ON THE HOMOTOPY RELATION FOR c.s.s. MAPS[1]

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1. Introduction

A c.s.s. complex (see [2]) may be considered as a collection of sets together with a collection of maps between them satisfying certain identities. Similarly a c.s.s. group may be considered as a collection of groups together with a collection of homomorphisms between them satisfying the same identities. This suggests the notion of a c.s.s. *object* over an arbitrary category $\mathcal{C}$.

Let $\mathcal{C}$ be a category and let $\mathcal{C}^V$ denote the category of a c.s.s. objects over $\mathcal{C}$. Then it will be shown that if in $\mathcal{C}$ a notion of sum is defined, it is possible to introduce in $\mathcal{C}^V$ a homotopy relation in a rather natural way.

Let $\mathcal{C}$ and $\mathfrak{D}$ be such categories with sums. Then we will show that under certain conditions a functor $\Gamma:\mathcal{C}^V \to \mathfrak{D}^V$ preserves homotopies. This generalizes a result of A. Dold ([1]).

As an application we prove an analogue for c.s.s. groups of a theorem of J. H. C. Whitehead.

2. C.s.s. categories

For every integer $n \geqq 0$ let $[n]$ denote the ordered set $(0, \cdots, n)$. By a map $\alpha:[m] \to [n]$ we mean a monotone function, i.e., $\alpha(i) \leqq \alpha(j)$ for $0 \leqq i \leqq j \leqq m$. Clearly the sets $[n]$ and the maps $\alpha:[m] \to [n]$ form a category. This category will be denoted by $\mathcal{V}$.

Definition (2.1). Let $\mathcal{C}$ be a category. The function category $\mathcal{C}^V$ (see [3]) will be called the c.s.s. *category over* $\mathcal{C}$; its objects and maps will be called c.s.s. *objects* and c.s.s. *maps over* $\mathcal{C}$. We recall that an object of $\mathcal{C}^V$ is any *contravariant* functor $K:\mathcal{V} \to \mathcal{C}$ and that for two objects $K, L \in \mathcal{C}^V$ a map $f:K \to L$ is a natural transformation. Instead of $K[n]$, $K\alpha$ and $f[n]$ we usually write K_n, K_α and f_n.

Examples (2.2). (a) Let $\mathfrak{M}$ be the category of sets. Then $\mathfrak{M}^V$ is the category of c.s.s. complexes ([2]).

(b) Let $\mathfrak{N}$ be the category of sets with a distinguished element. Then $\mathfrak{N}^V$ is the category of c.s.s. complexes with a base point ([4], §2).

(c) Let $\mathfrak{L}$ be the category of modules (over a ring Λ). Then $\mathfrak{L}^V$ is the category of c.s.s. modules over Λ ([1]). In particular if $\Lambda = Z$, then $\mathfrak{L}^V$ is the category of c.s.s. abelian groups.

(d) Let $\mathcal{G}$ be the category of groups. Then $\mathcal{G}^V$ is the category of c.s.s. groups ([4]).

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3. Categories with sums

Definition (3.1). Let $\mathcal{C}$ be a category and let M be a set. Let $C \in \mathcal{C}$ be an object and for every element $\mu \in M$ let be given an object $C_\mu \in \mathcal{C}$ and a map $j_\mu : C_\mu \to C$. Then C is called the *sum of the objects* C_μ *under the maps* j_μ if for every object $D \in \mathcal{C}$ and every set of maps $d_\mu : C_\mu \to D$, $\mu \in M$ there is a *unique* map $d : C \to D$ such that for every $\mu \in M$ commutativity holds in the diagram

$$\begin{array}{ccc} C_\mu & \xrightarrow{j_\mu} & C \\ & \searrow^{d_\mu} & \downarrow d \\ & & D \end{array}$$

We then write $C = \sum_{\mu \in M} j_\mu C_\mu$ or $C = j_{\mu_1} C_{\mu_1} + j_{\mu_2} C_{\mu_2} + \cdots$

This definition is a special case of the definition of direct limit of [3], chapter II.

Definition (3.2). A category $\mathcal{C}$ is called a *category with sums* if for every set M and function Γ which assigns to the elements of M an object of $\mathcal{C}$, there are *given* an object $M \cdot \Gamma \in \mathcal{C}$ and maps $\mu \cdot \Gamma(\mu) : \Gamma(\mu) \to M \cdot \Gamma$ $(\mu \in M)$ such that $M \cdot \Gamma = \sum_{\mu \in M} (\mu \cdot \Gamma(\mu)) \Gamma(\mu)$.

Examples (3.3). All categories in example (2.2) are categories with sums. Using the same notation we have

(a) The sum of a collection of objects of $\mathcal{M}$ is what is usually called their union.

(b) The sum of a collection of objects of $\mathcal{N}$ is their union with identification of all the distinguished elements.

(c) The sum of a collection of objects of $\mathcal{L}$ is their direct sum.

(d) The sum of a collection of objects of $\mathcal{G}$ is their free product.

Definition (3.4). Let $\mathcal{C}$ be a category with sums. Then we define a functor $\otimes : \mathcal{M}, \mathcal{C} \to \mathcal{C}$ as follows. Let $M \in \mathcal{M}$ and $C \in \mathcal{C}$ be objects and let Γ be the function given by $\Gamma(\mu) = C$ for all $\mu \in M$. We then define $M \otimes C$ by

$$M \otimes C = M \cdot \Gamma$$

(i.e., $M \otimes C$ is the sum of as many copies of C as there are elements in M). For maps $g : M \to N \in \mathcal{M}$ and $f : C \to D \in \mathcal{C}$ let $g \otimes f : M \otimes C \to N \otimes D$ be the (unique) map such that for every $\mu \in M$ commutativity holds in the diagram

$$\begin{array}{ccc} C & \xrightarrow{\mu \cdot C} & M \otimes C \\ \downarrow f & & \downarrow g \otimes f \\ D & \xrightarrow{g\mu \cdot D} & N \otimes D \end{array}$$

It is readily verified that the function $\otimes$ so defined is a *covariant* functor.

It should be noted that definition (3.2) and hence definition (3.4) contains an element of choice. However it follows from [3], chapter II, that if in definition (3.2) the *given* object $C \in \mathcal{C}$ and maps $j_\mu : C_\mu \to C$ are changed, then the functor $\otimes$ gets changed by a unique natural equivalence.

DEFINITION (3.5). Let $\mathcal{C}$ be a category with sums. Then a covariant functor $\otimes : \mathfrak{M}^V, \mathcal{C}^V \to \mathcal{C}^V$ may be defined by

$$(K \otimes A)_n = K_n \otimes A_n$$

$$(K \otimes A)_\alpha = K_\alpha \otimes A_\alpha$$

$$(g \otimes f)_n = g_n \otimes f_n$$

for every object $K \in \mathfrak{M}^V$ and $A \in \mathcal{C}^V$ and map $g \in \mathfrak{M}^V$ and $f \in \mathcal{C}^V$.

It is clear that the use of the symbol $\otimes$ for two different functors will not cause any trouble. In both cases we often write $g \otimes A$ and $K \otimes f$ instead of $g \otimes i_A$ and $i_K \otimes f$.

EXAMPLES (3.6). (a) Let $K, L \in \mathfrak{M}^V$. Then $K \otimes L$ is usually called their cartesian product.

(b) Let $K \in \mathfrak{M}^V$, $A \in \mathcal{G}^V$. The product $K \otimes A$ then is as in [5], §3.

4. The homotopy relation

DEFINITION (4.1). Let $\mathcal{C}$ be a category with sums. Let the standard simplices $P = \Delta[0]$ and $I = \Delta[1]$ and the c.s.s. maps $\Delta\varepsilon^i : P \to I$ $(i = 0, 1)$ be as in [5], §2. Then two maps $f_0, f_1 : A \to B \in \mathcal{C}^V$ are called *homotopic* (*over* $\mathcal{C}$) if there exists a map $f_I : I \otimes A \to B \in \mathcal{C}^V$ (called *homotopy*) such that commutativity holds in the diagram

$$\begin{array}{ccccc}
P \otimes A & \xrightarrow{\Delta\varepsilon^0 \otimes A} & I \otimes A & \xleftarrow{\Delta\varepsilon^1 \otimes A} & P \otimes A \\
\approx \downarrow i & & \downarrow f_I & & \approx \downarrow i \\
A & \xrightarrow{f_0} & B & \xleftarrow{f_1} & A
\end{array}$$

where $i : P \otimes A \approx A$ is the natural isomorphism. Notation $f_I : f_0 \sim f_1$ (over $\mathcal{C}$) or $f_0 \sim f_1$ (over $\mathcal{C}$).

A map $f : A \to B \in \mathcal{C}^V$ is called a *homotopy equivalence* (*over* $\mathcal{C}$) if there exists a map $g : B \to A \in \mathcal{C}^V$ such that the composite maps $g \circ f$ and $f \circ g$ are homotopic (over $\mathcal{C}$) to the identity maps of A and B. The objects A and B then are said to have the same *homotopy type* (*over* $\mathcal{C}$).

EXAMPLES (4.2). Using the notation of example (2.2) we have:

(a) Two maps of $\mathfrak{M}^V$ are homotopic over $\mathfrak{M}$ if and only if they are homotopic in the usual sense ([4] §2).

(b) Two maps of $\mathfrak{N}^V$ are homotopic over $\mathfrak{N}$ if and only if they are homotopic rel. the base point ([4], §2).

(c) Two maps of $\mathfrak{L}^V$ are homotopic over $\mathfrak{L}$ if and only if they are homotopic in the sense of [1].

(d) Two maps of $\mathfrak{G}^V$ are homotopic over $\mathfrak{G}$ if and only if they are loop homotopic in the sense of [5].

Examples (4.3). Sometimes several homotopy relations may be defined over different categories.

(a) As a c.s.s. group also may be regarded as a c.s.s. complex it follows that on $\mathfrak{G}^V$ there is a homotopy relation over $\mathfrak{G}$ and one over $\mathfrak{M}$. Clearly two maps homotopic over $\mathfrak{G}$ are also homotopic over $\mathfrak{M}$, but the converse need not be true. This may be seen from the following example: Let $z:K(\pi, n) \rightarrow K(\phi, q)$ be a map representing a non-zero element of $H^q(\pi, n; \phi)$ which suspends into zero. Then (see [5]) the c.s.s. homomorphism $Gz:G(K(\pi, n)) \rightarrow G(K(\phi, g))$ is homotopic over $\mathfrak{M}$ to the trivial map, but by [5], §11 this is not the case over $\mathfrak{G}$.

(b) Let $\mathfrak{L}$ momentarily denote the category of abelian groups. As a c.s.s. abelian group may also be regarded as a c.s.s. group or a c.s.s. complex there corresponds for the category $\mathfrak{L}^V$ three homotopy relations (over $\mathfrak{G}$, $\mathfrak{L}$ and $\mathfrak{M}$). It is however readily verified that the homotopy relations over $\mathfrak{G}$ and $\mathfrak{L}$ are equivalent. Clearly maps of $\mathfrak{L}^V$ homotopic over $\mathfrak{L}$ are also homotopic over $\mathfrak{M}$, but the converse need not be true.

Remark (4.4). It should be noted that the homotopy relation defined in (4.1) need *not* be an equivalence relation. For c.s.s. complexes counter examples can easily be found. However the homotopy relation always has the following property.

Proposition (4.5). *Let $\mathfrak{C}$ be a category with sums. Let $f:A \rightarrow B$, $g_0, g_1:B \rightarrow C$ and $h:C \rightarrow D$ be maps of $\mathfrak{C}^V$ and let $g_0 \sim g_1$ over $\mathfrak{C}$. Then $h \circ g_0 \circ f \sim h \circ g_1 \circ f$ over $\mathfrak{C}$.*

Proof. Let $g_I:g_0 \sim g_1$ over $\mathfrak{C}$. Then it follows immediately from the definitions that the composite map

$$I \otimes A \xrightarrow{I \otimes f} I \otimes B \xrightarrow{g_I} C \xrightarrow{h} D$$

is the desired homotopy.

Special cases of Proposition (4.5) are [5], Proposition (2.5) and (3.4).

5. C.s.s. functors

We now define a class of functors involving c.s.s. categories (roughly speaking: functors such that "dimension n of the range" only depends on "dimension n of the domain") and show that these functors map homotopic maps into homotopic maps. This generalizes a result of A. Dold ([1]).

Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A covariant functor $\Gamma:\mathfrak{C} \to \mathfrak{D}^V$ induces a functor $D(\Gamma):\mathfrak{C}^V \to \mathfrak{D}^V$ given by

$$(D(\Gamma)A)_n = (\Gamma A_n)_n$$

$$(D(\Gamma)A)_\alpha = (\Gamma A_\alpha)_\alpha$$

$$(D(\Gamma)g)_n = (\Gamma g_n)_n$$

for every object $A \in \mathfrak{C}^V$ and map $g \in \mathfrak{C}^V$. Denote by $\Xi:\mathfrak{C} \to \mathfrak{C}^V$ the constant functor, i.e. for every object $C \in \mathfrak{C}$ and map $c \in \mathfrak{C}$

$$(\Xi C)_n = C$$

$$(\Xi C)_\alpha = i_C$$

$$(\Xi c)_n = c.$$

Then with every covariant functor $\Theta:\mathfrak{C}^V \to \mathfrak{D}^V$ one may associate the composite functor $\Theta \circ \Xi:\mathfrak{C} \to \mathfrak{D}^V$ and it is readily verified that

PROPOSITION (5.1). *There exists a natural equivalence* $n:\Gamma \to D(\Gamma) \circ \Xi$.

However in general the functors Θ and $D(\Theta \circ \Xi)$ do not differ by a natural equivalence. We therefore define

DEFINITION (5.2). A covariant functor $\Theta:\mathfrak{C}^V \to \mathfrak{D}^V$ is called a *c.s.s. functor* if there exists a natural equivalence $t:\Theta \to D(\Theta \circ \Xi)$

The theorem now may be stated as follows

THEOREM (5.3). *Let* $\mathfrak{C}$ *and* $\mathfrak{D}$ *be categories with sums and let* $\Theta:\mathfrak{C}^V \to \mathfrak{D}^V$ *be a* c.s.s. *functor. Then* Θ *maps maps homotopic over* $\mathfrak{C}$ *into maps homotopic over* $\mathfrak{D}$.

PROOF. It clearly suffices to show that for every object $A \in \mathfrak{C}^V$ there exists a map $a:I \otimes \Theta A \to \Theta(I \otimes A) \in \mathfrak{D}^V$ such that commutativity holds in the diagram

$$\begin{array}{ccccc} P \otimes \Theta A & \xrightarrow{\Delta\epsilon^0 \otimes \Theta A} & I \otimes \Theta A & \xleftarrow{\Delta\epsilon^1 \otimes \Theta A} & P \otimes \Theta A \\ \approx\downarrow i & & \downarrow a & & \approx\downarrow i \\ \Theta(P \otimes A) & \xrightarrow{\Theta(\Delta\epsilon^0 \otimes A)} & \Theta(I \otimes A) & \xleftarrow{\Theta(\Delta\epsilon^1 \otimes A)} & \Theta(P \otimes A) \end{array} \tag{5.4}$$

where $i:P \otimes \Theta A \approx \Theta(P \otimes A)$ is the natural isomorphism. For each integer $n \geqq 0$ let $a_n:I_n \otimes (\Theta A)_n \to (\Theta(I \otimes A))_n \in \mathfrak{D}$ be the (unique) map such that for every element $\mu \in I_n$ commutativity holds in the diagram

$$\begin{array}{ccccc} (\Theta A)_n & \xrightarrow{\mu\cdot(\Theta A)_n} & I_n \otimes (\Theta A)_n & \xrightarrow{a_n} & (\Theta(I \otimes A))_n \\ \downarrow t_n & & & & \updownarrow t_n \\ ((\Theta \circ \Xi)A_n)_n & & \xrightarrow{((\Theta \circ \Xi)(\mu\cdot A_n))_n} & & ((\Theta \circ \Xi)(I_n \otimes A_n))_n. \end{array}$$

It then follows from the naturality of all functions involved that the function $a: I \otimes \Theta A \to \Theta(I \otimes A)$ so defined is a map in $\mathfrak{D}^V$ and is such that commutativity holds in the diagram (5.4).

REMARK (5.5). The above result also holds for categories with *finite* sums i.e. if in definition (3.2) M is restricted to be finite.

6. An application

We now prove the following analogue for free c.s.s. groups (see [5], §4) of a theorem of J. H. C. Whitehead.

THEOREM (6.1). *Let A and A' be connected free* c.s.s. *groups and let $f: A \to A'$ be a* c.s.s. *homomorphism. Let $B = A/[A, A]$ and $B' = A'/[A', A']$ be their abelianizations and let $g: B \to B'$ be the map induced by f. Then f is a homotopy equivalence (over $\mathcal{G}$) if and only if g is so.*

PROOF. If f is a homotopy equivalence, then by Theorem (5.3) so is g. In order to prove the converse consider the commutative diagram

$$\begin{array}{ccc} G\overline{W}A & \xrightarrow{\alpha'(i)} & A \\ \downarrow p & & \downarrow q \\ G\overline{W}A/[G\overline{W}A, G\overline{W}A] & \xrightarrow{b} & B = A/[A, A] \end{array}$$

where $\alpha'(i)$ is as in [5], Theorem (11.3) and p and q are the projections. By [5], Theorem (11.3) $\alpha'(i)$ is a homotopy equivalence over $\mathcal{G}$ and hence by Theorem (5.3) so is b. Repeating this for A' we get (see [4], §15) a commutative diagram

$$\begin{array}{ccc} H_n(\overline{W}A) = \pi_{n-1}(G\overline{W}A/[G\overline{W}A, G\overline{W}A]) & \overset{\pi_{n-1}(b)}{\approx} & \pi_{n-1}(B) \\ \downarrow (\overline{W}f)_* & & \downarrow g_* \\ H_n(\overline{W}A') = \pi_{n-1}(G\overline{W}A'/[G\overline{W}A', G\overline{W}A']) & \overset{\pi_{n-1}(b')}{\approx} & \pi_{n-1}(B') \end{array}$$

If g is a homotopy equivalence, then clearly $(\overline{W}f)_*: H_n(\overline{W}A) \to H_n(\overline{W}A')$ is an isomorphism for all n. The connectedness of A and A' implies the simply connectedness of $\overline{W}A$ and $\overline{W}A'$. Hence the theorem of J. H. C. Whitehead ([6], Theorem 3) yields that $\overline{W}f: \overline{W}A \to \overline{W}A'$ is a homotopy equivalence over $\mathfrak{M}$. That f is a homotopy equivalence over $\mathcal{G}$ now follows from [5], §11.

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