ON c.s.s. CATEGORIES[1]

By Daniel M. Kan

1. Introduction

A c.s.s. complex (see [2]) may be considered as a collection of sets together with a collection of maps between them satisfying certain identities. Similarly a c.s.s. group may be considered as a collection of groups together with a collection of homomorphisms between them satisfying the same identities. This suggests the notion of a c.s.s. object over an arbitrary category $\mathcal{C}$ (see [8]). The purpose of this note is to obtain for these c.s.s. objects analogues of

(i) the homotopy extension and/or covering theorems

(ii) a theorem of J. H. C. Whitehead ([10], Theorem 1; that, under certain conditions, a map is a homotopy equivalence if and only if it induces isomorphisms of all homotopy groups).

It is often necessary in statements concerning maps of one c.s.s. complex into another to require that the range complex satisfies the extension condition ([6], §2), while in the corresponding statements for maps of one c.s.s. group into another the domain has to be a free c.s.s. group ([7], §4). It appears that in the general case both kind of notions are needed; the domain has to be free in some sense, while the range has to satisfy an extension condition.

Free use will be made of the notation and definitions of [8]. Most results remain valid for c.s.s. objects over a category with finite sums (see [8], remark (5.5)).

2. Function complexes

Throughout this paper let $\mathcal{C}$ denote a category with sums ([8], definition (3.2)). The category of sets will be denoted by $\mathfrak{M}$.

Definition (2.1). For every two objects $A, B \in \mathcal{C}^V$ let Hom (A, B) be the c.s.s. complex (called *function complex*) defined as follows. For every integer $n \geqq 0$ and every map $\alpha:[m] \to [n]$ let the standard n-simplex $\Delta[n]$ and the c.s.s. map $\Delta\alpha:\Delta[m] \to \Delta[n]$ be as in [4], §2. An n-simplex of Hom (A, B) then is any map $\sigma:\Delta[n] \otimes A \to B \in \mathcal{C}^V$ and for every map $\alpha:[m] \to [n]$ the n-simplex $\sigma\alpha$ is the composition

$$\Delta[m] \otimes A \xrightarrow{\Delta\alpha \otimes A} \Delta[n] \otimes A \xrightarrow{\sigma} B.$$

Similarly for every two maps $a:A' \to A$, $b:B \to B' \in \mathcal{C}^V$ let

$$\text{Hom}\,(a, b):\text{Hom}\,(A, B) \to \text{Hom}\,(A', B')$$

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be the c.s.s. map which assigns to an n-simplex $\sigma \in \text{Hom}(A, B)$, i.e. map

$$\sigma : \Delta[n] \otimes A \to B \in \mathcal{C}^V$$

the composition

$$\Delta[n] \otimes A' \xrightarrow{\Delta[n] \otimes a} \Delta[n] \otimes A \xrightarrow{\sigma} B \xrightarrow{b} B'.$$

It is readily verified that the function Hom: $\mathcal{C}^V, \mathcal{C}^V \to \mathfrak{M}^V$ so defined is a functor contravariant in the first variable and covariant in the second.

Choice (2.2). We now choose an arbitrary but fixed object $X_0 \in \mathcal{C}$.

Let $X \in \mathcal{C}^V$ denote the object given by $X_n = X_0$ and $X_\alpha = i_{X_0}$ for all n and α. An important role then will be played by the restricted functor

$$\text{Hom}(X, \quad) : \mathcal{C}^V \to \mathfrak{M}^V.$$

For $A \in \mathcal{C}^V$ and $\sigma \in \text{Hom}(X, A)_n$ let $\xi(\sigma) : X_n = X_0 \to A_n \in \mathcal{C}$ be the composition

$$X_n \xrightarrow{\varepsilon_n \cdot X_n} (\Delta[n] \otimes X)_n \xrightarrow{\sigma_n} A_n$$

where $\varepsilon_n \in \Delta[n]$ is the non-degenerate n-simplex. Then a simple computation yields

Proposition (2.3). *The function ξ establishes a one to one correspondence between the elements of* $\text{Hom}(X, A)_n$ *and the maps* $X_0 \to A_n \in \mathcal{C}$.

It follows immediately from Proposition (2.3) that $\text{Hom}(X, \quad)$ is a c.s.s. functor ([8], definition (5.2)) and hence by [8], Theorem (5.3).

Proposition (2.4). *The functor* $\text{Hom}(X, \quad) : \mathcal{C}^V \to \mathfrak{M}^V$ *maps maps homotopic over $\mathcal{C}$ into maps homotopic over $\mathfrak{M}$.*

The functor $\text{Hom}(X, \quad)$ is closely related to the restricted functor

$$\otimes X : \mathfrak{M}^V \to \mathcal{C}^V.$$

For every map $f : K \otimes X \to A \in \mathcal{C}^V$ define a c.s.s. map $\gamma(f) : K \to \text{Hom}(X, A)$ as follows: For every $\sigma \in K_n$ the simplex $\gamma(f)\sigma \in \text{Hom}(X, A)_n$, i.e. the map $\gamma(f)\sigma : \Delta[n] \otimes X \to A$ is the composition

$$\Delta[n] \otimes X \xrightarrow{\phi_\sigma \otimes X} K \otimes X \xrightarrow{f} A$$

where $\phi_\sigma : \Delta[n] \to K$ is the (unique) c.s.s. map such that $\phi_\sigma \alpha = \sigma\alpha$ for all $\alpha \in \Delta[n]$. A straightforward computation now yields

Proposition (2.5). *The function γ establishes (in a natural manner) a one to one correspondence between the maps* $K \otimes X \to A \in \mathcal{C}^V$ *and the maps* $K \to \text{Hom}(X, A) \in \mathfrak{M}^V$.

In the terminology of [5] this means that the functor $\otimes X$ is a left adjoint of the functor Hom $(X, \quad)$.

EXAMPLES (2.6). (a) Let $K, L \in \mathfrak{M}^V$, then Hom (K, L) is the function complex in the sense of [7], §2. Choose $P_0 \in \mathfrak{M}$ as a set consisting of one element. Then $P \approx \Delta[0]$ and clearly the functors $\otimes P:\mathfrak{M}^V \to \mathfrak{M}^V$ and

$$\text{Hom}\,(P, \quad):\mathfrak{M}^V \to \mathfrak{M}^V$$

are (up to a natural equivalence) equal to the identity functor.

(b) Let $\mathcal{G}$ be the category of groups, and let $A, B \in \mathcal{G}^V$. Then Hom (A, B) is the function complex as defined in [7], §5. Let $Z_0 \in \mathcal{G}$ denote the groups of the integers, then the functor Hom $(Z, \quad):\mathcal{G}^V \to \mathfrak{M}^V$ assigns to every c.s.s. group its underlying c.s.s. complex.

(c) Let $\mathcal{A}$ be the category of abelian groups and let $Z_0 \in \mathcal{A}$ denote the group of integers. Then the functor Hom $(Z, \quad):\mathcal{A}^V \to \mathfrak{M}^V$ assigns to every c.s.s. abelian group its underlying c.s.s. complex, while for every c.s.s. complex K, $K \otimes Z$ is the free c.s.s. abelian group generated by K (see [1]).

3. The extension condition and fibre maps

The extension condition ([6], definition (2.2)) and the notion of a fibre map ([6], definition (3.1)) will be generalized. The definitions will depend on the choice of the object $X_0 \in \mathcal{C}$ (choice (2.2)).

DEFINITION (3.1). An object $A \in \mathcal{C}^V$ is said to satisfy the *extension condition* if the c.s.s. complex Hom (X, A) satisfies the extension condition in the sense of [6], definition (2.2).

DEFINITION (3.2). A map $p:E \to B \in \mathcal{C}^V$ is called a *fibre map* if the c.s.s. map Hom (X, p):Hom $(X, E) \to$ Hom (X, B) is a fibre map in the sense of [6], definition (3.1).

It should be noted that a fibre map as defined in (3.2) need not have fibres.

The extension condition and the notion of a fibre map for c.s.s. complexes are closely related. In fact it follows easily from the definitions that

PROPOSITION (3.3). *An object $K \in \mathfrak{M}^V$ satisfies the extension condition if and only if the* (*unique*) c.s.s. *map $K \to \Delta[0]$ is a fibre map.*

This may be generalized as follows. Suppose there exists an object $Q_0 \in \mathcal{C}$ such that for every object $C \in \mathcal{C}$ there is exactly one map $C \to Q_0 \in \mathcal{C}$. (This is no real restriction on $\mathcal{C}$, as such an object always may be added to $\mathcal{C}$). Let $Q \in \mathcal{C}$ be given by $Q_n = Q_0$ and $Q_\alpha = i_{Q_0}$ for all n and α. Then definition (3.1) and (3.2) and Proposition (3.3) imply

PROPOSITION (3.4). *An object $A \in \mathcal{C}^V$ satisfies the extension condition if and only if the* (*unique*) *map $A \to Q \in \mathcal{C}^V$ is a fibre map.*

EXAMPLES (3.5). Using the choices of examples (2.6) we have

(a) An object of $\mathfrak{M}^V$ satisfies the extension condition in the sense of definition (3.1) if and only if it does so in the sense of [6], definition (2.2). The same holds for the notion of a fibre map.

(b) It was shown by J. C. Moore ([9]) that every object of $\mathcal{G}^V$ satisfies the extension condition. It can be shown by a similar argument that every epimorphism in $\mathcal{G}^V$ is a fibre map.

§4. Fibre squares

In view of Propositions (3.3) and (3.4) the notion "fibre map" may be considered as a kind of relativization of the notion "object satisfying the extension condition". Repeating the process we obtain the notion of a *fibre square*. Only the case of c.s.s. complexes will be considered here.

DEFINITION (4.1). A commutative diagram

$$\begin{tikzcd} K \arrow[r, "p"] \arrow[d, "r"] & L \arrow[d, "s"] \\ M \arrow[r, "q"] & N \end{tikzcd}$$

of maps in $\mathfrak{M}^V$ is called a *fibre square* if for every pair of integers (k, n) with $0 \leqq k \leqq n$, for every n $(n-1)$-simplices $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots \sigma_n \in K$ such that $\sigma_i\varepsilon^{j-1} = \sigma_j\varepsilon^i$ for $i < j$ and $i \neq k \neq j$ and for every n-simplex $\lambda \in L$ and $\mu \in M$ such that $s\lambda = q\mu$ and $p\sigma_i = \lambda\varepsilon^i$ and $r\sigma_i = \mu\varepsilon^i$ for $i \neq k$, there exists an n-simplex $\sigma \in K$ such that $p\sigma = \lambda$, $r\sigma = \mu$ and $\sigma\varepsilon^i = \sigma_i$ for $i \neq k$.

The following generalization of Proposition (3.3) then is immediate.

PROPOSITION (4.2). *A map $p: K \to L \in \mathfrak{M}^V$ is a fibre map if and only if the diagram*

$$\begin{tikzcd} K \arrow[r, "p"] \arrow[d] & L \arrow[d] \\ \Delta[0] \arrow[r] & \Delta[0] \end{tikzcd} \quad \textit{or equivalently the diagram} \quad \begin{tikzcd} K \arrow[r] \arrow[d, "p"] & \Delta[0] \arrow[d] \\ L \arrow[r] & \Delta[0] \end{tikzcd}$$

is a fibre square.

The following proposition describes the behavior of fibre square with respect to inverse limits. The proof is straightforward.

PROPOSITION (4.3). *Let*

$$\begin{tikzcd} K^\infty \arrow[r, "s^\infty"] \arrow[d, "p^\infty"] & \cdots \arrow[r, "s^3"] & K^2 \arrow[r, "s^2"] \arrow[d, "p^2"] & K^1 \arrow[r, "s^1"] \arrow[d, "p^1"] & K^0 \arrow[d, "p^0"] \\ L^\infty \arrow[r, "t^\infty"] & \cdots \arrow[r, "t^3"] & L^2 \arrow[r, "t^2"] & L^1 \arrow[r, "t^1"] & L^0 \end{tikzcd}$$

be a commutative diagram in $\mathfrak{M}^V$, *let the map* $p^\infty : K^\infty \to L^\infty$ *be the inverse limit of the map* $p^i : K^i \to L^i$, *and let all small squares be fibre squares. Then the big square is a fibre square.*

5. Free objects and maps

We now generalize and relativize the notion of a free c.s.s. group ([7]), definition (5.1)). Again the definitions will depend on the choice made in (2.2).

DEFINITION (5.1). An object $A \in \mathfrak{C}^V$ is called *free* if for every integer $n \geqq 0$ there exists a (possibly empty) set M_n of maps $X_0 \to A_n \in \mathfrak{C}$ such that

(i) $A_n = \sum_{\nu \in M_n} \nu X_0$

(ii) for every *epimorphism* $\beta : [n] \to [z]$ and every element $\zeta \in M_z$ there exists an element $\nu \in M_n$ such that commutativity holds in the diagram

(5.1a)
$$\begin{tikzcd} X_0 \arrow[r, "\nu"] \arrow[dr, "\zeta"'] & A_n \\ & A_z \arrow[u, "A_\beta"'] \end{tikzcd}$$

The graded set $M = \bigcup_{n=0}^{\infty} M_n$ then is called a *basis* of A.

EXAMPLES (5.2). (a) the object $X \in \mathfrak{C}^V$ is free; the identity maps $i : X_0 \to X_n$ form a basis.

(b) If $A \in \mathfrak{C}^V$ is free and $K \in \mathfrak{M}^V$, then $K \otimes A$ is free; if M is a basis of A, then the composite maps

$$X_0 \xrightarrow{\nu} A_n \xrightarrow{\sigma \cdot A_n} K_n \otimes A_n$$

where $\nu \in M_n$ and $\sigma \in K_n$, form a basis.

(c) Let $\varnothing_0 \in \mathfrak{M}$ denote the empty set and let $\varnothing \in \mathfrak{M}^V$ be the empty c.s.s. complex, i.e. $\varnothing_n = \varnothing_0$ and $\varnothing_\alpha = i_{\varnothing_0}$ for all n and α. Then $\varnothing \otimes X$ is free and has an empty basis.

DEFINITION (5.3). A map $f : B \to A \in \mathfrak{C}^V$ is called *free* if for every integer $n \geqq 0$ there exists a (possibly empty) set M_n of maps $X_0 \to A_n \in \mathfrak{C}$ such that

(i) $A_n = f_n B_n + \sum_{\nu \in M_n} \nu X_0$

(ii) for every *epimorphism* $\beta : [n] \to [z]$ and every element $\zeta \in M_n$ there exists an element $\nu \in M_n$ such that commutativity holds in diagram (5.1a).

The graded set $M = \bigcup_{n=0}^{\infty} M_n$ then is called a *basis* of f.

EXAMPLES (5.4). (a) Let $A \in \mathfrak{C}^V$. Then the identity map i_A is free; it has empty basis.

(b) Let $A \in \mathfrak{C}^V$ be free, let $K \in \mathfrak{M}^V$, let $H \subset K$ be a subcomplex and let $i : H \to K$ denote the inclusion map. Then the map $i \otimes A : H \otimes A \to K \otimes A$

is free; if M is a basis of A, then the composite maps

$$X_0 \xrightarrow{\ \nu\ } A_n \xrightarrow{\sigma \cdot A_n} K_n \otimes A_n$$

where $\nu \in M_n$, $\sigma \in K_n$, $\sigma \notin H_n$, form a basis of $i \otimes A$.

The notions "free map" and "free object" are similarly related as the notions "fibre map" and "object satisfying the extension condition" (see §3). Let $\emptyset_0$ and $\emptyset$ be as in example (5.2c). It follows from [8], definition (3.2) that for every object $C \in \mathcal{C}$ there exists *exactly one* map $\emptyset_0 \otimes X_0 \to C \in \mathcal{C}$. Hence for every object $A \in \mathcal{C}^V$ there exists *exactly one* map $\emptyset \otimes X \to A \in \mathcal{C}^V$ and it now follows easily from the definition that

PROPOSITION (5.5). *An object $A \in \mathcal{C}^V$ is free if and only if the (unique) map $\emptyset \otimes X \to A \in \mathcal{C}^V$ is free.*

EXAMPLES (5.6). Using the choices of example (2.6) we have

(a) Every object in $\mathfrak{M}^V$ is free and every monomorphism in $\mathfrak{M}^V$ is a free map. Clearly $\emptyset \otimes P = \emptyset$.

(b) An object in $\mathcal{G}^V$ is free in the above sense if and only if it is free in the the sense of [7], definition (5.1). The c.s.s. group $\emptyset \otimes Z$ consists in every dimension of one element.

(c) A c.s.s. abelian group $A \in \mathcal{A}^V$ is free in the above sense if and only if for every integer $n \geqq 0$ the group A_n is a free abelian group.

6. Homotopy extension and/or covering theorems

We state a general theorem on function complexes and derive from this several homotopy extension and/or covering theorems.

THEOREM (6.1). *Let $f:B \to A \in \mathcal{C}^V$ be a free map and let $g:C \to D \in \mathcal{C}^V$ be a fibre map. Then the diagram*

$$\begin{array}{ccc}
\operatorname{Hom}(A,C) & \xrightarrow{\operatorname{Hom}(A,\,g)} & \operatorname{Hom}(A,D) \\
\Big\downarrow{\scriptstyle \operatorname{Hom}(f,\,C)} & & \Big\downarrow{\scriptstyle \operatorname{Hom}(f,\,D)} \\
\operatorname{Hom}(B,C) & \xrightarrow{\operatorname{Hom}(B,\,g)} & \operatorname{Hom}(B,D)
\end{array}$$

is a fibre square.

The proof will be given in §10.

COROLLARY (6.2). (Homotopy extension covering theorem). *For every map $\sigma_0:A \to C$ and homotopies $\tau:\sigma_0 \circ f \sim \tau_1$ and $\rho:g \circ \sigma_0 \sim \rho_1$ such that $g \circ \tau = \rho \circ (I \otimes f)$, there exists a homotopy $\sigma:\sigma_0 \sim \sigma_1$ such that $\tau = \sigma \circ (I \otimes f)$ and $\rho = g \circ \sigma$.*

This follows immediately from Theorem (6.1) by observing that $\sigma_0 \in \operatorname{Hom}(A, C)_0$, $\tau \in \operatorname{Hom}(B, C)_1$ and $\rho \in \operatorname{Hom}(A, D)_1$.

THEOREM (6.3). *Let $f:B \to A \in \mathcal{C}^V$ be a free map and let $C \in \mathcal{C}^V$ satisfy the extension condition. Then the map* $\operatorname{Hom}(f, C):\operatorname{Hom}(A, C) \to \operatorname{Hom}(B, C)$ *is a fibre map.*

PROOF. This follows immediately from Theorem (6.1) by taking $D = Q$ (see §3) and applying proposition (3.4) and (4.2).

COROLLARY (6.4). (Homotopy extension theorem). *For every map $\sigma_0:A \to C$ and homotopy $\tau:\sigma_0 \circ f \sim \tau_1$, there exists a homotopy $\sigma:\sigma_0 \sim \sigma_1$ such that $\tau = \sigma \circ (I \otimes f)$.*

THEOREM (6.5). *Let $A \in \mathcal{C}^V$ be free and let $g:C \to D \in \mathcal{C}^V$ be a fibre map. Then the map* $\operatorname{Hom}(A, g):\operatorname{Hom}(A, C) \to \operatorname{Hom}(A, D)$ *is a fibre map.*

PROOF. This follows immediately from Theorem (6.1) by taking $B = \varnothing \otimes X$ (see §5) and applying Proposition (5.5) and (4.2).

COROLLARY (6.6). (Homotopy covering theorem). *For every map $\sigma_0:A \to C$ and homotopy $\rho:g \circ \sigma_0 \sim \rho_1$ there exists a homotopy $\sigma:\sigma_0 \sim \sigma_1$ such that $\rho = g \circ \sigma$.*

THEOREM (6.7). *Let $A \in \mathcal{C}^V$ be free and let $C \in \mathcal{C}^V$ satisfy the extension condition. Then* $\operatorname{Hom}(A, C)$ *satisfies the extension condition.*

PROOF. Theorem (6.7) follows from Theorem (6.5) in the same manner as Theorem (6.3) followed from Theorem (6.1).

COROLLARY (6.8). The relation $\sim$ is an equivalence relation on the maps $A \to C \in \mathcal{C}^V$.

Most of the above theorems for c.s.s. complexes are contained in [3]. Theorem (6.7) for c.s.s. group may be found in [7], §5.

7. J. H. C. Whitehead's theorem

We first define the homotopy groups of an object $A \in \mathcal{C}^V$ as those of the c.s.s. complex $\operatorname{Hom}(X, A)$ (see [6]).

DEFINITION (7.1). Let $A \in \mathcal{C}^V$ and let $\phi:X \to A \in \mathcal{C}^V$ be a map. Then we define $\pi_n(A; \phi)$, the n^{th} *homotopy group of* A rel. ϕ by

$$\pi_n(A; \phi) = \pi_n(\operatorname{Hom}(X, A); \phi).$$

Clearly a map $f:A \to B \in \mathcal{C}^V$ induces a map $\pi_n(f):\pi_n(A; \phi) \to \pi_n(B; f \circ \phi)$ defined by

$$\pi_n(f) = \pi_n(\operatorname{Hom}(X, f)).$$

We now state the analogue of J. H. C. Whitehead's theorem ([10] theorem 1).

THEOREM (7.2). *Let A, $B \in \mathcal{C}^V$ be free and satisfy the extension condition. Then a map $f: A \to B \in \mathcal{C}^V$ is a homotopy equivalence (over $\mathcal{C}$) if and only if f induces isomorphisms of all homotopy groups, i.e. if for every map $\phi: X \to A$ and every integer $n \geqq 0$*

$$\pi_n(f): \pi_n(A; \phi) \approx \pi_n(B; f \circ \phi).$$

PROOF. Using corollary (6.8) and [8], proposition (4.5) it may be shown by same argument as was used in [7], that Theorem (7.2) is a consequence of the following lemma.

LEMMA (7.3). *Let A, $B \in \mathcal{C}^V$ satisfy the extension condition, let B be free and let $f: A \to B \in \mathcal{C}^V$ be a map which induces isomorphisms of all homotopy groups. Then there exists a map $g: B \to A \in \mathcal{C}^V$ together with a homotopy $h: g \circ f \sim i_B$.*

The proof of Lemma (7.3) will be given in §11.

8. Skeletons

Throughout this section let $f: B \to A \in \mathcal{C}^V$ be a free map and let M be a basis of f.

DEFINITION (8.1). An element $\nu \in M_n$ is called *degenerate* if there exists an integer $z < n$, an epimorphism $\beta: [n] \to [z]$ and an element $\zeta \in M_z$ such that commutativity holds in diagram (5.1a); otherwise it is called *non-degenerate.*

PROPOSITION (8.2). *Let $\nu \in M_n$. Then there exists a unique integer $z \leqq n$, a unique epimorphism $\beta: [n] \to [z]$ and a unique non-degenerate element $\zeta \in M_z$ such that commutativity holds in diagram* (5.1a).

The proof is similar to that of the corresponding result for c.s.s. complex (see [2], §8).

In view of Proposition (8.2) one may define

DEFINITION (8.3). An element $\nu \in M_n$ is said to have *rank* z if there exists an epimorphism $\beta: [n] \to [z]$ and a non-degenerate element $\zeta \in M_z$ such that commutativity holds in diagram (5.1a).

DEFINITION (8.4). We now define for every integer $q \geqq -1$

(i) an object $A^q \in \mathcal{C}^V$ (*q-skeleton* of A rel. f)

(ii) a free map $a^q: A^q \to A \in \mathcal{C}^V$

(iii) a free map $a^{q,q+1}: A^q \to A^{q+1} \in \mathcal{C}^V$

such that commutativity holds in the diagram

(8.4a)
$$\begin{tikzcd} A^q \arrow[rr, "a^{q,q+1}"] \arrow[dr, "a^q"'] & & A^{q+1} \arrow[dl, "a^{q+1}"] \\ & A & \end{tikzcd}$$

Let $M_n^q \subset M_n$ be the subset of the elements of rank $\leqq q$. Then we define (see [8] §3)

$$A_n^q = (f_n \cup M_n^q)\cdot\Gamma$$

where $\Gamma(f_n) = B_n$ and $\Gamma(\nu) = X_0$ for $\nu \in M_n^q$ (i.e. A_n^q is the sum of B_n with as many copies of X_0 as there are elements in M_n^q). The maps $a_n^q : A_n^q \to A_n$ are the unique maps such that commutativity holds in the diagrams

$$\begin{tikzcd} B_n \arrow[rr, "f_n\cdot B_n"] \arrow[dr, "f_n"'] & & A_n^q \arrow[dl, "a_n^q"] \\ & A_n & \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[rr, "\nu\cdot X_0"] \arrow[dr, "\nu"'] & & A_n^q \arrow[dl, "a_n^q"] \\ & A_n & \end{tikzcd}$$

where $\nu \in M_n^q$. For every map $\alpha:[m] \to [n]$ the maps $A_\alpha^q : A_n^q \to A_m^q$ are uniquely determined by the condition that commutativity should hold in the diagram

$$\begin{tikzcd} A_n^q \arrow[r, "A_\alpha^q"] \arrow[d, "a_n^q"] & A_m^q \arrow[d, "a_m^q"] \\ A_n \arrow[r, "A_\alpha"] & A_m \end{tikzcd}$$

and finally the maps $a^{q,q+1} : A_n^q \to A_n^{q+1}$ are the unique ones such that commutativity holds in diagram (8.4a). It is readily verified that the collections A^q, a^q and $a^{q,q+1}$ so defined are indeed objects and maps in $\mathcal{C}^V$.

As M_n^{-1} is empty for all n, it follows that

PROPOSITION (8.5). *There is a unique isomorphism $i : B \approx A^{-1}$ such that*

$$a^{-1} \circ i = f.$$

REMARK (8.6). The above definition of q-skeleton involves the basis M. It may, however, be shown that the q-skeleton is essentially independent of the basis chosen; in fact there exists a canonical isomorphism between any two q-skeletons corresponding to different bases.

In view of Proposition (5.5) one may define

DEFINITION (8.7). Let $A \in \mathcal{C}^V$ be free. Then the *q-skeleton* of A is defined as the q-skeleton of A rel. the unique map $\varnothing \otimes X \to A$.

EXAMPLES (8.8). Using the choices of example (2.6) we have

(a) The q-skeleton of a c.s.s. complex K in the sense of definition (8.7) is canonically isomorphic with the usual q-skeleton of K, i.e. the subcomplex generated by K_q.

(b) The q-skeleton of a free c.s.s. group A in the sense of definition (8.7) is canonically isomorphic with the usual q-skeleton of A, i.e. the subgroup generated by A_q.

9. Two lemmas

We now state two lemmas which will be used in the proofs of Theorem (6.1) and Lemma (7.3). Their proofs are straightforward, although rather long, and are left to the reader. We use the notation of §8.

LEMMA (9.1). *Let $C \in \mathcal{C}^V$, let $K \in \mathcal{M}^V$ and for every integer $q > -1$ let be given a map $k^{q-1}: K \otimes A^{q-1} \to C \in \mathcal{C}^V$ such that commutativity holds in the diagram*

(9.1a)
$$\begin{tikzcd} K \otimes A^{q-1} \arrow[rr, "K \otimes a^{q-1,q}"] \arrow[dr, "k^{q-1}"'] & & K \otimes A^q \arrow[dl, "k^q"] \\ & C & \end{tikzcd}$$

Then there exists a unique map $k: K \otimes A \to C$ such that commutativity also holds in the diagram

$$\begin{tikzcd} K \otimes A^{q-1} \arrow[rr, "K \otimes a^{q-1}"] \arrow[dr, "k^{q-1}"'] & & K \otimes A \arrow[dl, "k"] \\ & C & \end{tikzcd}$$

Let $\dot{\Delta}[q] \subset \Delta[q]$ denote the $(q-1)$-skeleton in the usual sense (see example (8.8a)) and let $j: \dot{\Delta}[q] \to \Delta[q]$ be the inclusion map. For an element $\nu \in M_q$ (i.e. map $\nu: X_0 \to A_q$) let $\xi^{-1}(\nu): \Delta[q] \otimes X \to A$ be as in §2. Then we shall denote by $\nu': \Delta[q] \otimes X \to A^q$ and $\nu'': \dot{\Delta}[q] \otimes X \to A^{q-1}$ the unique maps such that the following diagram is commutative

$$\begin{tikzcd} \dot{\Delta}[q] \otimes X \arrow[r, "j \otimes X"] \arrow[d, "\nu''"] & \Delta[q] \otimes X \arrow[d, "\nu'"] \arrow[dr, "\xi^{-1}(\nu)"] & \\ A^{q-1} \arrow[r, "a^{q-1,q}"] & A^q \arrow[r, "a^q"] & A \end{tikzcd}$$

LEMMA (9.2). *Let $K \in \mathcal{M}^V$, let $H \subset K$ be a subcomplex and let $i: H \to K$ denote the inclusion map. Let $C \in \mathcal{C}^V$ and let $h: H \otimes A \to C$ and $k^{q-1}: K \otimes A^{q-1} \to C$ be such that commutativity holds in the diagram*

(9.2a)
$$\begin{tikzcd} H \otimes A^{q-1} \arrow[r, "i \otimes A^{q-1}"] \arrow[d, "H \otimes a^{q-1}"] & K \otimes A^{q-1} \arrow[d, "k^{q-1}"] \\ H \otimes A \arrow[r, "h"] & C \end{tikzcd}$$

For every non-degenerate element $\nu \in M^q$ let be given a map $k_\nu: K \otimes \Delta[q] \otimes X \to C$ such that commutativity holds in the diagram

(9.2b)
$$\begin{tikzcd} K \otimes \dot\Delta[q] \otimes X \arrow[r, "K \otimes j \otimes X"] \arrow[d, "K \otimes \nu''"] & K \otimes \Delta[q] \otimes X \arrow[d, "k_\nu"] & H \otimes \Delta[q] \otimes X \arrow[l, "i \otimes \Delta[q] \otimes X"'] \arrow[d, "H \otimes \xi^{-1}(\nu)"] \\ K \otimes A^{q-1} \arrow[r, "k^{q-1}"] & C \arrow[r, dash, "h"] & H \otimes A \end{tikzcd}$$

Then there exists a unique map $k^q: K \otimes A^q \to C$ such that commutativity holds

(i) *in diagram* (9.1a)

(ii) *in diagram* (9.2a) *with q instead of $q - 1$*

(iii) *in the diagram*

$$\begin{tikzcd} K \otimes \Delta[q] \otimes X \arrow[rr, "K \otimes \nu'"] \arrow[dr, "k_\nu"'] & & K \otimes A^q \arrow[dl, "k^q"] \\ & C & \end{tikzcd}$$

10. Proof of Theorem (6.1)

It follows from Propositions (4.3) and (8.5) and Lemma (9.1) that it suffices to consider the case that $A = A^q$, $B = A^{q-1}$ and $f = a^{q-1} = a^{q-1,q}$.

For every pair of integers (k, n) with $0 \leqq k \leqq n$ let $\Lambda_n^k \subset \Delta[n]$ denote the subcomplex generated by the n $(n - 1)$-simplices $\varepsilon_n^0, \cdots, \varepsilon_n^{k-1}, \varepsilon_n^{k+1}, \cdots, \varepsilon_n^n \in \Delta[n]$ (see [7], §2). It then follows from Lemma (9.2) that we must show that given

(i) commutativity holds in diagram (9.2a) with $H = \Lambda_n^k$ and $K = \Delta[n]$,

(ii) commutativity holds in the diagram

$$\begin{tikzcd} \Delta[n] \otimes A^{q-1} \arrow[r, "\Delta[n] \otimes a^{q-1,q}"] \arrow[d, "k^{q-1}"] & \Delta[n] \otimes A^q \arrow[d, "m"] & \Lambda_n^k \otimes A^q \arrow[l, "i \otimes A^q"'] \arrow[d, "h"] \\ C \arrow[r, dash, "g"] & D \arrow[r, dash, "g"] & C \end{tikzcd}$$

(iii) a non-degenerate element $\nu \in M_q$ there exists a map

$$k_\nu: \Delta[n] \otimes \Delta[q] \otimes X \to C$$

such that commutativity holds

(a) in diagram (9.2b) with $H = \Lambda_n^k$ and $K = \Delta[n]$

(b) in the diagram

$$\begin{tikzcd}
\Delta[n] \otimes \Delta[q] \otimes X \arrow[r, "\Delta[n] \otimes \nu'"] \arrow[d, "k_\nu"] & \Delta[n] \otimes A^q \arrow[d, "m"] \\
C \arrow[r, dash, "g"] & D.
\end{tikzcd}$$

In view of Proposition (2.5) this is equivalent with the existence of a map

$$\gamma(k_\nu):\Delta[n] \times \Delta[q] \to \text{Hom}\,(X, C) \in \mathfrak{M}^V$$

satisfying the following condition:

Let $\Lambda = (\Delta[n] \times \dot{\Delta}[q]) \cup (\Lambda_n^k \times \Delta[q])$, let $j:\Lambda \to \Delta[n] \times \Delta[q]$ denote the inclusion map and let $t:\Lambda \to \text{Hom}\,(X, C)$ be the map given by

$$t \mid (\Delta[n] \times \dot{\Delta}[q]) = \gamma(k^{q-1} \circ (\Delta[n] \otimes \nu''))$$

$$t \mid (\Lambda_n^k \times \Delta[q]) = \gamma(h \circ (\Lambda_n^k \otimes \xi^{-1}(\nu)))$$

then commutativity holds in the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "j"] \arrow[d, "t"] & \Delta[n] \times \Delta[q] \arrow[dl, "\gamma(k_\nu)"] \arrow[d, "\gamma(m \circ (\Delta[n] \otimes \nu'))"] \\
\text{Hom}\,(X, C) \arrow[r, "\text{Hom}\,(X, g)"] & \text{Hom}\,(X, D)
\end{tikzcd}$$

As Hom (X, g) is a fibre map, the existence of such a c.s.s. map $\gamma(k_\nu)$ follows from the main lemma of [3]. This completes the proof.

11. Proof of Lemma (7.3)

Use will be made of Theorem (7.2) for c.s.s. complexes (see [7]).

Let N be a basis of B. In view of Lemma (9.1) it suffices to show that, given a map $g^{q-1}:B^{q-1} \to A$ and a homotopy $h^{q-1}:f \circ g^{q-1} \sim b^{q-1}$, there exists a map $g^q:B^q \to A$ and a homotopy $h^q:f \circ g^q \sim b^q$ such that commutativity holds in the diagrams

(11.1)
$$\begin{tikzcd}
B^{q-1} \arrow[rr, "b^{q-1,q}"] \arrow[dr, "g^{q-1}"'] & & B^q \arrow[dl, "g^q"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes B^{q-1} \arrow[rr, "I \otimes b^{q-1,q}"] \arrow[dr, "h^{q-1}"'] & & I \otimes B^q \arrow[dl, "h^q"] \\
& B &
\end{tikzcd}$$

This is done as follows. Let $\nu \in N_q$ be non-degenerate. Because the map

$$\mathrm{Hom}\,(X, f) : \mathrm{Hom}\,(X, A) \to \mathrm{Hom}\,(X, B) \in \mathfrak{M}^V$$

is a homotopy equivalence it readily follows that there exist maps $a_\nu : \Delta[q] \to \mathrm{Hom}\,(X, A)$ and $b_\nu : I \times \Delta[q] \to \mathrm{Hom}\,(X, B)$ such that commutativity holds in the diagrams

$$\begin{tikzcd}
\Delta[q] \approx P \times \Delta[q] \arrow[r, "\Delta\epsilon^0 \times \Delta[q]"] \arrow[d, "a_\nu"] & I \times \Delta[q] \arrow[d, "b_\nu"] \\
\mathrm{Hom}\,(X, A) \arrow[r, "\mathrm{Hom}\,(X, f)"] & \mathrm{Hom}\,(X, B)
\end{tikzcd}$$

$$\begin{tikzcd}
I \times \dot{\Delta}[q] \arrow[r, "I \times j"] \arrow[d, "\gamma(I \otimes \nu'')"] & I \times \Delta[q] \arrow[d, "b_\nu"] \\
\mathrm{Hom}\,(X, I \otimes B^{q-1}) \arrow[r, "\mathrm{Hom}\,(X, h^{q-1})"] & \mathrm{Hom}\,(X, B).
\end{tikzcd}$$

Define maps $g_\nu : \Delta[q] \otimes X \to A$ and $h_\nu : I \otimes \Delta[q] \otimes X \to B$ by $g_\nu = \gamma^{-1}(a_\nu)$ and $h_\nu = \gamma^{-1}(b_\nu)$. Doing this for all non-degenerate elements $\nu \in M_q$ and applying Lemma (9.2) we get maps $g^q : B^q \to A$ and $h^q : I \otimes B^q \to B$ such that commutativity holds in the diagrams (11.1). A straightforward computation now yields that $h^q : f \circ g^q \sim b^q$.

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