiplication

$$(a,b).(a',b') = (aa' - bb', ab' + a'b).$$

In the latter case, (ii), we should think of K(i) as an object not of Sets but of $\mathrm{Sets}^{\mathbb{Z}_2}$, the topos of sets-with-an-involution, where of course the involution on K(i) is given by complex conjugation $(a,b) \mapsto (a,-b)$.

In a general Topos ε, matters are not so straightforward. Let I denote $\{a \in K \mid a^2 = -1\}$, and let U be the support of I, i.e. $U = \mathrm{im}(I \to 1)$. We will adapt the usual abuse of language by writing U for the open suptopos of ε given by ε/U, and we write ε-U for its complementary closed suptopos. It is clear that K/U, the restriction of K to U, falls into case (i), so that over U only the trivial extension is needed. Over ε-U we are in case (ii), so that we shall want to consider $K(i)/(\varepsilon\text{-}U)$ in $(\varepsilon\text{-}U)^{\mathbb{Z}_2}$. Then we must somehow glue these two cases back together. A slight complication arises here because K/U may contain no global $\sqrt{-1}$ to which the i of $K(i)/(\varepsilon\text{-}U)$ should be attached on the boundary of U.

We will see that I/U is in general a $\mathbb{Z}_2$-torsor, whose non-triviality tells us that the two $\sqrt{-1}$'s in K/U can be distinguished locally but not globally. We shall need to use this torsor to twist matters straight.

We define an involution on I by $a \mapsto -a$.

<u>Proposition</u> I/U is a $\mathbb{Z}_2$-torsor in U.

Proof. Since $I \to U$ is epic, by definition, I/U has global support. Since 2 is invertible in K the involution has no fixed points. Since in K we have

$$a^2 = b^2 \Rightarrow (a=b) \vee (a=-b)$$

we deduce that $\mathbb{Z}_2$ acts transitively on I/U, and hence I/U is a $\mathbb{Z}_2$-torsor.

Now we recall some facts about glueing. Let

$$U \xrightarrow{d} \varepsilon - U$$

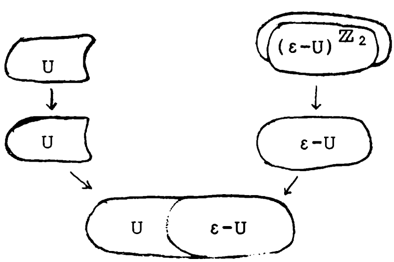
be the fringe functor, that is to say the composite of the direct image functor for the open inclusion $U \subseteq \varepsilon$ followed by the inverse image functor for the closed inclusion $\varepsilon - U \subseteq \varepsilon$. Every object X of ε is uniquely determined up to isomorphism by the three pieces of data $(X/U,\ X/(\varepsilon-U)\ ,\ \alpha(X))$ where

$$X/(\varepsilon-U) \xrightarrow{\alpha(X)} d(X/U)$$

is the attaching map of X.

We have already argued informally that over U we want the trivial extension, and that over $\varepsilon-U$ we want the extension $K \subseteq K(i)$, thought of as living in $(\varepsilon-U)^{\mathbb{Z}_2}$, the topos of objects-with-involution in $\varepsilon-U$.
This means we wish to glue U and $(\varepsilon-U)^{\mathbb{Z}_2}$ together along a left exact functor, which we will now pull out of the hat and justify later.

$K/U \subseteq K/U$



$K/(\varepsilon-U) \subsetneq K(i)/(\varepsilon-U)$

For any object Y of U, the involution on I/U defines an involution on

$$Y^{(I/U)}$$

and hence we get an involution on $d(Y^{(I/U)})$. We denote this object of $(\varepsilon\text{-}U)^{\mathbb{Z}_2}$ by $\Delta(Y)$, and thus get a left exact functor

$$U \xrightarrow{\Delta} (\varepsilon\text{-}U)^{\mathbb{Z}_2} .$$

We denote by $\mathcal{G}$ the topos obtained by glueing along Δ. It is an ε-topos because Δ is a locally internal functor over ε.

Now we want to construct a T-model in $\mathcal{G}$. First note that since $(-)^I$ is left exact, K^I has a canonical ring structure, and that the map $K \xrightarrow{c} K^I$ adjoint to the projection $K \times I \longrightarrow K$ is a ring homomorphism. Furthermore c/U is injective since I/U has global support, so that we may identify K/U with a subring of $(K/U)^{(I/U)}$ by means of c/U.

We denote by $1 \xrightarrow{j} K^I$ the map adjoint to the inclusion $I \subseteq K$. It should be clear that $j^2 = -1$.

Let L be the object of $\mathcal{G}$ defined by

$$L = (K/U\ ,\ K(i)/(\varepsilon\text{-}U)\ ,\ l)$$

where $K(i)/(\varepsilon\text{-}U)$ has the standard involution $(a,b) \mapsto (a,-b)$, i.e. complex conjugation, and l is given by

$$K(i)/(\varepsilon\text{-}U) \xrightarrow{l} \Delta(K/U) \quad : \quad (a,b) \longmapsto \alpha(K^I)\ (a+jb) .$$

It is easily verified that l is a $\mathbb{Z}_2$-equivariant ring homomorphism, from which it follows that L is a ring in $\mathcal{G}$. We have a ring homomorphism $K \longrightarrow L$ in $\mathcal{G}$ given by the commuting diagram in $\varepsilon\text{-}U$

$$\begin{array}{ccccc} K/(\varepsilon\text{-}U) & \xrightarrow{\alpha(K)} & d(K/U) & \qquad & K \\ \downarrow & & \downarrow d(c/U) & & \downarrow \\ K(i)/(\varepsilon\text{-}U) & \xrightarrow{l} & d(K^I/U) & & L \end{array}$$

Since we have a surjection of toposes

$$U + (\varepsilon-U) \longrightarrow U + (\varepsilon-U)^{\mathbb{Z}_2} \longrightarrow \mathcal{G} ,$$

to verify that L is a T-model it is enough to verify that K/U is a T-model in U and that $K(i)/(\varepsilon-U)$ is a T-model in $\varepsilon-U$.

It remains to show that L is the generic T-model. So suppose that

$$\mathcal{F} \xrightarrow{p} \varepsilon$$

is an ε-topos and that $p^*(K) \subseteq F$ is a T-model in $\mathcal{F}$. The splitting of ε into the open and closed complementary pieces U and $\varepsilon-U$ pulls back along p to split $\mathcal{F}$ into open and closed complementary subtoposes $\mathcal{F}/U$ and $\mathcal{F}/(\varepsilon-U)$.
Let us write

$$\mathcal{F}/U \xrightarrow{\delta} \mathcal{F}/(\varepsilon-U)$$

for the fringe functor. We have the following diagram of functors commuting up to natural isomorphisms:

$$\begin{array}{ccccc} \mathcal{F}/U & \xrightarrow{(p/U)_*} & U & \xrightarrow{(p/U)^*} & \mathcal{F}/U \\ \delta \downarrow & & d \downarrow & & \delta \downarrow \\ \mathcal{F}/(\varepsilon-U) & \xrightarrow{p/(\varepsilon-U)_*} & \varepsilon-U & \xrightarrow{p/(\varepsilon-U)^*} & \mathcal{F}/(\varepsilon-U) \end{array}$$

Let us write $J = \{x \in F \mid x^2=-1\}$, with involution given by $x \mapsto -x$. Since F is a T-model J has global support and is a $\mathbb{Z}_2$-torsor in $\mathcal{F}$. Hence $J/(\varepsilon-U)$ is a $\mathbb{Z}_2$-torsor in $\mathcal{F}/(\varepsilon-U)$. Let

$$\mathcal{F}/(\varepsilon-U) \xrightarrow{g} (\varepsilon-U)^{\mathbb{Z}_2}$$

be its classifying map. This means that for any object V of $(\varepsilon-U)^{\mathbb{Z}_2}$ we have

$$g^*(V) = (p/(\varepsilon-U))^*(V) \otimes_{\mathbb{Z}_2} J/(\varepsilon-U) .$$

If a and b are variables of type K , then in $F/(\varepsilon-U)$ we have

$$\forall x \quad \forall a \quad \forall b \quad (x^2 = -1 \wedge a+xb = o) \Rightarrow (a = b = o) \ .$$

Hence $a + xb \longmapsto a - xb$, for $x^2 = -1$, defines an involution on $F/(\varepsilon-U)$. Now if we twist $F/(\varepsilon-U)$ by the $\mathbb{Z}_2$-torsor $J/(\varepsilon-U)$ we obtain a locally isomorphic field with a global $\sqrt{-1}$, which must therefore be isomorphic to $p^*(K(i)) \ / \ (\varepsilon-U)$.

We deduce that

$$F/(\varepsilon-U) \ \simeq \ g^*(L/(\varepsilon-U) \ .$$

It is clear that $F/U \simeq p^*(K)/U$, and that $J/U \simeq p^*(I)/U$. From the latter isomorphism, and from the commuting diagrams of functors above, we find that we have a diagram of functors commuting up to natural isomorphisms

$$\begin{array}{ccccc}
\mathcal{F}/U & \xrightarrow{(p/U)_*} & U & \xrightarrow{(p/U)^*} & \mathcal{J}/U \\
\delta \downarrow & & \Delta \downarrow & & \delta \downarrow \\
\mathcal{F}/(\varepsilon-U) & \xrightarrow{g_*} & (\varepsilon-U)^{\mathbb{Z}_2} & \xrightarrow{g^*} & J/(\varepsilon-U)
\end{array}$$

It follows that $\mathcal{F}/U \xrightarrow{p/U} U$ and $\mathcal{F}/(\varepsilon-U) \xrightarrow{g} (\varepsilon-U)^{\mathbb{Z}_2}$ glue together to give a map of ε-toposes

$$\mathcal{F} \xrightarrow{f} \mathcal{G}$$

uniquely determined up to natural isomorphism by the property that $f^*(L) = F$. It follows that $\mathcal{G}$ is the classifying ε-topos of T , and that L is the generic T-model.

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G. C. Wraith. Artin Glueing. Journal of Pure and Applied Algebra. 4 (1974), p 345-348.

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Talks given:

May 10: A. Kock: Opening talk.

May 11: A. Kock: Lie's synthetic theory of differential equations (~ this volume no. 6).

G.C. Wraith: On Chou's iterated path integrals.

A. Joyal: Real Algebraic geometry.

May 12: G.E. Reyes: Subtoposes of the ring classifier (~ this volume no. 4).

G.C. Wraith: Recent work of Kennison.

D.van Osdol: An exposition of virtual groups.

May 13: C. Rousseau: Complex structure on topoi (~ this volume no. 8)

May 15: G.E. Reyes: Dubuc's models for formal differential geometry.

A. Joyal: Recent work on real number systems.

M. Fourman: Logic in Chen's topos.

May 16: A. Kock: Universally solving differential equations.

Problem Session 1.

A. Kock: More on Lie's synthetic theory.

May 17: M. Coste: The generic model of an ε-stable theory is of line type (~ this volume no. 2).

M. Tierney: On Schanuel's work.

A. Joyal and F.W. Lawvere: Discussion on Philosophy.

May 18: Problem Session 2.

C. Rousseau: Parameters versus logic.

F.W. Lawvere: Algebraic Theory of classical thermostatics.

May 19: M.-F. Coste: On real algebraic geometry (~ this volume no. 3).

R. Bkouche: Frobenius Theorem in Differential Algebra.

F.W. Lawvere: Is category theory useful in learning thermo-mechanics ?

May 20: M. Coste: On real algebraic geometry (~ this volume no. 3).

May 22: G. Cifoletti: Hegel and differential calculus.

M. Foruman: $\mathbb{C}$ is separably closed.

F.W. Lawvere: Discussion on Physics.

May 23: J. Beck: Simplicial Methods in Foundations of Analysis.

P. Johnstone: Gleason Cover.

M. Tierney: Beck conditions in Topoi.

May 24: F.W. Lawvere: Category of dimensions.

M. Coste: Solutions to some of the problems. (~ Appendix 1 and 2 in no. 3).