EXTENDED C*-ALGEBRAS AND W*-ALGEBRAS

by

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Introduction : One of the most successful and attractive branches of modern functional analysis is the theory of Banach algebras and its application to the theory of bounded operators in Banach and Hilbert spaces. As is well known, it is necessary, for many applications, to consider unbounded operators and, as far as we know, there is no completely satisfactory axiomisation of this theory, corresponding to that for bounded operators. The starting point for generalisations of the theory of Banach algebras has usually been a topological algebra with some additional properties (e.g. locally convexity). In order to define a spectrum of an element of the algebra which would correspond to the classical spectrum of an unbounded operator, it is necessary to introduce a subalgebra whose elements correspond to bounded operators. Essentially equivalent definitions of such bounded elements (in slightly different contexts) were given by Warner, Waelbroeck, and Allan. In this lecture, we would like to follow a different course - we begin with a "core"Banach algebra of "bounded elements" and consider certain algebraic extension. We then use the normed structure of the core algebra to define a suitable structure on its extendsion. It turns out that the convenient structure is a bornology and not a topology. Our results and methods are close to those of Allan and indeed it was the reading of his work that motivated us. However, by abandoning the original topology used by Allan (which we believe to be irrelevant), one obtains a certain simplification and a number of results become more precise.

In this lecture, we consider only commutative *-algebras. The methods we use are essentially a formalisation of familiar techniques used to extend the spectral theory for bounded operators to unbounded operators. We have concentrated on obtaining representations of our algebras as algebras of functions. One can then easily develop the usual theory (definition of the spectrum of an element, a functional calculus, spectral mapping theorems, etc.). We have also given what we believe to be the natural extension of the concept of a W^*-algebra (algebra of operators) to algebras of unbounded operators and have given, in this context, an abstract version of the spectral theorem for unbounded operators in Hilbert space.

Notation : Let E be a vector space over $\mathscr{C}$. A subset B of E is a *ball* if it is absolutely convex and contains no line. We then define E_B to be the linear span of $B(:= \bigcup_{n=1}^{\infty} nB)$ and regard it as a normed space under the norm

$$\|\ \|_B : x \mapsto \inf \{\lambda : x \in \lambda B\}$$

B is a *Banach ball* if $(E_B, \|\ \|_B)$ is a Banach space.

A convex bornological space (abbreviated CBS) is a vector space E together with a family $\mathfrak{B}$ of balls which is directed on the right by inclusion and whose union is E. If $(E, \mathfrak{B})$ is a CBS, $B \subseteq E$ is defined to be bounded if $B \subseteq B_1$ for some B_1 in $\mathfrak{B}$. $(E, \mathfrak{B})$ is a complete CBS if there is a subset $\mathfrak{B}_1$ of $\mathfrak{B}$ consisting of Banach balls which contains an overset of each $B \in \mathfrak{B}$. If $(E, \mathfrak{B})$, $(F, \mathfrak{B}_1)$ are CBS's, a linear mapping $T : E \to F$ is *bounded* if it maps bounded sets of E into bounded sets in F. If E has a locally convex space structure, the set of bounded, absolutely convex subsets of E defines a bornology on E - it is called the *von Neumann bornology.*

Let A be an algebra over $\mathscr{C}$. An *involution on A* is an antilinear mapping $x \to x^*$ from A into itself so that $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for each $x, y \in A$. An algebra with involution is called a **-algebra.* A *-algebra A with unit e is *symmetric* if $(e + x^* x)$ is invertible for each $x \in E$. If X is a set, the space $\mathfrak{F}(X)$ of all mappings from X into $\mathscr{C}$ has a natural *-algebra structure (given by pointwise algebraic operations and complex conjugation as involution). We will be concerned with subalgebras of $\mathfrak{F}(X)$, closed under complex conjugation, and will always regard them as *-algebras with the induced structure.

A *C^*-algebra* is a *-algebra A together with a norm so that

(*i*) $(A, \|\ \|)$ is a Banach algebra;

(*ii*) $\|x^* x\| = \|x\|^2$ for each $x \in A$.

We shall only be intereseted in commutative C^*-algebras with a unit and so *shall use the name* "C^*-algebra" *in this restricted sense.* If X is a compact space, then $C(X)$, the space of continuous functions in $\mathfrak{F}(X)$ is a C^*-algebra with the supremum norm

$$x \mapsto \sup \{ |x(t)| : t \in X \}$$

If A is a C^*-algebra, the *spectrum* $M(A)$ of A is defined to be the set of linear multiplicative functions f on A which map e into 1. $M(A)$ is a $\sigma(A', A)$-compact subset of A' and we regard it as a compact space with the induced topology. If $x \in A$, then the mapping

$$\hat{x} : f \mapsto f(x)$$

from $M(A)$ into $\mathscr{C}$ is continuous, i.e. is an element of $C(M(A))$. We have thus constructed a mapping $x \mapsto \hat{x}$ from A into $C(M(A))$. The theory of C^*-algebras can be summarised in the theorem :

The Gelfand-Neumark transform is a C^*-algebra isomorphism from A onto $C(M(A))$.

A C^*-algebra is said to be a *W^*-algebra* if, as a Banach space, it is the dual of a Banach space. Let M be a locally compact space which is the topological sum of a family of compact sets, μ a positive Radon measure on M. Then the Lebesgue spaces $L^1(M; \mu)$ and $L^\infty(M; \mu)$ are Banach spaces and $L^\infty(M; \mu)$ can be identified with the dual of $L^1(M; \mu)$. $L^\infty(M; \mu)$ has a natural C^*-algebra structure and so is a W^*-algebra. The representation theory for W^*-algebras is as follows :

Let A be a W^*-algebra. Then A is isomorphic (as a C^*-algebra) to a W^*-algebra $L^\infty(M; \mu)$ (M, μ as above).

§ 1. Extended C^*-algebras.

1.1. **Definition** : Let $(A_0, \| \ \|)$ be a C^*-algebra and suppose that A_0 is embedded as a subalgebra of an algebra A with involution so that

(*i*) e is a unit for A ;

(*ii*) for each $x \in A$, $(e+x^*x)$ is invertible and $(e+x^*x)^{-1}$ and $x(e+x^*x)^{-1}$ lie in A_0 .

Then we call A *an extended C^*-algebra over A_0* .

We note that this implies that every element of A is expressible in the form yz^{-1} where $y, z \in A_0$, i.e. A is a quotient algebra over A_0 and so A is commutative. We define a bornology $\mathfrak{B}$ on A as follows : Let $B(A_0)$ denote the unit ball of $(A_0, \| \ \|)$. Then a subset B of A is in $\mathfrak{B}$ if there is a $z \in A_0$ (z invertible in A) so that $B \subseteq B(A_0) z^{-1}$.

1.2. **Proposition** : (*i*) $(A, \| \ \|)$ *is a complete CBS* ;

(*ii*) $x \mapsto x^*$ *is a bounded mapping* ;

(*iii*) *multiplication is a jointly bounded mapping.*

Proof. (*i*) It is clear from the above remark that $A = \bigcup_{B \in \mathfrak{B}} B$. Also the mapping $x \mapsto xz^{-1}$ is linear and maps $B(A_0)$ onto $B_1 := B(A_0) z^{-1}$ and so the latter is a ball. The same mapping defines an isomorphism between $(A_0, \| \ \|)$ and the normed space A_{B_1} and so B_1 is even a Banach ball.

(*ii*) Involution maps $B(A_0)z^{-1}$ onto $B(A_0) (z^*)^{-1}$.

(*iii*) Multiplication maps $B(A_0) z_1^{-1} \times B(A_0) z_2^{-1}$ into $B(A_0) (z_1 z_2)^{-1}$.

1.3. **Examples**: *A*. Let X be a completely regular topological space and let A_0 be the C^*-algebra $C^\infty(X)$ of bounded, continuous, complex-valued functions on X with the supremum norm. If A is the algebra $C(X)$ of all continuous, complex-valued functions on X (with complex conjugation as involution), then 1(*i*) and (*ii*) are obviously satisfied and so we can regard $C(X)$ as an extended C^*-algebra over $C^\infty(X)$. The bornology of $C(X)$ defined above is easily seen to be the bornology with a basis $\{B_x : x \in C(X)\}$ where

$$B_x := \{y \in C(X) : |y| \leq |x|\}$$

If X is locally compact and countable at infinity, then this is the von Neumann bornology associated with the structure of compact convergence on X.

B. Let A be a commutative algebra of unbounded operators in a Hilbert space H. That is, A is a set of closed, densely defined partial linear operators in H so that

(*i*) if $S \in A$, then $S^* \in A$;

(*ii*) if $S, T \in A$, then $S+T$ and ST are densely defined and closeable and their closures lie in A (we write $S+T$ and $S.T$ for their closures);

(*iii*) if $S, T \in A$, then $S.T = T.S$;

(*iv*) $I \in A$ and if $T \in A$, then $(I+T^*T)^{-1}$ and $T(I+T^*T)^{-1}$ are in A;

(*v*) $A \cap L(H)$ is norm-closed.

Then A is an extended C^*-algebra over $A \cap L(H)$.

C. Let M be a locally compact space, μ a positive Radon measure on M. We consider the C^*-algebra $L^\infty(M;\mu)$ of equivalence classes of complex-valued functions which are bounded (on the complement of a locally μ-negligeable set). Let $S(M;\mu)$ denote the $*$-algebra of equivalence classes of μ-measurable complex-valued functions. Then 1(*i*) and (*ii*) are satisfied and we can regard $S(M;\mu)$ as an extended C^*-algebra over $L^\infty(M;\mu)$. The bornology defined on $S(M;\mu)$ has as basis the sets $\{B_x : x \in S(M;\mu)\}$ where

$$B_x := \{y \in S(M;\mu) : |y| \leq |x| \text{ (locally } \mu\text{—a.e.)}\}$$

D. Let X be a compact space. We consider pairs (Y, A_y) where

(*i*) Y is an open dense subspace of X;

(*ii*) A_Y is a symmetric $*$-subalgebra of $C(Y)$ so that $A_Y \cap C^\infty(Y) = C(X)$.

Now we consider a family $\mathcal{S}$ of open, dense subsets of X which contains X and is closed under the formation of finite intersections and suppose that for each $Y \in \mathcal{S}$, we are given an algebra A_Y as above. In addition, we suppose that if $Y, Z \in \mathcal{S}$, $Y \subseteq Z$, then $A_Z \subseteq A_Y$. Then the system $\{A_Y : Y \in \mathcal{S}\}$ with the natural injection mappings form an inductive system of *-algebras. Their union is an extended C^*-algebra over $C(X)$-we denote it by $C_{\mathcal{S}}(X)$. The bornology of $C_{\mathcal{S}}(X)$ has as basis the sets $\{B_{Y,x} : Y \in \mathcal{S}, x \in A_Y\}$ where.

$$B_{Y,x} := \{y \in A_Y : |y| \leqslant |x| \text{ on } Y\}.$$

(Note that example A corresponds to the special case where $\mathcal{S}$ is the pair $\{X, \beta X\}$ and A_X is $C(X)$).

1.4. **Definition.** Let A (resp. B) be an extended C^*-algebra over A_0 (resp. B_0). Then an (*extended* C^*-) *morphism* from A into B is a unit-preserving *-algebra morphism from A into B which maps A_0 into B_0. If the morphism is bijective and maps A_0 onto B_0, it is called an (*extended* C^*-) *isomorphism.*

1.5. **Proposition**: *Let A, B be as in 4, Φ a morphism from A into B. Then Φ is bounded.*

Proof. Since Φ maps A_0 into B_0, it maps the unit ball $B(A_0)$ of A_0 into the unit ball $B(B_0)$ of B_0. Hence it maps $B(A_0)z^{-1}$ into $B(B_0)\{\Phi(z)\}^{-1}$ (z invertible in A).

1.6. **Proposition.** *Let A be an extended C^*-algebra over A_0, x the spectrum $M(A_0)$ of A_0. Then there is a family $\mathcal{S}$ of dense, open subsets of X with associated algebras $\{A_Y : Y \in \mathcal{S}\}$ so that A is isomorphic tu $C_{\mathcal{S}}(X)$.*

Proof. Let M denote the multiplicative semigroup in A_0 consisting of those elements which are invertible in A. If $z \in M$, define

$$Y_Z := \{f \in M(A_0) : f(z) \neq 0\}.$$

Then Y_Z is open in X. It is dense. For suppose that $X \setminus Y_Z$ contains an open set. Then we can find a positive function in $C(X)$ which vanishes on Y_Z and we can suppose that this function is the image under the Gelfand-Neumark transform (for A_0) of an element x of A_0. Then $xz = 0$ and so z is a divisor of zero in A_0 and so also in A which is impossible (z is invertible in A).

Let $\mathcal{S} := \{Y_Z : z \in M\}$. Then $Y_Z \cap Y_{Z_1} = Y_{ZZ_1}$ and so $\mathcal{S}$ is closed under finite intersections. If $z \in M$, put

$$M_Z := \{z_1 \in M : S_{Z_1} \supseteq S_Z\}$$

and define $\bar{A}_Z$ to be the set of those elements of A which can be written in the form xy^{-1} ($x \in A_0$, $y \in M_Z$). $\bar{A}_Z$ is a *-subalgebra of A and the Gelfand-Neumark transform $x \to \widehat{x}$ from A_0 onto $C(X)$ can be extended to a *-algebra morphism from $\bar{A}_Z$ onto an algebra of functions in $Y := Y_Z$ by mapping xy^{-1} onto $(\widehat{x}|_Y)/(\widehat{y}|_Y)$. We denote this algbra by A_Y. A_Y is symmetric since $\bar{A}_Z$

is. We show that $A_Y \cap C^\infty(Y) = C(X)$, i.e. that each bounded function in A_Y can be extended continuously to X. Such a function is the image $\hat{x}$ under the extended Gelfand-Neumark transform of an element x of A. Then the function $\{(e+x^*x)^{-1}\}^\frown$ is bounded away from zero on Y and so on X. Hence $(e+x^*x)^\frown$ is bounded on X and the product $\{x(e+x^*x)^{-1}\}^\frown(e+x^*x)^\frown$ is the required extension.

1.7. **Corollary**: *Let A be an extended C^*-algebra over A_0. Then $B(A_0)$ is the largest idempotent, self-adjoint, absolutely convex bounded subset of A.*

1.8. **Corollary**: *If A (resp. B) is an extended C^*-algebra over A(resp. B) and Φ is a bounded unit-preserving $*$-algebra morphism from A into B, then Φ is a morphism.*

1.9. **Corollary**: *Let A be an extended C^*-algebra over A_0. Then A is isomorphic to the algebra of continuous functions on a completely regular space* (cf. 3A) *if the following conditions hold:*

(i) $Y := \bigcap \mathscr{S}$ *is a dense subset of* $X := M(A_0)$ *and X is the Stone-Čech compactification of Y;*

(ii) if $x \in A$ does not vanish on Y, then x is invertible in A.

1.10. **Definition**: A linear form f on an extended C^*-algebra is *positive* if $f(x^*x) \geqslant 0$ for each $x \in A$.

1.11. **Proposition**: *(i) A positive linear functional on an extended C^*-algebra is bounded;*

(ii) if $\bigcap \mathscr{S}$ is dense in $M(A_0)$, then the set of positive linear forms on A separates A.

1.12. **Proposition**: *Let A be an extended C^*-algebra. Then every multiplicative linear form f on A is bounded.*

Proof. The restriction of f to A_0 is bounded in absolute value by one on $B(A_0)$. Hence if $y \in B(A_0) . z^{-1}$ then $|f(y)| \leqslant |f(z)^{-1}|$.

1.13. **Remark**: We mention some categorical constructions in the category of extended C^*-algebras. If A is an extended C^*-algebra over A_0, a *subobject of A* is a $*$-subalgebra B of A containing e so that

(*i*) $B \bigcap A_0$ is closed in A_0;

(*ii*) $(e+x^*x)^{-1}$ is in B for each $x \in B$.

Then B is an extended C^*-algebra over $B \cap A_0$.

Now let I be a self-adjoint ideal in A so that $I \cap A_0$ is closed in A_0 (note that this is equivalent to the closedness of I in the bornology of A). Then A/I (with its natural structure as a $*$-algebra) is an extended C^*-algebra over $A_0/I \cap A_0$ and the resulting bornology is the quotient bornology.

If $\{A^i : i \in I\}$ is a family of extended C^*-algebras, then the product $A := \prod_{i \in I} A^i$ has a natural structure of a $*$-algebra with a unit. Consider the subset $B := \prod_{i \in I} B(A^i_0)$. Then this is the unit ball of a C^*-algebra $A_0 \subseteq A$ (A_0 is the C^*-algebra product of the C^*-algebras A^i_0) and A is an extended C^*-algebra over A_0. Similarly, the unrestricted projective limit of a projective system of extended C^*-algebras is an extended C^*-algebra.

1.14. **Example**: Let $\{j_{\beta\alpha} : A_\beta \to A_\alpha, \alpha \leqslant \beta, \alpha, \beta \in I\}$ be a projective system of C^*-algebras and let A be its projective limit (in the category of sets). Then A has the structure of a commutative $*$-algebra with a unit. Let A_0 be the C^*-algebra projective limit of the system. Then $A_0 \subseteq A$ and A is an extended C^*-algebra over A. The bornology of A is precisely the projective limit of the norm bornologies on A_α and so is the von Neumann bornology of the projective locally convex structure on A.

1.15. **Example**: Let H be a Hilbert space, $\mathscr{K}$ a family of closed subspaces with the following properties :

(*i*) $\mathscr{K}$, ordered by inclusion, is a directed set;

(*ii*) $\bigcup_{\mathscr{K}} K$ is dense in H.

Suppose given a family $\{A_K : K \in \mathscr{K}\}$ where, A_K is a commutative C^*-algebra of operators on K so that if $K \subseteq K_1$ then K is invariant under A_{K_1} and A_K coincides with the set of restrictions of the operators of A_{K_1} to K. Then if we write $j_{K_1, K}$ for the natural mapping from A_{K_1} onto A_K,

$$\{j_{K_1, K} : A_{K_1} \to A_K, \; K \subseteq K_1, \; K, K_1 \in \mathscr{K}\}$$

is a projective system. The C^*-algebra projective limit of this system can be identified with a C^*-algebra A_0 of operators on H and the projective limit described in 14 can be identified with an algebra of unbounded operators in H.

1.16. **Remark**: We comment briefly on the relation between the above and the results of Allan. If A is commutative C_B-$*$algebra (see the papers of Allan quoted below for the definition), then the subalgebra A_0 of bounded elements defined by Allan is a C^*-algebra and A is an extended C^*-algebra over A_0. The bornology defined in 1 is finer than the von Neumann bornology of A.

§ 2. Extended W^*-algebras.

2.1. **Definition**: Let A, A_1 be extended C^*-algebras over A_0. We say that A_1 is *greater than* A (written $A_1 \geqslant A$) if there is an injective morphism Φ from A into A_1 extending the identity on A_0.

$$\begin{tikzcd} A_0 \arrow[r, "\subseteq"] \arrow[dr, "\subseteq"'] & A \arrow[d, "\Phi"] \\ & A \end{tikzcd}$$

2.2. **Proposition**: *Let A and A_1 be extended C^*-algebras over A_0 and let* $\mathrm{inv}(A_0)$ (resp. $\mathrm{inv}_1(A_0)$) *denote the set of elements of A_0 which are invertible in A (resp. in A_1). Then $A_1 \geqslant A$ if and only if* $\mathrm{inv}(A_0) \subseteq \mathrm{inv}_1(A_0)$.

Proof. The necessity of the condition is clear. Suppose that it is fulfilled. Then Φ can be defined by mapping xz^{-1} ($x \in A$, $z \in \mathrm{inv}(A_0)$ — z^{-1} denotes the inverse in A) into xz^{-1} (inverse in A_1).

2.3. **Definition**: Let A_0 be a W^*-algebra. Then an *extended W^*-algebra over A_0* is an extended C^*-algebra over A_0 which is maximal under the above ordering.

2.4. **Lemma**: *Let A be an extended C^*-algebra over a W^*-algebra A_0. Then A is an extended W^*-algebra over A_0 if every non zero divisor in A_0 is invertible in A.*

2.5. **Example**: Let M be a locally compact space which is the direct sum of compact spaces and let μ be a positive Radon measure on M. Then $L^\infty(M;\mu)$ is a W^*-algebra and $S(M;\mu)$ is an extended W^*-algebra over $L^\infty(M;\mu)$. (For x is a non zero divisor in $L^\infty(M;\mu)$ if and only if $\mu\{t \in M : x(t) = 0\} = 0$ and then x is invertible in $S(M;\mu)$).

2.6. **Proposition**: *Let A be an extended W^*-algebra over A_0. Then there is a pair $(M;\mu)$ as in 5 so that A is isomorphic to $S(M;\mu)$.*

Proof. By the representation theorem for W^*-algebras, A_0 is isomorphic to a space $L^\infty(M;\mu)$. Then, up to isomorphism, $S(M;\mu)$ and A are extended W^*-algebras over $L^\infty(M;\mu)$. But it follows immediately from 2 that there is at most one extended W^*-algebra over a given W^*-algebra and so A and $S(M;\mu)$ are isomorphic.

2.7. **Remark**: Let A be an extended W^*-algebra over A_0. We give A an ordering as follows:

$x \geqslant 0$ if and only if $x = y^*y$ for some $y \in A$.

Then A, with this ordering, is a Dedekind complete vector lattice (this follows immediately from 6).

2.8. **Definition**: Let A be an extended W^*-algebra over A_0. We consider the basis

$$\mathfrak{B}_1 := \{B(A_0)z^{-1} : z \in \text{inv}(A_0)\}$$

for the bornology $\mathfrak{B}$. If $B \in \mathfrak{B}_1$, then A_B is isomorphic as a Banach space to A_0 and so we can transfer the Mackey topology $\mathscr{T}(A_0, F)$ to a topology on A_B (F denotes the predual of A_0). We say that a net (x_α) in A *converges bounded ultrastrongly* to x_0 in A if there is a $B \in \mathfrak{B}_1$ so that $x_\alpha \to x_0$ in this topology on A_B. This is equivalent to the existence of a $z \in \text{inv}(A_0)$ so that $\{x_\alpha z\}$ is bounded in A_0 and $x_\alpha z \to x_0 z$ in $\mathscr{T}(A_0, F)$.

2.9. **Example**: In the extended W^*-algebra $S(M; \mu)$ (see 5) then $x_\alpha \to x_0$ bounded ultrastrongly if and only if there is a strictly positive measurable function z so that $\{x_\alpha z\}$ is bounded and $x_\alpha z \to x_0 z$ in $L^1(K; \mu_K)$ for each compact subset K of M (μ_K is the restriction of μ to K).

2.10. **Definition**: Let A be an extended W^*-algebra over A_0. An element p of A is said to be a projection if it is selfadjoint and idempotent (then $p \in A_0$ and is a projection in A_0). If x is an element of A, there exists a smallest projection p so that $px = x$. We call this *the support of* x. (If A is the extended W^*-algebra $S(M; \mu)$, then the projections are precisely the functions of the form χ_N - the characteristic function of the measurable subset N of M. If $x \in S(M; \mu)$, the support of x is χ_N where $N := \{t \in M : x(t) \neq 0\}$).

2.11. **Notation**: Let A be an extended W^*-algebra over A_0, x a self-adjoint element of A. Then for each $\lambda \in \mathscr{K}$, we denote by $e(\lambda)$ the support of $(\lambda e - x)^+$ ($:= \sup\{(\lambda e - x), 0\}$).

2.12. **Proposition**: *Let x be a self-adjoint element of an extended W^*-algebra A. Then*

(i) *if $\lambda \leqslant \mu$ in $\mathscr{K}$, $e(\lambda) \leqslant e(\mu)$;*

(ii) $\lim\limits_{\lambda \to -\infty} e(\lambda) = 0, \quad \lim\limits_{\lambda \to \infty} e(\lambda) = e$ *(in $\sigma(A_0, F)$);*

(iii) *if $\lambda_n \uparrow \lambda$ in $\mathscr{K}$, then $e(\lambda_n) \to e(\lambda)$ in $\sigma(A_0, F)$;*

(iv) $x = \int_{\mathscr{K}} \lambda \, de(\lambda)$ *in the sense that the natural Riemann-Stieltjes sums for this integral converge to x bounded ultrastrongly,*

Proof. This follows from the representation theorem 6.

BIBLIOGRAPHICAL REMARKS

The theory of C^*-algebras and W^*-algebras is presented in

S. SAKAI, C^*-algebras and W^*-algebras (Berlin, 1971).

See also

J. DIXMIER, Les C^*-algèbres et leurs représentations (Paris, 1964).

J. DIXMIER, Les algèbres d'opérateurs dans l'espace Hilbertien (Paris, 1969).

The Lebesgue spaces used above are defined in

N. BOURBAKI, Eléments de Mathématique. Livre VI, Intégration. (Paris, 1965).

Allan's papers are as follows :

G.R. ALLAN, A spectral theory for locally convex algebras, Proc. London Math. Soc. **15** (3) : 399-421 (1965).

G.R. ALLAN, On a class of locally convex algebras, Proc. London Math. Soc. **17**(3): 91-114 (1967).

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