STABLE HOMOTOPY THEORY

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A GENERAL SUMMARY

We set out here, under lettered heads, the general properties of CW-spectra, which are designed to overcome the objections to previous theories of stable homotopy. They are closely analogous to CW-complexes in ordinary homotopy theory; and it is this analogy that gives them their favourable properties.

A. Topological categories

In any category $\underset{\approx}{A}$ we write $\mathrm{Mor}_{\underset{\approx}{A}}(X, Y)$ or $\mathrm{Mor}(X, Y)$ for the set of morphisms from X to Y.

A.1. We say that $\underset{\approx}{A}$ is a topological category if

a) $\mathrm{Mor}(X, Y)$ is a topological space for all objects X, Y,

b) The composition map

$$\mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z),$$

which we write $f \times g \rightsquigarrow g \circ f$, is separately continuous for all objects X, Y, Z.

In practice our topological categories satisfy the extra axiom:

A.2. The composite $g \circ f$ is jointly continuous in $f \times g$ if we restrict f or g to lie in a compact subset.

For example, the category $\underset{\sim}{T}$ of topological spaces and continuous maps becomes itself a topological category if we endow $\mathrm{Mor}(X, Y)$ with the compact-open topology. In this example composition is not always jointly continuous.

If $\underset{\sim}{A}$ has a zero object, we may take the zero morphism (which we write as o) as base point of $\mathrm{Mor}(X, Y)$.

Let $F:\underset{\sim}{A} \to \underset{\sim}{B}$ be a functor between topological categories.

A.3. We call F continuous if $F:\mathrm{Mor}(X, Y) \to \mathrm{Mor}(FX, FY)$ is continuous for all objects X, Y in $\underset{\sim}{A}$.

Now let $\underset{\sim}{A}$ be a topological category.

A.4. A homotopy from the morphism $f:X \to Y$ to the morphism $g:X \to Y$ is a path from f to g in $\mathrm{Mor}(X, Y)$. We write $f \simeq g$.

This enables us to define the usual homotopy-theoretic concepts, such as homotopy equivalence (written $f:X \simeq Y$), deformation retract, and homotopy type. In particular assume $\underset{\sim}{A}$ has a zero object. Then the object X is contractible if either of the equivalent conditions holds:

A.5. X has the homotopy type of a zero object,

A.6. The identity and zero morphisms of X are homotopic.

In general,

A.7. The homotopy category $\underset{\sim}{A}_h$ has the same objects as $\underset{\sim}{A}$, its morphisms are the homotopy classes of morphisms of $\underset{\sim}{A}$, and composition is induced from that in $\underset{\sim}{A}$.

Let $F:\underset{\sim}{A} \to \underset{\sim}{B}$ be a continuous functor.

<u>A.8.</u> Then F induces the homotopy functor, $F_h:\underset{\sim}{A}_h \to \underset{\sim}{B}_h$.

<u>B. CW-complexes</u>

Our CW-complexes are assumed to be given a particular cell structure and a base point o, which is a 0-cell. We consider only those maps that respect base points; in particular, we consider only those subcomplexes that contain the base point.

<u>B.1.</u> We have various categories $\underset{\sim}{C}$, $\underset{\sim}{F}$, $\underset{\sim}{I}(\underset{\sim}{C})$, $\underset{\sim}{I}(\underset{\sim}{F})$ of CW-complexes, with objects and morphisms as follows:

$\underset{\sim}{C}$: arbitrary CW-complexes, continuous maps.

$\underset{\sim}{F}$: finite CW-complexes, continuous maps.

$\underset{\sim}{I}(\underset{\sim}{C})$: arbitrary CW-complexes, inclusions of subcomplexes.

$\underset{\sim}{I}(\underset{\sim}{F})$: finite CW-complexes, inclusions of subcomplexes.

Thus $\mathbf{I}(\mathbf{F}) = \mathbf{I}(\mathbf{C}) \cap \mathbf{F}$, and $\mathbf{F}$ and $\mathbf{I}(\mathbf{F})$ are full subcategories of $\mathbf{C}$ and $\mathbf{I}(\mathbf{C})$ respectively. We write an inclusion $A \to X$ as $A \subset X$.

We generalize this situation. Suppose given a category $\mathbf{A}$ and any subcategory $\mathbf{I}(\mathbf{A})$ satisfying

B.2. a) $\mathbf{I}(\mathbf{A})$ has the same objects as $\mathbf{A}$,

b) Every morphism in $\mathbf{I}(\mathbf{A})$ is a monomorphism in $\mathbf{A}$.

We can construct new categories, by a double limit process,

B.3. $\mathbf{A}_W$ and $\mathbf{I}(\mathbf{A}_W) \subset \mathbf{A}_W$, satisfying B.2., and containing the pair $\mathbf{I}(\mathbf{A}) \subset \mathbf{A}$ as full subcategories.

Suffice it to say that the objects of $\mathbf{A}_W$ are the directed non-empty (commutative) diagrams over $\mathbf{I}(\mathbf{A})$, and that we recover (essentially) the pair $\mathbf{I}(\mathbf{C}) \subset \mathbf{C}$ from $\mathbf{I}(\mathbf{F}) \subset \mathbf{F}$ as follows. From $\mathbf{I}(\mathbf{F}) \subset \mathbf{F}$ we construct $\mathbf{I}(\mathbf{F}_W) \subset \mathbf{F}_W$. We assign to a CW-complex X the diagram of all its finite subcomplexes; this yields a functor $\mathbf{C} \to \mathbf{F}_W$ taking $\mathbf{I}(\mathbf{C})$ to $\mathbf{I}(\mathbf{F}_W)$.

B.4. The functors $\mathbf{C} \to \mathbf{F}_W$ and $\mathbf{I}(\mathbf{C}) \to \mathbf{I}(\mathbf{F}_W)$ are equivalences of categories.

B.5. Moreover, if $\mathbf{A}$ is a topological category, so is $\mathbf{A}_W$.

B.6. $\mathbf{F}$ is a topological category, under the compact-open topology, and by B.4. and B.5., $\mathbf{C}$ is also a topological category. (However, the topology received in this way by $\mathbf{C}$ is not the compact-open topology.)

We therefore have

B.7. The homotopy category $\underset{\sim}{C}_h$. We write [X, Y] for the set of morphisms from X to Y in $\underset{\sim}{C}_h$.

B.8. An equivalence of categories $\underset{\sim}{C}_h \to \underset{\sim}{F}_{Wh}$.

The category $\underset{\sim}{C}_h$ is the subject of homotopy theory. Equivalently we may study $\underset{\sim}{F}_{Wh}$. Note that we have not even defined $\underset{\sim}{F}_{hW}$.

We review briefly those constructions in homotopy theory that we need for our present purposes. The requisite formal properties are well known, and omitted, and will be reflected in the properties of CW-spectra.

B.9. The CW-complex consisting of one point, which we also write as o, is a zero object in $\underset{\sim}{F}$ or $\underset{\sim}{C}$.

B.10. For any $n \geqslant 0$, we define an n-sphere Σ^n as any CW-complex having just one n-cell and no others, apart from o. This determines Σ^n up to isomorphism in $\underset{\sim}{I}(\underset{\sim}{F})$.

B.11. Given $A \subset X$, we have the identification map $p: X \to X/A$, and the natural cell structure on X/A.

Let (X_λ) be any family of CW-complexes.

B.12. We have the wedge, or one-point-union, $\bigvee_\lambda X_\lambda$, in $\underset{\sim}{I}(\underset{\sim}{C})$ or $\underset{\sim}{C}$. It is a sum in the category $\underset{\sim}{C}$, and contains each X_λ as a subcomplex. We write $A \vee B$ if the family has two members A and B.

The smash product, or reduced join, $X \wedge Y$ of X and Y is defined as $X \wedge Y = (X \times Y)/(X \vee Y)$, retopologized as a CW-complex, and given the usual cell structure.

B.13. We have the smash product functor $\underset{\sim}{C} \times \underset{\sim}{C} \to \underset{\sim}{C}$, which is separately continuous. It induces functors $\underset{\sim}{F} \times \underset{\sim}{F} \to \underset{\sim}{F}$, $\underset{\sim}{I}(\underset{\sim}{C}) \times \underset{\sim}{I}(\underset{\sim}{C}) \to \underset{\sim}{I}(\underset{\sim}{C})$, and $\underset{\sim}{I}(\underset{\sim}{F}) \times \underset{\sim}{I}(\underset{\sim}{F}) \to \underset{\sim}{I}(\underset{\sim}{F})$.

The suspension SX of X is defined by $SX = \Sigma^1 \wedge X$, and we put $Sf = 1 \wedge f$ for a map f.

B.14. We have the suspension functor $S:\underset{\sim}{C} \to \underset{\sim}{C}$. It is continuous, and induces functors $S:\underset{\sim}{F} \to \underset{\sim}{F}$, $S:\underset{\sim}{I}(\underset{\sim}{C}) \to \underset{\sim}{I}(\underset{\sim}{C})$, and $S:\underset{\sim}{I}(\underset{\sim}{F}) \to \underset{\sim}{I}(\underset{\sim}{F})$.

We denote by S^n the functor S iterated n times. It is not to be confused with Σ^n, the n-sphere.

C. The stable categories

Take a copy $\underset{\sim}{F}_n$ of the category $\underset{\sim}{F}$ for each integer n. Then we consider the sequence of categories and functors

C.1. $$\cdots \to \underset{\sim}{F}_{-2} \vec{S} \underset{\sim}{F}_{-1} \vec{S} \underset{\sim}{F}_0 \vec{S} \underset{\sim}{F}_1 \vec{S} \underset{\sim}{F}_2 \cdots$$

In terms of this sequence, we obtain the *suspension category* $\mathbf{F}_S$ (a 'limit' in a highly technical sense only).

C.2. An object of $\mathbf{F}_S$ is uniquely an object of some $\mathbf{F}_n$. If $X \in \mathbf{F}_m$ and $Y \in \mathbf{F}_n$, the morphisms from X to Y in $\mathbf{F}_S$ form the set

$$\varinjlim_k \operatorname{Mor}_{\mathbf{F}}(S^{k-m}X,\ S^{k-n}Y).$$

We note that in $\mathbf{F}_S$, $X \in \mathbf{F}_n$ is isomorphic to $SX \in \mathbf{F}_{n+1}$; it is not necessary for categorical purposes to identify these two objects.

C.3. The subcategory $\mathbf{I}(\mathbf{F}_S) \subset \mathbf{F}_S$ is obtained similarly, with $\mathbf{I}(\mathbf{F})$ in place of $\mathbf{F}$ throughout.

C.4. The pair of categories $\mathbf{I}(\mathbf{F}_{SW}) \subset \mathbf{F}_{SW}$ is obtained from the pair $\mathbf{I}(\mathbf{F}_S) \subset \mathbf{F}_S$ by means of B.3.

C.5. The *category of CW-spectra* $\mathbf{S}$ is defined as $\mathbf{F}_{SW}$. Its objects are called *CW-spectra* or simply *spectra*. We shall call its morphisms *maps* of spectra.

C.6. The maps in the subcategory $\mathbf{I}(\mathbf{S}) = \mathbf{I}(\mathbf{F}_{SW})$ are called *inclusions* of spectra; we write an inclusion $A \to X$ as $A \subset X$, and say (by abuse of language) that A is a *subspectrum* of X.

C.7. A *finite spectrum* is a spectrum which is isomorphic in $\mathbf{I}(\mathbf{S})$ to some object of $\mathbf{F}_S$.

C.8. The category $\mathbf{S}$ is a topological category. The space $\operatorname{Mor}_{\mathbf{S}}(X, Y)$ is Hausdorff, and normal when X is a finite spectrum.

C.9. The homotopy category of CW-spectra is the homotopy category $\underset{\sim}{S}_h$ of $\underset{\sim}{S}$. We write $\{X, Y\}$ for the set of morphisms from X to Y in $\underset{\sim}{S}_h$; these are the homotopy classes of maps in $\underset{\sim}{S}$.

The category $\underset{\sim}{S}_h$ is the ultimate object of study in our stable homotopy theory.

Remark $\underset{\sim}{S}_h$ has as subcategory $\underset{\sim}{F}_{Sh}$. The latter appears to be absent from the literature, in spite of the fact that it is the most natural category for expressing Spanier-Whitehead duality, and that it is easy to set up directly: in obvious notation $\underset{\sim}{F}_{Sh} = \underset{\sim}{F}_{hS}$.

Suppose $X \in \underset{\sim}{F}_m$ and $Y \in \underset{\sim}{F}_n$ are objects of $\underset{\sim}{F}_S$. Then

C.10. $$\{X, Y\} = \varinjlim_k \, [S^{k-m}X, S^{k-n}Y].$$

The inclusion of $\underset{\sim}{F}$ in $\underset{\sim}{F}_S$ as the copy $\underset{\sim}{F}_0$ of $\underset{\sim}{F}$ induces functors

C.11. $$\underset{\sim}{F} \subset \underset{\sim}{F}_S, \quad \underset{\sim}{I}(\underset{\sim}{F}) \subset \underset{\sim}{I}(\underset{\sim}{F}_S), \quad \underset{\sim}{C} \subset \underset{\sim}{F}_W \subset \underset{\sim}{F}_{SW} = \underset{\sim}{S}, \quad \underset{\sim}{C}_h \to \underset{\sim}{S}_h,$$

which are usually omitted from the notation.

In terms of the functor $\underset{\sim}{C} \subset \underset{\sim}{S}$ in C.11., let X and Y be CW-complexes, and suppose that X is finite-dimensional.

C.12. Then $$\{X, Y\} = \varinjlim_k \, [S^kX, S^kY].$$

This result is false in general if X has infinite dimension, which shows where our theory diverges from the S-category as originally proposed.

The sequence C.1. has an obvious automorphism, given by moving one step to the left; for each object X in $\underset{\sim}{F}$ we take its copy in $\underset{\sim}{F}_n$ to its copy in $\underset{\sim}{F}_{n-1}$, for each n. This automorphism induces the following translation suspension functors, which all have the obvious inverses:

C.13. $S':\underset{\sim}{F}_S \to \underset{\sim}{F}_S$, $\underset{\sim}{I}(\underset{\sim}{F}_S) \to \underset{\sim}{I}(\underset{\sim}{F}_S)$, $\underset{\sim}{S} \to \underset{\sim}{S}$, $\underset{\sim}{I}(\underset{\sim}{S}) \to \underset{\sim}{I}(\underset{\sim}{S})$, $\underset{\sim}{S}_h \to \underset{\sim}{S}_h$.

We have the point CW-complex o in $\underset{\sim}{F}$, from B.9.

C.14. By C.11., the CW-complex o gives rise to a spectrum, also written o. We call this, and any isomorphic spectrum, a point spectrum. The point spectra are the zero objects of $\underset{\sim}{S}$.

Suppose given spectra and subspectra, $A \subset X$, $B \subset Y$.

C.15. The subspace $\mathrm{Mor}((X, A), (Y, B))$ of $\mathrm{Mor}_{\underset{\sim}{S}}(X, Y)$ is defined as the set of all maps $f:X \to Y$ such that $f|A:A \to Y$ factors through B, and is given the subspace topology.

C.16. We write $\{(X, A), (Y, B)\}$ for the set of homotopy classes (path components) of $\mathrm{Mor}((X, A), (Y, B))$.

C.17. In particular, $\mathrm{Mor}_{\underset{\sim}{S}}(X, Y) = \mathrm{Mor}((X, o), (Y, B))$, and

$$\{X, Y\} = \{(X, o), (Y, B)\}.$$

Thus C.17. includes the absolute case in the relative. It is clear that C.15., C.16., and C.17. extend to more complicated configurations, so that we can define maps of triads, triples, etc.

C.18. Every inclusion $A \subset X$ has the homotopy extension property.

C.19. If the inclusion $A \subset X$ is a homotopy equivalence, then A is a deformation retract of X.

Now let $(X_\lambda)_{\lambda\in\Lambda}$ be any family of spectra.

C.20. We can define the wedge $\bigvee_\lambda X_\lambda$, uniquely in $\underset{\sim}{I}(\underset{\sim}{S})$. It is a sum in $\underset{\sim}{S}$ or in $\underset{\sim}{S}_h$. It contains each X_λ as a subspectrum. The inclusion $\underset{\sim}{C} \subset \underset{\sim}{S}$ respects wedges.

C.21. If $A_\lambda \subset X_\lambda$ for all $\lambda \in \Lambda$, then $\bigvee_\lambda A_\lambda \subset \bigvee_\lambda X_\lambda$.

C.22. We can define the product spectrum $\Pi_\lambda X_\lambda$ up to homotopy type, as a product in $\underset{\sim}{S}_h$.

C.23. The homotopy category $\underset{\sim}{S}_h$ has an additive structure.

This additive structure is unique, as always. Apart from C.14., C.20., C.22., this assertion amounts to C.24:

C.24. $\{X, Y\}$ can be given a natural abelian group structure, for any spectra X, Y, for which composition is bilinear.

C.25. Hence wedges and products of finite families of spectra coincide in $\underset{\sim}{S}_h$.

C.26. If A is finite, we have, generally, $\{A, \bigvee_\lambda X_\lambda\} \cong \oplus_\lambda \{A, X_\lambda\}$.

C.27. If $A \subset X$, $B \subset Y$, $\{(X, A), (Y, B)\}$ can be given a natural abelian group structure. Similarly for more complicated configurations.

D. Smash products

The smash product functor $\wedge : \underset{\sim}{C} \times \underset{\sim}{C} \to \underset{\sim}{C}$ can be extended to give a smash product functor on $\underset{\sim}{S}$:

D.1. We have a separately continuous smash product functor $\wedge : \underset{\sim}{S} \times \underset{\sim}{S} \to \underset{\sim}{S}$, which takes $\underset{\sim}{I}(\underset{\sim}{S}) \times \underset{\sim}{I}(\underset{\sim}{S})$ to $\underset{\sim}{I}(\underset{\sim}{S})$.

D.2. The homotopy functor $\wedge : \underset{\sim}{S}_h \times \underset{\sim}{S}_h \to \underset{\sim}{S}_h$ is bilinear, and coherently commutative and associative.

Note. The functor D.1. is not quite canonical, since it involves certain choices; but these choices become equivalent when we pass to homotopy.

For any spectrum X, we have the natural isomorphisms in $\underset{\sim}{I}(\underset{\sim}{S})$:

D.3. $$o \wedge X \approx o, \quad X \wedge o \approx o.$$

D.4. $\Sigma^0 \wedge X \approx X$, $X \wedge \Sigma^0 \approx X$, where the 0-sphere Σ^0 is inherited from $\underset{\sim}{I}(\underset{\sim}{F})$.

Take any family (X_λ) of spectra, and any spectrum A. Then we have natural isomorphisms in $\underset{\sim}{I}(\underset{\sim}{S})$

D.5. $$\bigvee_\lambda (A \wedge X_\lambda) \approx A \wedge \bigvee_\lambda X_\lambda, \text{ and } \bigvee_\lambda (X_\lambda \wedge A) \approx (\bigvee_\lambda X_\lambda) \wedge A.$$

Choose a 1-sphere Σ^1, inherited from $\underset{\sim}{I}(\underset{\sim}{F})$. Parallel to B.14, we define the <u>suspension</u> SX of the spectrum X by $SX = \Sigma^1 \wedge X$, and $Sf = 1 \wedge f : \Sigma^1 \wedge X \to \Sigma^1 \wedge Y$ for a map $f : X \to Y$.

D.6. This defines the <u>suspension functor</u> $S : \underset{\sim}{S} \to \underset{\sim}{S}$, which takes $\underset{\sim}{I}(\underset{\sim}{S})$ to $\underset{\sim}{I}(\underset{\sim}{S})$, and induces $S : \underset{\sim}{S}_h \to \underset{\sim}{S}_h$.

We have already defined an invertible translation suspension functor S'. Exceptionally, let us write $J:\underset{\sim}{C} \subset \underset{\sim}{S}$ for the functor in C.11.

D.7. There is an isomorphism $SJ \approx S'J$ of functors from $\underset{\sim}{C}$ to $\underset{\sim}{S}$.

Unfortunately, this natural isomorphism cannot be extended to the whole category $\underset{\sim}{S}$. However, we do have:

D.8. There is an isomorphism $S \approx S'$ of the homotopy functors from $\underset{\sim}{S}_h$ to $\underset{\sim}{S}_h$, compatible with D.7. We write it in the form $\sigma:S'^{-1}S \approx 1$.

D.9. The functor $S:\underset{\sim}{S}_h \to \underset{\sim}{S}_h$ is an equivalence of categories, and induces isomorphisms $S:\{X, Y\} \cong \{SX, SY\}$ for all spectra X, Y.

D.10. The additive structure on $\underset{\sim}{S}_h$ may be induced by track addition and the isomorphisms D.9.

Let B be the closed interval $[0, 1]$ of the real line, with 0 as base point and two other cells. We can repeat D.6. with B instead of Σ^1.

D.11. The cone functor T is defined by $TX = B\wedge X$, $Tf = 1\wedge f$.

D.12. The cone TX is contractible, for all spectra X.

D.13. We have the canonical natural inclusion $X \subset TX$, for any spectrum X, by D.4.

Let X and Y be spectra, and A a CW-complex. Then we have the natural isomorphism

D.14. $$[A, \mathrm{Mor}_{\underset{\sim}{S}}(X, Y)] \approx \{A\wedge X, Y\}.$$

On the left we have the set of homotopy classes in the ordinary sense of maps from A to the space $\mathrm{Mor}_{\underset{\sim}{S}}(X, Y)$. This property has a well-known analogue for CW-complexes.

This result is useful for constructing secondary operations. For example, one can define Toda brackets directly by it.

E. The graded category

We have the additive category $\underset{\sim}{S}_h$, on which the translation suspension functor S' is an automorphism. Take any spectra, X and Y. For each integer n, we put

E.1. $\{X, Y\}_n = \{X, Y\}^{-n} = \{S'^n X, Y\}$, and call the elements the graded homotopy classes from X to Y of degree n, or alternatively of codegree - n.

E.2. The graded category $\underset{\sim}{S}_{h*}$ has the same objects, spectra, as $\underset{\sim}{S}_h$, or $\underset{\sim}{S}$. The morphisms from X to Y form the graded group $\{X, Y\}_* = \{X, Y\}^*$, whose components are the abelian groups $\{X, Y\}_n = \{X, Y\}^{-n}$. Composition is evident.

E.3. The bigraded category $\underset{\sim}{S}_{h*} \otimes \underset{\sim}{S}_{h*}$ has as objects the ordered pairs $X \otimes Y$, where X and Y are objects of $\underset{\sim}{S}_{h*}$. The morphisms from $X \otimes Y$ to $X' \otimes Y'$ of bidegree (m, n) form the group $\{X, X'\}_m \otimes \{Y, Y'\}_n$, which is thus generated by the morphisms $\alpha \otimes \beta$, where $\alpha \in \{X, X'\}_m$ and $\beta \in \{Y, Y'\}_n$. We also give $\alpha \otimes \beta$ the total degree $m + n$. Composition is defined by the formula

$$(\alpha' \otimes \beta') \circ (\alpha \otimes \beta) = (-)^{mn'} (\alpha' \circ \alpha) \otimes (\beta' \circ \beta),$$

where α and β' have degrees m and n'.

E.4. The bigraded category $\underset{\sim}{S}_{h*} \otimes \underset{\sim}{S}_{h*}$ has the canonical involution taking $X \otimes Y$ to $Y \otimes X$ and $\alpha \otimes \beta$ to $(-)^{mn} \beta \otimes \alpha$, where m and n are the degrees of α and β.

We can obviously repeat the construction in E.3.

E.5. We have canonically $(\underset{\sim}{S}_{h*} \otimes \underset{\sim}{S}_{h*}) \otimes \underset{\sim}{S}_{h*} \cong \underset{\sim}{S}_{h*} \otimes (\underset{\sim}{S}_{h*} \otimes \underset{\sim}{S}_{h*})$, where $(X \otimes Y) \otimes Z$ corresponds to $X \otimes (Y \otimes Z)$ and $(\alpha \otimes \beta) \otimes \gamma$ to $\alpha \otimes (\beta \otimes \gamma)$.

Our purpose in introducing these multigraded categories is to express the properties of the smash product more succinctly.

E.6. By use of D.2. and D.8., the smash product functor $\wedge : \underset{\sim}{S}_h \times \underset{\sim}{S}_h \to \underset{\sim}{S}_h$ extends to an additive graded functor

$$\wedge : \underset{\sim}{S}_{h*} \otimes \underset{\sim}{S}_{h*} \to \underset{\sim}{S}_{h*}.$$

E.7. The functor E.6. is commutative and associative with respect to E.4. and E.5.

This formulation takes care of the signs introduced in computations involving smash products and suspensions.

We may also rewrite D.8. in graded form.

E.8. There is a natural isomorphism (in the graded sense)

$$\sigma X : SX \approx X,$$

of degree -1. It satisfies

$$f \circ \sigma X = (-)^{n} \sigma Y \circ Sf$$

for a morphism $f : X \to Y$ of degree n.

More generally than in E.1., given pairs $A \subset X$ and $B \subset Y$ of spectra, we define, for any integer n,

E.9. $\{(X, A), (Y, B)\}_{n} = \{(X, A), (Y, B)\}^{-n} = \{(S'^{n}X, S'^{n}A), (Y, B)\}$.

These form a graded group. It is clear that this definition extends to triads and other configurations.

F. Cells

We need to express the fact that CW-complexes have cells in a categorical form, so as to apply also to spectra. They are needed, for example, in proofs by induction on cells. We do this by introducing an auxiliary space.

F.1. A *cell space* consists of a topological space V, whose points are called *cells*, in which each cell is assigned an integer (possibly negative), called its *dimension*. A cell of dimension n is called an *n-cell*. These are subject to the axioms:

a) The closure of a single cell is a finite subset of V.

b) For any n-cell a, every cell in its closure $\bar{a}$, other than a itself, has dimension strictly less than n.

c) A subset of V is closed if it contains the closure of each of its points.

Let X be a CW-complex, with base point o.

F.2. The cell space QX of X is obtained from the space X-o by identifying each open cell in X-o to a point, and giving QX the identification topology. An n-cell is assigned dimension n. Thus o does not count as a cell.

F.3. The category $\underset{\sim}{V}$ of cell spaces has cell spaces as objects, and a morphism of $\underset{\sim}{V}$ is a dimension-preserving embedding onto a closed subspace.

F.4. From F.2. and F.3. we deduce the functor $Q:\underset{\sim}{I}(\underset{\sim}{C}) \to \underset{\sim}{V}$.

In any category, denote by Sub(X) the set of equivalence classes of subobjects of X. Then for any CW-complex X, the functor Q in F.4. induces an isomorphism

F.5. $$Q:\mathrm{Sub}(X) \cong \mathrm{Sub}(QX).$$

This is the formulation we seek.

F.6. The functor Q on $\underset{\sim}{I}(\underset{\sim}{C})$ extends canonically to a functor $Q:\underset{\sim}{I}(\underset{\sim}{S}) \to \underset{\sim}{V}$ which satisfies F.5. for any spectrum X.

This expresses in a very precise and accessible form the information we need about the possible subspectra of a spectrum. It also enables us to extend more of the language of CW-complexes to spectra.

F.7. For any spectrum X, Sub(X) is a complete distributive lattice.

F.8. Given any family (A_λ) of subspectra of X, we can use F.7. to define the union $\bigcup_\lambda A_\lambda$ and intersection $\bigcap_\lambda A_\lambda$, uniquely in Sub(X). If the family consists of A and B only, we write $A \cup B$ and $A \cap B$.

Conversely, we can also build new spectra out of inclusions.

F.9. Given any directed non-empty diagram (A_λ) of inclusions of spectra, we can extend the diagram so as to include a spectrum X containing the A_λ as subspectra, such that X is the union of the A_λ. The spectrum X is unique up to isomorphism in $\underset{\sim}{I}(\underset{\sim}{S})$.

<u>F.10.</u> Given inclusions of spectra $A \subset B$, and $A \subset C$, there exists a spectrum D containing B and C, unique up to isomorphism in $\underset{\sim}{I}(\underset{\sim}{S})$, in which $A = B \cap C$ and $D = B \cup C$.

The process in F.10. is called <u>gluing</u> B to C along A to form D.

Let (X_λ) be a directed system of subspectra of the spectrum X whose union is X. Let A be any finite spectrum. Then

<u>F.11.</u> $$\{A, X\} = \varinjlim \{A, X_\lambda\}.$$

There are various ways of constructing new cell spaces from old.

<u>F.12.</u> Given a family (V_λ) of cell spaces, their <u>disjoint union</u> $\cup_\lambda V_\lambda$ is the topological disjoint union, with the obvious dimension function.

<u>F.13.</u> The <u>product</u> $V_1 \times V_2$ of the cell spaces V_1 and V_2 is the topological product; the cell $a \times b$ is given dimension $m + n$, where m and n are the dimensions of a and b.

<u>F.14.</u> The <u>n-fold suspension</u> S^nV of the cell space V is the same topological space, with the dimension function increased by n, for any integer n.

<u>F.15.</u> Given $V_2 \subset V_1$, the <u>difference</u> cell space V_1/V_2 is the space $V_1 - V_2$ with the subspace topology, and the restricted dimension function.

All these occur for spectra. Let (X_λ) be a family of subspectra of X.

F.16. Then $X = \bigvee_\lambda X_\lambda$ if and only if QX is the disjoint union of the QX_λ.

F.17. We have $Q(X \wedge Y) \approx QX \times QY$, for any spectra X and Y.

F.18. For any spectrum X, $QSX \approx SQX$, and $QS'^n X \approx S^n QX$.

F.19. X is a point spectrum if and only if QX is empty. All point spectra are isomorphic in $\underset{\sim}{I}(\underset{\sim}{S})$.

F.20. The spectrum X is finite if and only if QX is finite.

F.21. We call the spectrum X an n-sphere if and only if QX consists of a single n-cell.

F.22. For each n, n-spheres exist and are all isomorphic in $\underset{\sim}{I}(\underset{\sim}{S})$. We therefore write Σ^n for any n-sphere. We inherit an n-sphere from $\underset{\sim}{I}(\underset{\sim}{F})$ if $n \geqslant 0$.

F.23. For any m and n, we have $S\Sigma^n \cong \Sigma^{n+1}$, $S'^n\Sigma^m \cong \Sigma^{m+n}$, and $\Sigma^m \wedge \Sigma^n \cong \Sigma^{m+n}$.

G. Identification

Let $i:A \subset X$ be an inclusion of spectra.

G.1. We can construct canonically a map $p:X \to X/A$ of spectra, such that

a) p is a cokernel of i, in $\underset{\sim}{S}$,

b) i is a kernel of p, in $\underset{\sim}{S}$,

c) This extends the notion of identification for CW-complexes in B.11.

G.2. The subspectra of X/A are the spectra Y/A, where $A \subset Y \subset X$.

G.3. We have $Q(X/A) \approx QX/QA$, which was defined in F.15.

G.4. Suppose A is contractible. Then p is a homotopy equivalence.

Suppose A, B, C are subspectra of X.

G.5. Then $(A \cup B)/A \cong A/(A \cap B)$.

G.6. Suppose $A \supset B \supset C$. Then $(A/C)/(B/C) \cong A/B$ (excision).

Suppose $A \subset X$ and $B \subset Y$ are subspectra.

G.7. Then $(X \vee Y)/(A \vee B) = (X/A) \vee (Y/B)$.

G.8. Then $(X/A) \wedge (Y/B) = (X \wedge Y)/(X \wedge B \cup A \wedge Y)$, and $(X \wedge B) \cap (A \wedge Y) = A \wedge B$.

G.9. We have $S(X/A) \cong SX/SA$, and $S'^n(X/A) \cong S'^n X/S'^n A$.

G.10. The inclusion D.13. $X \subset TX$ yields $TX/X \approx SX$.

There is a natural exact sequence of abelian groups

G.11. $\cdots \to \{X/A,\ Y\}_n \to \{(X,\ A),(Y,\ B)\}_n \to \{A,\ B\}_n \to \{X/A,\ Y\}_{n-1} \to \cdots$

Suppose X is contractible. Then identification induces an isomorphism

G.12. $$\{(X, A), (Y, B)\} \cong \{X/A, Y/B\}.$$

Thus in one very important case the relative groups are easily expressed in terms of the absolute groups.

H. Filtrations

Filtrations of spectra are of crucial importance both in the abstract theory and in applications.

H.1. A filtration of a spectrum X is an increasing sequence (X_n) $(n \in \mathbb{Z})$ of subspectra of X whose union is X.

Thus a filtration of X corresponds precisely to a filtration of the cell space QX of X by a sequence of closed subspaces whose union is QX, and conversely. Note that we do not insist on $\bigcap_n X_n = o$, a condition that has no significance in homotopy theory.

Let (X_n) be a filtration of X, and Y any spectrum. Then

H.2. $\{A, X\} = \varinjlim \{A, X_n\}$, for any finite spectrum A.

<u>H.3.</u> $$\varprojlim \{X/X_n, Y\} = 0.$$

<u>H.4.</u> We have a natural short exact sequence

$$0 \to R\varprojlim \{SX_n, Y\} \to \{X, Y\} \to \varprojlim \{X_n, Y\} \to 0,$$

where $R\varprojlim$ denotes the first right derived functor of $\varprojlim$, as applied to a sequence of abelian groups and homomorphisms.

Suppose given any sequence of spectra and maps

$$\ldots Y_{-2} \to Y_{-1} \to Y_0 \to Y_1 \to Y_2 \ldots .$$

<u>H.5.</u> Then there exists a filtration (X_n) of a spectrum X and a homotopy equivalence $X_n \simeq Y_n$ for each n, such that the diagram

$$\begin{array}{ccccccccccc}
\ldots & X_{-2} & \subset & X_{-1} & \subset & X_0 & \subset & X_1 & \subset & X_2 & \ldots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\ldots & Y_{-2} & \to & Y_{-1} & \to & Y_0 & \to & Y_1 & \to & Y_2 & \ldots
\end{array}$$

commutes up to homotopy. Moreover, the homotopy type of X is uniquely determined (but <u>not</u> up to unique homotopy equivalence).

Suppose given filtrations (X_n) of X and (Y_n) of Y. The <u>product filtration</u> (Z_n) of $Z = X \wedge Y$ is defined by

<u>H.6.</u> $$Z_n = \cup_{i+j=n} \; X_i \wedge Y_j.$$

<u>H.7.</u> Then $Z_n/Z_{n-1} = \bigvee_{i+j=n} (X_i/X_{i-1}) \wedge (Y_j/Y_{j-1})$.

Take any spectra X and Y.

<u>H.8.</u> The <u>n-skeleton</u> X^n of X is the subspectrum of X such that QX^n is the set of all cells in QX having dimension at most n. The skeletons form the <u>skeleton filtration</u> of X.

H.9. Then X^n/X^{n-1} is a wedge of n-spheres.

H.10. The product filtration of the skeleton filtrations on X and Y is the skeleton filtration on $X \wedge Y$.

H.11. We call the map $f:X \to Y$ skeletal (or cellular) if $f|X^n : X^n \to Y$ factors through Y^n for all n.

H.12. Every map is homotopic to a skeletal map.

H.13. Given a subspectrum $A \subset X$ and a skeletal map $f:A \to C$, there exists a spectrum Z and a map $g:X \to Z$ such that:

a) C is a subspectrum of Z,

b) g extends f,

c) The inclusion maps, with f and g, form a pushout diagram in $\underset{\sim}{S}$,

d) The isomorphism $X/A \cong Z/C$ induced by f and g lies in $\underset{\sim}{I}(\underset{\sim}{S})$.

H.14. Given a skeletal map $f:X \to Y$, we have the mapping cylinder M of f, and maps $i:X \subset M$, $j:Y \subset M$, $p:M \to Y$ such that

a) By j, Y is a deformation retract of M, with retraction map p,

b) $f = p \circ i$.

H.15. In H.14., the mapping cone of f is the spectrum M/X.

H.16. Any map f can be expressed as a composite, $f = h \circ g$, where g is an inclusion and h is a homotopy equivalence.

The above mapping cylinder and cone correspond to the reduced mapping cylinder and cone in ordinary homotopy theory.

Older theories of spectra are based essentially on the following definitions:

H.17. A CW-prespectrum (or simply prespectrum) consists of a sequence (A_n) of CW-complexes and inclusion maps $a_n : SA_n \subset A_{n+1}$ of CW-complexes.

H.18. The prespectrum $A = (A_n;\, a_n)$ is a Ω-prespectrum if each adjoint map $\hat{a}_n : A_n \to \Omega A_{n+1}$ is a homotopy equivalence.

Given a prespectrum $(A_n;\, a_n)$, we may, by C.11., regard each A_n as a spectrum, then, by D.7., obtain an inclusion $a'_n : S'A_n \subset A_{n+1}$ of spectra.

H.19. Each prespectrum $(A_n;\, a_n)$ determines a spectrum X with filtration (X_n) such that, for each integer n,

a) $X_n = S'^{-n}A_n$,

b) The inclusion $X_n \subset X_{n+1}$ is $S'^{-n-1}a'_n$.

H.20. Any spectrum is isomorphic in $\underset{\sim}{I}(\underset{\sim}{S})$ to the spectrum determined by a suitable prespectrum.

H.21. Any spectrum has the homotopy type of a spectrum determined by a suitable Ω-prespectrum.

Let X be the spectrum determined by the prespectrum $A = (A_n;\, a_n)$.

H.22. Then $\{B, X\}_n = \varinjlim_k\, [S^{n+k}B, A_k]$ for any finite CW-complex B.

H.23. Further, $\{B, X\}_{-n} = \{B, X\}^n \cong [B, A_n]$ for any

CW-complex B, provided A is a Ω-prespectrum. (Homotopy classes are taken in $\underset{\sim}{S}_h$ on the left, in $\underset{\sim}{C}_h$ on the right.)

These are the properties that relate our spectra to previous notions of spectra. There is little worth saying about maps of prespectra.

<u>J. Exact triangles</u>

In this section, let us write $|f|$ for the degree of a morphism f in $\underset{\sim}{S}_{h*}$.

Suppose given an inclusion $i:A \subset X$ of spectra.

<u>J.1.</u> The <u>boundary</u> morphism $\delta : X/A \to A$ in $\underset{\sim}{S}_{h*}$, of degree -1, is defined as the composite

$$X/A = (X \cup TA)/TA \xrightarrow[p^{-1}]{} X \cup TA \xrightarrow[p]{} (X \cup TA)/X \cong TA/A \cong SA \xrightarrow[\sigma A]{} A.$$

(The morphism p^{-1} exists in $\underset{\sim}{S}_h$, by G.4. E.8. provides σ.)

<u>J.2.</u> The <u>standard exact triangle</u> of the inclusion $A \subset X$ is the triangle

$$A \xrightarrow[i]{} X \xrightarrow[p]{} X/A \xrightarrow[\delta]{} A.$$

(It has a distinguished vertex at A.)

Given two triangles in $\underset{\sim}{S}_{h*}$

$$\Delta : A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A$$

$$\Delta' : A' \xrightarrow[h']{} B' \xrightarrow[f']{} C' \xrightarrow[g']{} A',$$

we define a <u>morphism</u> from Δ to Δ' as a triple (a, b, c) of graded morphisms such that in the diagram

<u>J.3.</u>

$$\begin{tikzcd} A \arrow[r, "h"'] \arrow[d, "a"] & B \arrow[r, "f"'] \arrow[d, "b"] & C \arrow[r, "g"'] \arrow[d, "c"] & A \arrow[d, "a"] \\ A' \arrow[r, "h'"'] & B' \arrow[r, "f'"'] & C' \arrow[r, "g'"'] & A' \end{tikzcd}$$

the three squares commute up to the signs $(-)^u$, $(-)^v$, and $(-)^w$, where

$$|a| + |b| + |c| + u + v + w$$

is even.

Thus the triangles in $\underset{\sim}{S}_{h*}$ and these morphisms form a category, under the obvious composition.

<u>J.4.</u> We call the triangle

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A$$

an <u>exact triangle</u> if it is isomorphic in the sense of J.3. to some standard exact triangle.

A slight modification of exactness is frequently useful.

<u>J.5.</u> We say the triangle

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A$$

is an <u>anti-exact triangle</u> if and only if

$$A \xrightarrow[-h]{} B \xrightarrow[-f]{} C \xrightarrow[-g]{} A$$

is an exact triangle. By <u>$(\text{anti})^n$-exact</u> we mean exact if n is even, anti-exact if n is odd.

<u>J.6.</u> Suppose

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A$$

is an exact triangle. Then

$$A \xrightarrow[\pm h]{} B \xrightarrow[\pm f]{} C \xrightarrow[\pm g]{} A$$

is $(\text{anti})^n$-exact, if we choose n minus signs.

We have the class of exact triangles in $\underset{\sim}{S}_{h*}$. It satisfies the axioms of Puppe:

<u>J.7.</u> In any exact triangle

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A,$$

we have $|f| + |g| + |h| = -1$.

<u>J.8.</u> Any triangle isomorphic to an exact triangle is an exact triangle.

<u>J.9.</u> If

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A$$

is an exact triangle, so is

$$B \xrightarrow[f]{} C \xrightarrow[g]{} A \xrightarrow[h]{} B.$$

Hence exact triangles no longer need distinguished vertices.

J.10. For any spectrum A,

$$A \xrightarrow{1} A \longrightarrow o \longrightarrow A$$

is an exact triangle.

J.11. Every morphism $h:A \to B$ in $\underset{\sim}{S}_{h*}$ can be included in some exact triangle

$$A \xrightarrow{h} B \xrightarrow{f} C \xrightarrow{g} A,$$

in which the degree of f is arbitrary.

J.12. Given the diagram, in which the rows are exact triangles and the square commutes up to sign,

$$\begin{tikzcd}
A \arrow[r, "h"'] \arrow[d, "a"] & B \arrow[r, "f"'] \arrow[d, "b"] & C \arrow[r, "g"'] & A \arrow[d, "a"] \\
A' \arrow[r, "h'"'] & B' \arrow[r, "f'"'] & C' \arrow[r, "g'"'] & A',
\end{tikzcd}$$

we can fill in $c:C \to C'$ to form a morphism of triangles.

The axiom of Verdier also holds:

J.13. Given three exact triangles

$$B \xrightarrow{f} C \longrightarrow A' \longrightarrow B,$$

$$A \xrightarrow{g} C \longrightarrow B' \longrightarrow A,$$

$$A \xrightarrow{h} B \longrightarrow C' \longrightarrow A,$$

such that $g = f \circ h$, there exists a fourth exact triangle

$$A' \longrightarrow C' \longrightarrow B' \longrightarrow A'$$

which makes the diagram

$$\begin{tikzcd}[column sep=small]
& A \arrow[rr, "g"'] \arrow[dr, "h"'] & & C \arrow[rr] \arrow[dr] & & A' \arrow[rr] \arrow[dr] & & C' & \\
\cdots & & B \arrow[ur, "f"'] \arrow[dr] & & B' \arrow[ur] \arrow[dr] & & B \arrow[ur] \arrow[dr] & & \cdots \\
& A' \arrow[rr] \arrow[ur] & & C' \arrow[rr] \arrow[ur] & & A \arrow[rr] \arrow[ur] & & C &
\end{tikzcd}$$

commute up to sign; and these signs are such that this diagram yields four morphisms of exact triangles:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & A \arrow[d, equal] \\
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & B \arrow[d] \\
C' \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & C' \arrow[d, equal] \\
C' \arrow[r] & A \arrow[r] & B \arrow[r] & C'.
\end{tikzcd}$$

Exact triangles enjoy various properties, most of which are easy deductions from the above axioms. Take any exact triangle

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A.$$

Then for any spectrum X, the sequences

<u>J.14.</u> $\ldots \{X, A\}_* \xrightarrow[h^\circ]{} \{X, B\}_* \xrightarrow[f^\circ]{} \{X, C\}_* \xrightarrow[g^\circ]{} \{X, A\}_* \ldots$

<u>J.15.</u> $\ldots \{A, X\}^* \xrightarrow[{}^\circ g]{} \{C, X\}^* \xrightarrow[{}^\circ f]{} \{B, X\}^* \xrightarrow[{}^\circ h]{} \{A, X\}^* \ldots$

are exact sequences of graded abelian groups, in the ordinary sense.

<u>J.16.</u> For any n,

$$S'^n A \xrightarrow[S'^n h]{} S'^n B \xrightarrow[S'^n f]{} S'^n C \xrightarrow[S'^n g]{} S'^n A$$

is an $(\text{anti})^n$-exact triangle.

Also

J.17. $$X\wedge A \xrightarrow[1\wedge h]{} X\wedge B \xrightarrow[1\wedge f]{} X\wedge C \xrightarrow[1\wedge g]{} X\wedge A,$$

J.18. $$A\wedge X \xrightarrow[h\wedge 1]{} B\wedge X \xrightarrow[f\wedge 1]{} C\wedge X \xrightarrow[g\wedge 1]{} A\wedge X,$$

are exact triangles. In an obvious sense,

J.19. The smash product functor is exact in each variable.

Note J.16. and J.17. do not contradict D.8., because E.8. introduces a sign.

Suppose that we have a family of exact triangles

$$A_\lambda \xrightarrow[h_\lambda]{} B_\lambda \xrightarrow[f_\lambda]{} C_\lambda \xrightarrow[g_\lambda]{} A_\lambda,$$

in which the degrees $|f_\lambda|$, $|g_\lambda|$, and $|h_\lambda|$ are each independent of λ. Then the triangles

J.20. $$\bigvee_\lambda A_\lambda \xrightarrow[\bigvee_\lambda h_\lambda]{} \bigvee_\lambda B_\lambda \xrightarrow[\bigvee_\lambda f_\lambda]{} \bigvee_\lambda C_\lambda \xrightarrow[\bigvee_\lambda g_\lambda]{} \bigvee_\lambda A_\lambda$$

J.21. $$\Pi_\lambda A_\lambda \xrightarrow[\Pi_\lambda h_\lambda]{} \Pi_\lambda B_\lambda \xrightarrow[\Pi_\lambda f_\lambda]{} \Pi_\lambda C_\lambda \xrightarrow[\Pi_\lambda g_\lambda]{} \Pi_\lambda A_\lambda$$

are defined, and are exact triangles.

Finally, we give two methods of deciding whether a given morphism is an isomorphism.

J.22. Given the exact triangle

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A,$$

C is contractible if and only if h is an isomorphism (possibly of non-zero degree) in $\underset{\sim}{S}_{h*}$.

J.23. The 'five lemma'. If in the morphism of exact triangles

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "a"] & B \arrow[r, "f"] \arrow[d, "b"] & C \arrow[r, "g"] \arrow[d, "c"] & A \arrow[d, "a"] \\ A' \arrow[r, "h'"] & B' \arrow[r, "f'"] & C' \arrow[r, "g'"] & A' \end{tikzcd}$$

a and b are isomorphisms, then c is also an isomorphism.

K. Homology and cohomology

Let $\underset{\sim}{G}$ be the category of abelian groups and homomorphisms, and $\underset{\sim}{G}^{\infty}$ the category of graded abelian groups and graded homomorphisms.

K.1. A contravariant (respectively covariant) additive functor $K:\underset{\sim}{S}_h \to \underset{\sim}{G}$ is a *cohomology* (resp. *homology*) *theory* if:

a) K respects sums: for any wedge $X = \bigvee_\lambda X_\lambda$ of spectra, the inclusions $i_\lambda : X_\lambda \subset X$ of the factors induce an isomorphism

$$KX \cong \Pi_\lambda\ KX_\lambda \qquad\qquad (KX \cong \oplus_\lambda\ KX_\lambda),$$

b) For any inclusion $i:A \subset X$, the induced sequence

$$K(X/A) \xrightarrow[Kp]{} KX \xrightarrow[Ki]{} KA \qquad (KA \xrightarrow[Ki]{} KX \xrightarrow[Kp]{} K(X/A))$$

of abelian groups is exact.

K.2. Then, given any spectrum A, the functor K defined by

$$KX = \{X, A\} \qquad \text{(respectively } KX = \{\Sigma^0, X \wedge A\})$$

is a cohomology (homology) theory. For these theories, A is called the coefficient spectrum.

Given a contravariant (resp. covariant) functor $K:\underset{\sim}{S}_h \to \underset{\sim}{G}$, as in K.1., we extend to a functor $K^*:\underset{\sim}{S}_{h*} \to \underset{\sim}{G}^\infty$ ($K_*:\underset{\sim}{S}_{h*} \to \underset{\sim}{G}^\infty$) by setting

K.3. a) For any spectrum X, $K^nX = KS'^{-n}X$ ($K_nX = KS'^{-n}X$),

b) For any morphism $f:X \to Y$ in $\underset{\sim}{S}_{h*}$ of codegree p (resp. of degree q), so that $f:S'^{-p}X \to Y$ ($f:S'^{q}X \to Y$) in $\underset{\sim}{S}_h$,

$$K^*f = (-)^{np}KS'^{-n}f:K^nY \to K^{n+p}X \quad (K_*f = KS'^{-n-q}f:K_nX \to K_{n+q}Y).$$

K.4. If K is a cohomology (resp. homology) theory, and

$$A \xrightarrow[h]{} B \xrightarrow[f]{} C \xrightarrow[g]{} A$$

is any exact triangle, then the sequence of graded groups

$$\dots K^*A \xrightarrow[K^*g]{} K^*C \xrightarrow[K^*f]{} K^*B \xrightarrow[K^*h]{} K^*A \dots$$

$$\text{(resp. } \dots K_*A \xrightarrow[K_*h]{} K_*B \xrightarrow[K_*f]{} K_*C \xrightarrow[K_*g]{} K_*A \dots \text{)}$$

is exact. (Conversely, this condition clearly implies b) of K.1.)

K.5. In the case of the theories of K.2, we write

$$H^n(X;\,A) = K^n X = \{X,\,A\}^n \quad (\text{resp. } H_n(X;\,A) = K_n X = \{\Sigma^0,\,X\wedge A\}_n).$$

Cohomology theories can be classified:

K.6. Every cohomology theory on $\underset{\sim}{S}_h$ is representable, i.e. has the form $\{\ ,\,A\}$, for some spectrum A.

K.7. Every cohomology theory defined only for finite spectra, and such that $K^*\Sigma^0$ is countable, can be extended to a cohomology theory on the whole of $\underset{\sim}{S}_h$.

K.8. Given any spectra X and Y, there exists a spectrum F(X, Y), called the function spectrum of X and Y, and a natural isomorphism

$$\{Z,\,F(X,\,Y)\} \approx \{X\wedge Z,\,Y\}.$$

K.9. The spectrum F(X, Y) is functorial, and is anti-exact in X, and exact in Y. (Compare J.19.)

K.10. The functional dual DX of the spectrum X is defined as

$$DX = F(X,\,\Sigma^0).$$

Then K.8. yields the evaluation morphism

K.11. $$e: X\wedge DX \to \Sigma^0.$$

Assume from now on that X is a finite spectrum.

K.12. Then $X \simeq DDX$, and we may take DX to be finite.

K.13. The evaluation morphism e induces an isomorphism, for any spectra A and B,

$$\{A,\,B\wedge X\}_* \cong \{A\wedge DX,\,B\}_*,$$

which takes $f:A \to B \wedge X$ to $(1 \wedge e)\circ(f \wedge 1)$, followed by $B \wedge \Sigma^0 \approx B$.

K.14. Composition induces

$$F(A, B) \simeq DA \wedge B,$$

if A or B is finite.

K.15. By K.13., there is a canonical morphism

$$u:\Sigma^0 \to DX \wedge X.$$

K.16. Then u induces, for any spectra A and B, an isomorphism

$$\{X \wedge A, B\}_* \cong \{A, DX \wedge B\}_*.$$

K.17. Conversely, suppose given a map $v:\Sigma^0 \to Y \wedge X$ which induces an isomorphism

$$\{X, A\}_* \cong \{\Sigma^0, Y \wedge A\}_*$$

for all spectra A. Then we may take Y as DX, and v as u in K.15. It is sufficient to be given the isomorphism when $A = \Sigma^0$.

The last result enables us to recognise functional duals when we meet them.

L. Homotopy groups

L.1. For any integer n, the nth homotopy group $\pi_n(X)$ of the spectrum X is defined by

$$\pi_n(X) = \{\Sigma^0, X\}_n \cong \{\Sigma^n, X\}.$$

(Of course, if X is a CW-complex, this must be interpreted as the stable homotopy group, in the usual sense, by C.12.)

L.2. We say the spectrum X is n-connected if $\pi_i(X) = 0$ for all $i \leqslant n$. We say X is highly connected if it is n-connected for some finite n. (Of course, n may be large and negative.)

L.3. Any n-connected spectrum has the homotopy type of a spectrum whose n-skeleton is o.

L.4. If X is (m-1)-connected and Y is (n-1)-connected, then $X \wedge Y$ is (m+n-1)-connected, and

$$\pi_{m+n}(X \wedge Y) \cong \pi_m(X) \otimes \pi_n(Y).$$

L.5. If $\pi_n(X) = 0$ for all n, then X is contractible.

L.6. If the map $f: X \to Y$ of spectra induces isomorphisms $f_*: \pi_*(X) \cong \pi_*(Y)$, then f is a homotopy equivalence.

L.7. Given the family (X_λ) of spectra, the canonical morphism

$$\varkappa : \bigvee_\lambda X_\lambda \to \Pi_\lambda X_\lambda$$

is a homotopy equivalence if and only if for each integer n, the number of indices λ such that $\pi_n(X_\lambda) \neq 0$ is finite.

L.8. The filtration (Y_n) of the spectrum Y, with decreasing indices $(\ldots Y_n \supset Y_{n+1} \ldots)$ is called a Postnikov filtration of Y if for each n;

a) Y_n is $(n-1)$-connected,

b) $Y_n \subset Y$ induces $\pi_i(Y_n) \cong \pi_i(Y)$ for all $i \geqslant n$.

L.9. Any spectrum X has the homotopy type of a spectrum Y having a Postnikov filtration. The boundary morphism

$$Y/Y_n \xrightarrow{\delta} Y_n \xrightarrow{p} Y_n/Y_{n+1}$$

is called the (n + 1)th k-invariant $k^{n+1}(X)$ of X.

M. Eilenberg-MacLane spectra

Let G be an abelian group.

M.1. We say the spectrum X has type G if:

a) $\pi_n(X) = 0$ for all $n \neq 0$,

b) We are given an isomorphism $\pi_0(X) \cong G$.

M.2. Spectra of type G exist for any group G, and any two are canonically homotopy-equivalent. We therefore write $K(G)$ for any such spectrum.

M.3. The functor π_0 induces an isomorphism

$$\{K(G),\ K(H)\} \cong \mathrm{Hom}(G,\ H).$$

Hence $K(\alpha):K(G) \to K(H)$ is defined, for any homomorphism $\alpha:G \to H$.

M.4. The Steenrod algebra for the group G is the graded ring $\{K(G),\ K(G)\}^*$, with composition as multiplication.

M.5. Given a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow[u]{} B \xrightarrow[v]{} C \longrightarrow 0,$$

there is a unique morphism $\beta:K(C) \to K(A)$, of degree -1, such that

$$K(A) \xrightarrow[K(u)]{} K(B) \xrightarrow[K(v)]{} K(C) \xrightarrow[\beta]{} K(A)$$

is an exact triangle. The morphism β is called the Bockstein of the given exact sequence.

M.6. There is a natural morphism

$$K(G) \wedge K(H) \to K(G \otimes H).$$

M.7. Given the coefficient group G, we define the ordinary cohomology H^* and homology theory H_* of the spectrum X by:

$$H^n(X;\ G) = \{X,\ K(G)\}^n\ ; \quad H_n(X;\ G) = \{\Sigma^0,\ X \wedge K(G)\}_n;$$

so that in the notation of K.5.,

$$H^n(X;\ G) = H^n(X;\ K(G)); \quad H_n(X;\ G) = H_n(X;\ K(G)).$$

(When X is a CW-complex, these theories give the reduced cohomology and homology. To recover the 'absolute' theories, we adjoin a new base point o to X to form X^0, and use X^0 instead of X.)

Let X_{n-1} be the spectrum Y/Y_n in L.9., obtained from X by 'killing the homotopy groups above π_{n-1}'.

<u>M.8.</u> Then $k^{n+1}(X) \in H^{n+1}(X_{n-1}; \pi_n(X))$.

Take any spectrum X, and let (X^n) be its skeleton filtration. We define the <u>cochain complex</u> $(C^*(X; G); \delta)$ and <u>chain complex</u> $(C_*(X; G); \partial)$, for a given coefficient group G, by:

<u>M.9.</u> $C^n(X; G) = H^n(X^n/X^{n-1}; G)$; $C_n(X; G) = H_n(X^n/X^{n-1}; G)$,

and boundary homomorphisms

$$\delta: C^n(X; G) \to C^{n+1}(X; G); \qquad \partial: C_{n+1}(X; G) \to C_n(X; G)$$

induced by the boundary morphisms

$$X^{n+1}/X^n \xrightarrow[\delta]{} X^n \xrightarrow[p]{} X^n/X^{n-1},$$

with a sign $(-)^n$ in the case of cochains.

<u>M.10.</u> The chain groups $C_n(X; \mathbb{Z})$ are free abelian, and as complexes, we have

$$C^*(X; G) \cong \mathrm{Hom}(C_*(X; \mathbb{Z}), G); \qquad C_*(X; G) \cong C_*(X; \mathbb{Z}) \otimes G,$$

There are canonical isomorphisms

<u>M.11.</u> $H^n(X; G) \cong H^n(C^*(X; G), \delta)$; $H_n(X; G) \cong H_n(C_*(X; G), \partial)$

between the homology groups of these complexes and the cohomology and homology groups of X.

Let Y be another spectrum, and G and H abelian groups. Then there is a canonical isomorphism of chain complexes

<u>M.12.</u> $$C_*(X; G) \otimes C_*(Y; H) \cong C_*(X \wedge Y; G \otimes H),$$

where the left hand side is equipped with the usual differential.

Thus M.10., M.11., and M.12. entitle us to use the theory of chain complexes, and obtain results such as the Künneth formula. In particular, we have the universal coefficient theorems: there are natural short exact sequences, which split,

<u>M.13.</u> $0 \to \mathrm{Ext}(H_{n-1}(X;\ \underset{\sim}{Z}),\ G) \to H^n(X;\ G) \to \mathrm{Hom}(H_n(X;\ \underset{\sim}{Z}),\ G) \to 0,$

<u>M.14.</u> $0 \to H_n(X;\ \underset{\sim}{Z}) \otimes G \to H_n(X;\ G) \to \mathrm{Tor}(H_{n-1}(X;\ \underset{\sim}{Z}),\ G) \to 0.$

For any spectrum X, and any integer n, we have the natural Hurewicz homomorphism

<u>M.15.</u> $$h:\pi_n(X) \to H_n(X;\ \underset{\sim}{Z}).$$

<u>M.16.</u> Then $h:\pi_n(X) \cong H_n(X;\ \underset{\sim}{Z})$, whenever X is (n-1)-connected.

<u>M.17.</u> Suppose X is highly connected, and $H_*(X;\ \underset{\sim}{Z}) = 0$. Then X is contractible.

<u>M.18.</u> Suppose the map $f:X \to Y$ induces $f_*:H_*(X;\ \underset{\sim}{Z}) \cong H_*(Y;\ \underset{\sim}{Z})$, and that X and Y are highly connected. Then f is a homotopy equivalence.

<u>Note</u> M.17. and M.18. are false without the hypothesis that X and Y are highly connected. For example, let X be the spectrum that represents the modulo p complex K-theory; then $H_*(X;\ \underset{\sim}{Z}) = 0$, but X is clearly not contractible.

A slight generalization of the notion of Eilenberg-MacLane spectrum is frequently useful. Let G_* be a graded abelian group.

<u>M.19.</u> We say the spectrum X is a <u>graded Eilenberg-MacLane spectrum of type</u> G_* if:

a) X has the homotopy type of $\bigvee_n S'^n K(G_n) \simeq \Pi_n S'^n K(G_n)$,

b) We are given an isomorphism $\pi_n(X) \cong G_n$ for each n.

We write $K(G_*)$ for such a spectrum. Then $\pi_*(K(G_*)) \cong G_*$.

We have the somewhat surprising result:

M.20. For any spectrum X and any abelian group G, the spectrum $X \wedge K(G)$ is a graded Eilenberg-MacLane spectrum.

N. Spectral sequences

Let X be any spectrum, with arbitrary filtration (X_n), and Y any spectrum. Then we have two natural spectral sequences, which arise from H(p, q)-systems. In each, the differentials are homomorphisms

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

N.1. The 'contravariant' spectral sequence (contravariant in filtered spectra X), $(E_r(X_*, Y))$, in which

$$E_1^{p,q} = \{X_p/X_{p-1}, Y\}^{p+q} \qquad E_\infty^{p,q} = F^p\{X, Y\}^{p+q}/F^{p+1}\{X, Y\}^{p+q},$$

where the filtration $(F^p\{X, Y\}^*)$ is defined by

$$F^p\{X, Y\}^* = \mathrm{Ker}[\{X, Y\}^* \to \{X_{p-1}, Y\}^*].$$

N.2. The 'covariant' spectral sequence (covariant in filtered spectra X), $(E_r(Y, X_*))$, in which

$$E_1^{-p,-q} = \{Y, X_p/X_{p-1}\}_{p+q} \qquad E_\infty^{-p,-q} = F_p\{X, Y\}_{p+q}/F_{p-1}\{Y, X\}_{p+q},$$

where the filtration $(F_p\{X, Y\}_*)$ is defined by

$$F_p\{X, Y\}_* = \mathrm{Im}[\{Y, X_p\}_* \to \{Y, X\}_*].$$

In N.2., it is frequently (but not always) convenient to change the signs of the indices by writing $E^r_{p,q} = E_r^{-p,-q}$.

Let X' be another filtered spectrum, and Y' and Z be spectra. Then the above spectral sequences have natural products, with respect to which the differentials are derivations,

N.3. $E_r(X_*, Y) \otimes E_r(X'_*, Y') \to E_r((X\wedge X')_*, Y\wedge Y')$,

N.4. $E_r(Y, X_*) \otimes E_r(Y', X'_*) \to E_r(Y\wedge Y', (X\wedge X')_*)$,

N.5. $E_r(Y, X_*) \otimes E_r(X_*, Z) \to \{Y, Z\}_*$, by composition.

The spectral sequences N.1. and N.2. include all the usual types of spectral sequence in algebraic topology, which are obtained by constructing suitable filtrations of X, e.g. the Adams spectral sequence. For these particular cases, we can write down the E_2 term.

Let X have the skeleton filtration; then for any spectrum A we filter $X\wedge A$ by $(X\wedge A)_n = X_n\wedge A$. In this case,

N.6. $E_2^{p,q}(X_*, A) = H^p(X; \pi_{-q}(A))$,

N.7. $E^2_{p,q}(\Sigma^0, (X\wedge A)_*) = H_p(X; \pi_q(A))$.

Let $p:E \to B$ be a fibre bundle of CW-complexes, with fibre F, a CW-complex. Let B^n be the n-skeleton of B. Then filter E by putting $E_n = p^{-1}(B_n)$, a CW-complex. This filtration gives rise to the Leray-Serre spectral sequences. For these, we have

N.8. $E_2^{p,q}(E^0_*, A) = H^p(B^0; \{F^0, A\}^q) = H^p(B, \emptyset; H^q(F, \emptyset; A))$.

N.9. $E^2_{p,q}(\Sigma^0,(E^0\wedge A)_*) = H_p(B^0; \{\Sigma^0, F^0\wedge A\}_q) = H_p(B, \emptyset; H_q(F, \emptyset; A))$,

for any spectrum A. We had to take the disjoint union E^0 of E with a base point o, etc. The coefficient systems $H^*(F, \emptyset; A)$ and $H_*(F, \emptyset; A)$ are twisted in general.

There are also cap products in the spectral sequences, of which N.5. is a special case. Let X and Y be filtered spectra, and A and B any spectra. Then we have natural products, with respect to which the differentials are again derivations,

N.10. $E_r(A, (X\wedge Y)_*) \otimes E_r(Y_*, B) \to E_r(A, (X\wedge B)_*)$,

N.11. $E_r(A, X_*) \otimes E_r((X\wedge Y)_*, B) \to E_r((A\wedge Y)_*, B)$.

In using these products, it is frequently useful to filter a spectrum C trivially, with $C_0 = C$ and $C_{-1} = 0$.

STABLE HOMOTOPY THEORY

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CHAPTER V - DUALITY AND THOM SPECTRA

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CHAPTER V - DUALITY AND THOM SPECTRA

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In this chapter we consider the formal properties of Thom spectra, and how they arise in Spanier-Whitehead duality. There are many peculiar homomorphisms in algebraic topology, defined in widely differing ways; one of our objects is to unify several of these under the name 'transfer homomorphism'.

We also introduce the bordism homology and cobordism cohomology theories (see [C5]). We show how to define generally the cobordism characteristic classes of a vector bundle over a CW-complex; these take values in the cobordism cohomology ring of the base space.

This chapter comprises the sections:

1. Thom spectra
2. Combinatorial Poincaré duality
3. The Thom construction
4. Thom isomorphisms
5. Bordism and cobordism theories
6. Transfer homomorphisms
7. Riemann-Roch theorems
8. Characteristic cobordism classes
9. Some geometric homomorphisms.

§1. Thom spectra.

In [A6], Atiyah considered the Thom complex of a vector bundle over a finite CW-complex from the stable point of view, and observed that its stable homotopy type depended only on the stable class of the bundle. Now that we have the correct stable homotopy theory to work in, we can carry this through for vector bundles over arbitrary CW-complexes, and indeed for virtual vector bundles.

We shall assume that all our vector bundles have been given an orthogonal structural group. Those with fibre dimension n are classified by means of a universal bundle γ_n over a classifying space $B\underset{\sim}{O}(n)$. We shall assume that for the various n these fit together nicely:

(a) We have a CW-complex $B\underset{\sim}{O}$ filtered by subcomplexes

$$\ldots B\underset{\sim}{O}(n) \subset B\underset{\sim}{O}(n+1) \ldots,$$

(b) We have a universal vector bundle γ_n over $B\underset{\sim}{O}(n)$, with fibre dimension n,

(c) We have for each n a bundle isomorphism $\gamma_{n+1}|B\underset{\sim}{O}(n) \cong \gamma_n \oplus 1$, where 1 stands for the trivial line bundle,

(d) We have bundle isomorphisms $\mu^*\gamma_{m+n} \cong \gamma_m \times \gamma_n$ (cross product of vector bundles, over $B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)$), where $\mu: B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(m+n)$ is induced by $\underset{\sim}{O}(m) \times \underset{\sim}{O}(n) \subset \underset{\sim}{O}(m+n)$,

(e) The bundle isomorphisms in (c) and (d) are compatible. This can conveniently be done by using the universal bundles constructed by Milnor [M4].

Bundles are determined by their classifying maps. We shall work with spaces over $BO(n)$ rather than with vector bundles themselves. Let Z be the additive group of integers, with 0 as base point.

<u>1.1 Definition</u> The category A of finite CW-complexes over $BO \times Z$ has as objects pairs (X, f), where X is a finite CW-complex, without base point, and $f: X \to BO \times Z$ is a map. A morphism from (X, f) to (Y, g) is a map $h: X \to Y$ such that $g \circ h = f$. Composition is evident. We have also the subcategory $I(A)$, with the same objects, which contains the morphism h if and only if h is an inclusion of CW-complexes.

<u>1.2 Definition</u> The category A_W of CW-complexes over $BO \times Z$, with the subcategory $I(A_W)$, is the W-extension of the pair of categories $I(A) \subset A$ (see Chapter I).

We observe that A, and hence A_W, is a topological category. By means of $\mu: BO \times BO \to BO$ and group addition in Z, we have a multiplication on $BO \times Z$. The definition of μ is not obvious (see Chapter II). Given a vector bundle ξ over X, with fibre dimension n, we take $f: X \to BO \times Z$ to have a classifying map as first component, and n as second. If the fibre dimension varies, we treat each component of X separately.

1.3 Definition We define

$$KQ(X) = [X^0, BQ \times Z];$$

the set of unbased homotopy classes of maps from X to $BQ \times Z$. It is an abelian group. We call the elements virtual vector bundles over X. The projection $X \to Z$ is called the rank of the virtual vector bundle.

When X is finite, this is the usual Grothendieck group of vector bundles over X. When X is infinite, $KQ(X)$ is much bigger than the Grothendieck group - a virtual bundle is not in general the difference between two honest bundles, else the universal Stiefel-Whitney classes would not be algebraically independent.

Our object is to construct the Thom spectrum of a virtual vector bundle. Given $f:X \to BQ(n)$, the Thom complex of $\xi = f^*\gamma_n$ is obtained from the unit disk bundle in ξ by identifying the boundary sphere bundle to a base point o. It has a natural cell structure. We follow [A6], and write X^ξ for this space. Also, adding a trivial line bundle to ξ simply suspends X^ξ. In particular, X^0 is the disjoint union of X and o, as before! If we write n for the trivial bundle of fibre dimension n, and Σ for a point, we see that Σ^n is an n-sphere!

Write A_∞ for the category of finite CW-complexes over BQ, and A_n for the subcategory of complexes over $BQ(n)$. By compactness, $\underset{\sim}{A}_\infty$ is the union of the subcategories $\underset{\sim}{A}_n$. We can multiply $(X, f) \in \underset{\sim}{A}_m$ with $(Y, g) \in \underset{\sim}{A}_n$ by means of $\mu: \underset{\sim}{BO} \times \underset{\sim}{BO} \to \underset{\sim}{BO}$ to form $(X \times Y, \mu \circ (f \times g))$; this induces a functor $\underset{\sim}{A}_\infty \times \underset{\sim}{A}_\infty \to \underset{\sim}{A}_\infty$. The elementary information about Thom complexes is summarized in:

<u>1.4 Lemma</u> For each $n \geq 0$, we have the Thom complex functor

$$T_n : \underset{\sim}{A}_n \to \underset{\sim}{F}, \quad \underset{\sim}{I}(\underset{\sim}{A}_n) \to \underset{\sim}{I}(\underset{\sim}{F}),$$

where $\underset{\sim}{F}$ is the category of finite CW-complexes with base point. We have natural isomorphisms

a) $ST_n \approx T_{n+1} : \underset{\sim}{A}_n \to \underset{\sim}{F}$,

b) $T_m\alpha \wedge T_n\beta \approx T_{m+n}(\alpha \times \beta)$, $(\alpha \in \underset{\sim}{A}_m,\ \beta \in \underset{\sim}{A}_n)$

which yield the commutative diagram

$$\begin{array}{ccccc}
ST_m\alpha \wedge T_n\beta & \approx & S(T_m\alpha \wedge T_n\beta) & \approx & T_m\alpha \wedge ST_n\beta \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & ST_{m+n}(\alpha\times\beta) & & \\
 & & \Big\downarrow & & \\
T_{m+1}\alpha \wedge T_n\beta & \approx & T_{m+n+1}(\alpha\times\beta) & \approx & T_m\alpha \wedge T_{n+1}\beta \, .
\end{array} \qquad]]]$$

We now feed all this material into the categorical machinery developed in Chapters I and II. We recall that the suspension category $\underset{\sim}{F}_S$ was defined as the 'limit' of the sequence

$$\cdots\cdot \ \underset{\sim}{F}_{-2} \xrightarrow{S} \underset{\sim}{F}_{-1} \xrightarrow{S} \underset{\sim}{F}_0 \xrightarrow{S} \underset{\sim}{F}_1 \xrightarrow{S} \underset{\sim}{F}_2 \ \cdots\cdot,$$

in which each $\underset{\sim}{F}_n$ is a copy of $\underset{\sim}{F}$.

<u>1.5 Lemma</u> We have the Thom spectrum functor

$$T:\underset{\sim}{A} \to \underset{\sim}{F}_S, \quad \underset{\sim}{I}(\underset{\sim}{A}) \to \underset{\sim}{I}(\underset{\sim}{F}_S),$$

and a natural isomorphism

$$T(\alpha \times \beta) \approx T\alpha \wedge T\beta \qquad (\alpha, \beta \in \underset{\sim}{A}).$$

<u>Proof</u> Take a map $f:X \to B\underset{\sim}{O} \times \underset{\sim}{Z}$, where X is a finite CW-complex, which is an object of $\underset{\sim}{A}$. By compactness of X, there exists a least n such that f factors through $f':X \to B\underset{\sim}{O}(n) \times \underset{\sim}{Z}$. We assume X is connected, for the moment. Then f' has first component $f_1:X \to B\underset{\sim}{O}(n)$ and second component $r \in \underset{\sim}{Z}$, say. We define the functor T on this object of $\underset{\sim}{A}$ to be $T_n(X, f_1) \in \underset{\sim}{F}_{n-r}$, an object of $\underset{\sim}{F}_S$. Now suppose $g:Y \to X$, where Y is also finite, connected. Then $f \circ g:Y \to B\underset{\sim}{O} \times \underset{\sim}{Z}$ factors through $B\underset{\sim}{O}(m) \times \underset{\sim}{Z}$, say, where $m \leqslant n$. We require a map $Tg:T_m(Y, f_1 \circ g) \to T_n(X, f_1)$ in $\underset{\sim}{F}_S$. Now $T_m(Y, f_1 \circ g) \in \underset{\sim}{F}_{m-r}$ is isomorphic in $\underset{\sim}{F}_S$, canonically, to $S^{n-m}T_m(Y, f_1 \circ g) \in \underset{\sim}{F}_{n-r}$. Since T_n is a functor, we have a map $T_n(Y, f_1 \circ g) \to T_n(X, f_1)$ in $\underset{\sim}{F}_{n-r}$. Naturality in 1.4 yields a map $S^{n-m}T_m(Y, f_1 \circ g) \to T_n(Y, f_1 \circ g)$. The required map Tg is the composite of these three. One can verify that T is a functor, defined so far for connected X.

If X is not connected, we treat each component of X separately, and take the wedge in $\underset{\sim}{F}_S$, so that T respects sums.

If $\alpha, \beta \in \underset{\sim}{A}$, the natural isomorphisms of 1.4 yield a natural isomorphism of $T(\alpha \times \beta)$ with the smash product $T\alpha \wedge T\beta$ (see II). Care is needed at this stage, but the machinery of II is equal to the task.]]]

<u>1.6 Lemma</u> Suppose the maps $f, g: X \to \underset{\sim}{BO} \times \underset{\sim}{Z}$ are homotopic. Then the Thom spectra of (X, f) and (X, g) are isomorphic, in $\underset{\sim}{I}(\underset{\sim}{F}_S)$.

<u>Proof</u> This derives from the covering homotopy property for bundles.]]]

We can now take W-extensions of everything (see I). Also, our functors are all continuous, and we may take homotopy classes.

<u>1.7 Theorem</u> We have the Thom spectrum functor

$$T: \underset{\sim}{A}_W \to \underset{\sim}{F}_{SW} = \underset{\sim}{S}, \quad \underset{\sim}{I}(\underset{\sim}{A}_W) \to \underset{\sim}{I}(\underset{\sim}{S}), \quad \underset{\sim}{A}_{Wh} \to \underset{\sim}{S}_h.$$

We write the Thom spectrum of the virtual vector bundle α over the CW-complex X as X^α. There are canonical natural isomorphisms

$$(X \times Y)^{\alpha\times\beta} \approx X^\alpha \wedge Y^\beta, \qquad S^n X^\alpha \approx X^{\alpha+n}$$

for each of the above three functors. The first is coherently commutative and associative.]]]

In particular, when $Y = X$, the diagonal map $\Delta: X \to X \times X$ induces from $\alpha \times \beta$ over $X \times X$ the <u>Whitney sum</u> $\alpha + \beta$, which makes $K\underset{\sim}{O}(X)$ an abelian group.

<u>1.8 Corollary</u> There is a canonical natural <u>diagonal</u> map

$$\Delta: X^{\alpha+\beta} \to X^\alpha \wedge X^\beta,$$

which is commutative and associative.]]]

Given a topological group G and a continuous orthogonal representation $G \to \underset{\sim}{O}(n)$, or $G \to \underset{\sim}{O}$, we have the Borel map $BG \to B\underset{\sim}{O}(n)$ or $BG \to B\underset{\sim}{O}$ (see [B2] or [M4]). Hence a map $BG \to B\underset{\sim}{O} \times \underset{\sim}{Z}$, with second component n in the first case, 0 in the second.

<u>1.9 Definition</u> The <u>Thom spectrum</u> MG is the Thom spectrum of the virtual vector bundle $BG \to B\underset{\sim}{O} \times \underset{\sim}{Z}$. In particular $M\underset{\sim}{O}$ is the Thom spectrum of the identity representation of $\underset{\sim}{O}$.

<u>1.10 Definition</u> Denote by γ the universal virtual vector bundle of rank 0, so that $B\underset{\sim}{O}^{\gamma} = M\underset{\sim}{O}$. Then any virtual vector bundle over X, α say, of rank 0, is induced from γ by a <u>classifying map</u> $X \to B\underset{\sim}{O}$, unique up to homotopy. The Thom spectrum functor applied to the classifying map of α yields the <u>classifying map</u> $X^{\alpha} \to M\underset{\sim}{O}$.

More generally, a virtual vector bundle α over X of constant rank n has a classifying map of Thom spectra $X^{\alpha} \to M\underset{\sim}{O}$ of degree $-n$.

We have observed that a genuine vector bundle ξ over X gives rise to a (homotopy class of) virtual vector bundle over X whose rank is the fibre dimension of ξ. By convention, we write X^{ξ} for its Thom spectrum; this is consistent with what we already have.

<u>1.11 Theorem</u> Given a genuine vector bundle ξ over X, let N

be its unit disk bundle, ∂N its unit sphere bundle, and $\pi : N \to X$ the bundle projection. Then for any virtual vector bundle α over X, we have an isomorphism of Thom spectra

$$X^{\xi+\alpha} \approx N^{\pi^*\alpha}/\partial N^{\pi^*\alpha}$$

in $\underset{\sim}{S}$, $\underset{\sim}{I}(\underset{\sim}{S})$, or $\underset{\sim}{S}_h$.

Proof. Our categorical machinery requires simply a natural isomorphism of CW-complexes defined when X is a finite CW-complex and α is a genuine vector bundle, and this isomorphism must commute with the suspension operations on α, $N^{\pi^*\alpha}$, and $X^{\xi+\alpha}$. We do this canonically in each fibre. This amounts to finding for each p, q, a $\underset{\sim}{O}(p) \times \underset{\sim}{O}(q)$-equivariant homeomorphism $D^p \times D^q/\partial(D^p \times D^q) \cong D^{p+q}/\partial D^{p+q}$, which has to be associative. This becomes a trivial matter if we first choose for each p an equivariant homeomorphism of $\underset{\sim}{R}^p$ with the interior of D^p.]]]

As an application, suppose given genuine vector bundles ξ, η over X, Y respectively, having unit disk bundles M and N. Then $X^\xi = M/\partial M$, and $Y^\eta = N/\partial N$. Suppose we are given an embedding of N in M as a tubular neighbourhood of $Y \subset M$, not meeting ∂M. Then identification induces a map of Thom spaces $\varphi : X^\xi \to Y^\eta$.

1.12 Theorem Under these conditions, we have also a canonical map of Thom spectra

$$X^{\xi+\alpha} \to Y^{\eta+f^*\alpha}$$

for any virtual vector bundle α over X, where $f : Y \to X$ is the composite $Y \subset N \subset M \to X$. This map is compatible with the diagonal maps, in the sense that the diagram

$$\begin{tikzcd}
X^{\xi+\alpha} \arrow[rr, "\Delta"'] \arrow[d] & & X^{\xi}{}_{\wedge}X^{\alpha} \arrow[d, "\varphi\wedge 1"] \\
Y^{\eta+f^{*}\alpha} \arrow[r, "\Delta"'] & Y^{\eta}{}_{\wedge}Y^{f^{*}\alpha} \arrow[r, "1\wedge f_{*}"'] & Y^{\eta}{}_{\wedge}X^{\alpha}
\end{tikzcd}$$

commutes.]]]

§2. Combinatorial Poincaré Duality

In this section we translate G.W. Whitehead's duality theorem [W4] into our theory, with the various simplifications possible.

Let X be any finite triangulated simplicial complex. Given any subcomplex K, the <u>supplement</u> K^{-} of K is the union of all simplexes of the first derived complex X' that do not meet K. We observe that

$$(K \cup L)^{-} = K^{-} \cap L^{-}, \quad (K \cap L)^{-} = K^{-} \cup L^{-}, \quad K \subset L \text{ implies } L^{-} \subset K^{-}.$$

There is a unique simplicial map $X' \to K^{-}*K$ (the join) extending the inclusions of K^- and K. Let $s:K^-*K \to [0, 1]$ be the simplicial map taking K^- to 1 and K to 0. Then we set, as [W4],

$$N(K) = s^{-1}[0, \tfrac{1}{2}], \quad N(K^-) = s^{-1}[\tfrac{1}{2}, 1],$$

which are triangulable subspaces of X. We see that we have homotopy equivalences

2.1 $$K^- \subset N(K^-) \subset X - K, \quad K \subset N(K) \subset X - K^-,$$
$$K/L \simeq N(K)/N(L), \qquad K^-/L^- \simeq N(K^-)/N(L^-).$$

If $L \subset K$, we can define a map

$$\Delta:X^0 \to (N(L^-)/N(K^-)) \wedge (N(K)/N(L))$$

in the obvious way on $N(L^-) \cap N(K)$, and zero elsewhere.

2.2 Definition Given subcomplexes $L \subset K$ of X, the diagonal map

$$\Delta:X^0 \to (L^-/K^-) \wedge (K/L)$$

is defined from the above map up to homotopy, by using the homotopy equivalences 2.1.

2.3 Remark When $K = X$, $L = \emptyset$, Δ is the usual diagonal $\Delta:X^0 \to X^0 \wedge X^0$ (recall $X/\emptyset = X^0$).

The diagonal has the expected naturality properties. Given subcomplexes $K \supset L \supset M$ of X, we have $i:L/M \subset K/M$, $p:K/M \to K/L$, and in $\underset{\sim}{S}_{h*}$ the boundary map $\delta:K/L \to L \to L/M$ of degree -1, and similarly for K^-, L^-, M^-. We consider the diagrams

(a)
$$\begin{tikzcd} X^0 \arrow[r, "\Delta"'] \arrow[d, "\Delta"] & (M^-/L^-) \wedge (L/M) \arrow[d, "1\wedge i"] \\ (M^-/K^-) \wedge (K/M) \arrow[r, "p\wedge 1"'] & (M^-/L^-) \wedge (K/M) \end{tikzcd}$$

(b)
$$\begin{tikzcd} X^0 \arrow[r, "\Delta"'] \arrow[d, "\Delta"] & (M^-/K^-) \wedge (K/M) \arrow[d, "1\wedge p"] \\ (L^-/K^-) \wedge (K/L) \arrow[r, "i\wedge 1"'] & (M^-/K^-) \wedge (K/L) \end{tikzcd}$$

(c)
$$\begin{tikzcd} X^0 \arrow[r, "\Delta"'] \arrow[d, "\Delta"] & (L^-/K^-) \wedge (K/L) \arrow[d, "1\wedge \delta"] \\ (M^-/L^-) \wedge (L/M) \arrow[r, "\delta\wedge 1"'] & (L^-/K^-) \wedge (L/M) \end{tikzcd}$$

<u>2.4 Lemma</u> The diagrams (a) and (b) commute, and (c) anticommutes, up to homotopy.

<u>Proof</u> (a) and (b) are obvious. Although (c) looks forbidding, it is sufficient, by (a) and (b), to take $K = X$, $M = \emptyset$.]]]

The diagonal induces cap products. Given a map $\mu: A \wedge B \to C$ of spectra, and $z \in H_n(X^0; A)$, we have the cap product
$$z \cap : H^i(L^-/K^-; B) \to H_{n-i}(K/L; C).$$

The naturality of the diagonal in 2.4 shows that we have the diagram, for any subcomplexes $K \supset L \supset M$ of X, 2.5

$$\begin{array}{ccccccc} H^i(M^-/L^-;\ B) & \to & H^i(M^-/K^-;\ B) & \to & H^i(L^-/K^-;\ B) & \to & H^{i+1}(M^-/L^-;\ B) \ \dots \\ \downarrow z\cap & & \downarrow z\cap & & \downarrow z\cap & & \downarrow z\cap \\ H_{n-i}(L/M;\ C) & \to & H_{n-i}(K/M;\ C) & \to & H_{n-i}(K/L;\ C) & \to & H_{n-i-1}(L/M;\ C) \ \dots \end{array}$$

in which the first two squares commute and the third commutes up to the sign $(-)^{n+1}$.

Now suppose that K and L differ by one simplex P; $K = L \cup P$, and the boundary ∂P of P lies in L. Let c be the barycentre of P, and $V = V_c$ the closed star of c in X', the union of all simplexes containing c. Let Q be the union of all simplexes of X' that meet P only in c, and R the subcomplex of Q consisting of those simplexes not meeting P, the 'link' of P in X'. Then $K/L = P/\partial P$, $L^-/K^- = Q/R$, and $V/\partial V \cong (Q/R) \wedge (P/\partial P)$, where ∂V is the frontier of V.

2.6 Lemma Under these conditions, the diagonal

$$\Delta : X^0 \to (L^-/K^-) \wedge (K/L) \cong (Q/R) \wedge (P/\partial P) \cong V/\partial V$$

agrees, up to homotopy, with the identification map

$$p_c : X^0 \to X/Cl(X - V) = V/\partial V. \quad]]]$$

Homology manifolds

2.7 Definition We say X is a combinatorial homology n-manifold if it is a triangulable space having the same local homology groups as an n-sphere.

We shall always choose a fixed triangulation. Various facts are more or less immediate from the definition. Assume for simplicity that X is compact and <u>connected</u>, for the moment. Then $H^n(X; \underset{\sim}{Z}) \cong \underset{\sim}{Z}$ or $\underset{\sim}{Z}_2$. Every simplex is contained in some n-simplex. Let V_c be the closed star of c in X', for any vertex c of X', and $p_c:X^0 \to V_c/\partial V_c$ the identification map; then $V_c/\partial V_c$ is a homotopy n-sphere (being a suspension), and $p_c^*:H^n(V_c/\partial V_c; \underset{\sim}{Z}) \to H^n(X; \underset{\sim}{Z})$ is epi. We write $q_c:X^0 \to \Sigma^0$ for the desuspended composite $X^0 \to V_c/\partial V_c \simeq \Sigma^n$ of degree $-n$, determined up to sign. Then by a theorem of Hopf, $H^n(X; \underset{\sim}{Z}) \cong \{X^0, \Sigma^0\}^n$, and is generated by q_c for any c.

<u>Orientability</u>

Let X be any triangulated compact combinatorial homology manifold. Suppose given a spectrum A, and a 'unit' map $i:\Sigma^0 \to A$, of degree 0.

<u>2.8 Definition</u> We say $z \in H_n(X; A)$ is <u>a fundamental class</u> of X if for every vertex c of X', $\langle z, q_c\rangle = \pm i \in \pi_0(A)$. We then say X is <u>A-oriented</u>.

<u>Duality</u>

We suppose given $i:\Sigma^0 \to A$ as above.

<u>2.9 Definition</u> We say the spectrum B <u>has A-action</u> if we are given a morphism, $\mu:A \wedge B \to B$, such that the composite

$$B \approx \Sigma^0 \wedge B \xrightarrow[i\wedge 1]{} A \wedge B \to B$$

is the identity morphism of B.

Remark From IV, the only spectra with $K(\underset{\sim}{Z})$-action are the graded Eilenberg-MacLane spectra.

2.10 Theorem Let X be a triangulated compact combinatorial homology n-manifold, A-oriented by $z \in H_n(X; A)$. Then for any subcomplexes $K \supset L$ of X and any spectrum B with A-action,

$$z \cap : H^r(L^-/K^-; B) \cong H_{n-r}(K/L; B), \quad H^r(K/L; B) \cong H_{n-r}(L^-/K^-; B).$$

Proof If K and L differ by one simplex, the theorem holds by 2.6 and the definition 2.8 of orientability. If the result is true for K/L and L/M, it is true for K/M by exactness, commutativity of 2.5, and the five-lemma. The result therefore follows by induction.]]]

We may take $K = X$, $L = \emptyset$.

2.11 Corollary $z \cap : H^r(X^0; B) \cong H_{n-r}(X^0; B)$.]]]

There is no longer any need to work with a fixed triangulation.

2.12 Theorem Let X be a compact combinatorial homology n-manifold, A-oriented by $z \in H_n(X; A)$. Let $K \supset L$ be closed subsets of X which are subcomplexes in some triangulation of X, and $K' \subset L'$ a pair of closed subsets of X homeomorphic to a CW-complex and a subcomplex, such that K' and L' are deformation retracts of $X - K$ and $X - L$ respectively. Then $z \cap$ induces

$$H^r(L'/K'; B) \cong H_{n-r}(K/L; B), \quad H^r(K/L; B) \cong H_{n-r}(L'/K'; B),$$

for any spectrum B with A-action.]]]

Clearly the n-sphere Σ^n is Σ^0-orientable, and any spectrum B has a unique Σ^0-action. Then given subcomplexes $K \supset L$ such that $K \neq \Sigma^n$, $L \neq \emptyset$, the diagonal map $\Delta : \Sigma^n \to (L^-/K^-) \wedge (K/L)$ induces the isomorphisms of 2.10 for any B. After desuspending, our recognition result (see IV) for dual spectra shows that we have duals here. We have

<u>2.13 Theorem</u> Given subcomplexes $K \supset L$ of Σ^n, $K \neq \Sigma^n$, $L \neq \emptyset$, we have, as spectra, $L^-/K^- \simeq S^n D(K/L)$. The hypotheses may be weakened as in 2.12.]]]

In particular, take L to be a point; then $\delta : L^-/K^- \simeq SK^-$.

<u>2.14 Corollary</u> $K^- \simeq S^{n-1} DK$.]]]

Historically, 2.14 was used [S3] as the definition of the dual.

We may also add duality isomorphisms (see IV) $H^r(X^0; B) \cong H_{-r}(DX^0; B)$ and $H_r(X^0; B) \cong H^{-r}(DX^0; B)$ to 2.11 to give a new form to Poincaré duality.

<u>2.15 Theorem</u> Let X be a compact A-oriented combinatorial homology n-manifold. Then we have isomorphisms, for any spectrum B with A-action,

$$H^r(X^0; B) \cong H^{r-n}(DX^0; B), \quad H_r(X^0; B) \cong H_{r-n}(DX^0; B). \text{]]]}$$

<u>Remark</u> The proof of 2.10 did not make essential use of spectra. It could have been expressed entirely in terms of half-exact functors.

§3. The Thom construction

We give the elementary facts about the Pontryagin-Thom construction, expressed in a form suitable for our applications.

In this section, all manifolds shall be smooth C^∞. Triangulation theorems show that we may freely use the results in §2. We again write n for a trivial vector bundle of fibre dimension n.

3.1 Lemma (Milnor, Spanier [M5]). Let M be a smooth compact manifold, and α any virtual vector bundle over M. Then the dual spectrum $DM^\alpha \simeq M^{-\tau-\alpha}$, where τ is the tangent bundle of M. In particular, $DM^0 \simeq M^{-\tau}$. These equivalences are canonical.

Proof Since $DS'X = S'^{-1}DX$ for any spectrum X, and M is compact, by suspending we may assume α is a genuine vector bundle. If n is large enough, we can embed M in Σ^n smoothly, and the bundle α in the normal bundle ν to M in Σ^n, so that $\nu = \alpha \oplus \beta$, say. Then the disk bundle, with total space K, boundary L, of α, is embedded in a tubular neighbourhood of M in Σ^0. We have $M^\alpha = K/L$. We see geometrically, for suitable representatives, that in the notation of 2.12 $L'/K' = M^\beta$. Hence, by 2.13, $DM^\alpha = S'^{-n}M^\beta \simeq M^{\beta-n} = M^{-\tau-\alpha}$ provided $\alpha \neq 0$, since $\tau \oplus \alpha \oplus \beta = n$. If we take $\beta = 0$, we find $DM^{-\tau} = M^0$.]]]

Let $f:X \subset \mathbb{R}^m \times Y$ be a smooth embedding, where X and Y are compact, with normal bundle ν. Then the Thom construction [T1] yields a map $\underset{\sim}{R}^m \times Y \to X^\nu$, which, compactified, gives a map $S^mY^0 \to X^\nu$, or $Y^m \to X^\nu$.

<u>3.2 Definition</u> The <u>Thom map</u> $T(f)$ of f is this map $Y^m \to X^\nu$, or any map $T(f):Y^\alpha \to X^{\nu-m+f_1^*\alpha}$ constructed from it by 1.12, where α is a virtual vector bundle over Y, and $f_1:X \to Y$ is the composite of f with projection.

In particular take $-\alpha = \tau(Y)$, the tangent bundle of Y. Then over X we have $f_1^*\tau(Y) \oplus m = \tau(X) \oplus \nu$. Hence a Thom map $T(f):Y^{-\tau(Y)} \to X^{-\tau(X)}$.

<u>3.3 Lemma</u> Under the identifications of 3.1, the Thom map $T(f):Y^{-\tau(Y)} \to X^{-\tau(X)}$ agrees with the dual $Df_1:DY^0 \to DX^0$.

<u>Proof</u> Let N be the disk bundle over Y having $\underset{\sim}{R}^m \times Y$ as interior. Let M be a tubular neighbourhood of X in N. We embed N smoothly in Σ^k. Then we find tubular neighbourhoods Q of Y, P of X, in Σ^k, $P \subset Q$. By definition the identification map $N/\partial N \to M/\partial M$ is the Thom map $T(f)$, and we see from 1.12 that $Q/\partial Q \to P/\partial P$ is an associated Thom map $T(f)$. Comparison of 2.4(a) with the definition of Df_1 shows that the latter map, after desuspension, gives Df_1.]]]

<u>3.4 Corollary</u> The stable homotopy class of $T(f)$ depends only on the stable homotopy class of f.]]]

This was clear anyway.

<u>3.5 Lemma</u> Under the above hypotheses, the diagram commutes

for any virtual vector bundles α and β over Y:

$$\begin{tikzcd}
Y^{\alpha+\beta} \arrow[rr, "\Delta"'] \arrow[d, "T(f)"] & & Y^{\alpha} \wedge Y^{\beta} \arrow[d, "T(f) \wedge 1"] \\
X^{\nu+f_1^*\alpha+f_1^*\beta} \arrow[r, "\Delta"'] & X^{\nu+f_1^*\alpha} \wedge X^{f_1^*\beta} \arrow[r, "1 \wedge f_{1*}"'] & X^{\nu+f_1^*\alpha} \wedge Y^{\beta} .
\end{tikzcd}$$

<u>Proof</u> This is clear for genuine vector bundles. By compactness of Y the virtual bundles may be suspended to give genuine bundles.]]]

Now suppose we have a second smooth embedding, $g: Y \subset \underset{\sim}{R}^n \times Z$, with normal bundle μ. Then $(1 \times g) \circ f : X \subset \underset{\sim}{R}^{m+n} \times Z$ is a third, with normal bundle $\nu \oplus f_1^*\mu$.

<u>3.6 Lemma</u> Under these hypotheses,

$$T((1 \times g) \circ f) \simeq T(f) \circ T(g) : Z^{\alpha} \to X^{f_1^* g_1^* \alpha + \nu + f_1^* \mu},$$

for any virtual vector bundle α over Z.

<u>Proof</u> Directly, or from 3.3.]]]

§4. Thom isomorphisms

We give in this section the abstract theory behind the Thom isomorphisms, which applies regardless of the honesty or otherwise of the bundles involved. We deduce that a smooth manifold is A-orientable if and only if its stable normal bundle is A-orientable.

Let us take a CW-complex X, and a virtual vector bundle α over X, having constant rank n. Take also a spectrum A with a 'unit' $i:\Sigma^0 \to A$.

4.1 Definition We say $u:X^\alpha \to A$ (of codegree n, i.e. degree $-n$) is a fundamental class of X^α or of α if for every point $x \in X$, the composite $\Sigma^n \cong X^{\alpha|x} \subset X^\alpha \xrightarrow[u]{} A$ is $\pm i$. We then say that X^α and α are A-oriented.

4.2 Definition Given a fundamental class $u:X^\alpha \to A$ of α, and $\mu:A \wedge B \to C$, we have the Thom homomorphisms

$$\Phi^\alpha = u\cup:\{X^\beta, B\}^m \to \{X^{\beta+\alpha}, C\}^{m+n}$$
$$\Phi_\alpha = (-)^{mn}(\cap u):\{\Sigma^0, X^{\beta+\alpha}\wedge B\}_m \to \{\Sigma^0, X^\beta \wedge C\}_{m-n}$$

induced by the diagonal map (1.8)

$$X^{\alpha+\beta} \xrightarrow[\Delta]{} X^\beta \wedge X^\alpha \xrightarrow[1\wedge u]{} X^\beta \wedge A,$$

where β is any virtual vector bundle over X.

Also, by means of the diagonal $\Delta:K^{\beta+\alpha}/L^{\beta+\alpha} \to (K^\beta/L^\beta)\wedge X^\alpha$, we can define useful Thom homomorphisms

<u>4.3</u> $$\Phi^{\alpha}:\{K^{\beta}/L^{\beta},\ B\}^{m} \to \{K^{\beta+\alpha}/L^{\beta+\alpha},\ C\}^{m+n}$$
$$\Phi_{\alpha}:\{\Sigma^{0},\ (K^{\beta+\alpha}/L^{\beta+\alpha})\wedge B\}_{m} \to \{\Sigma^{0},\ (K^{\beta}/L^{\beta})\wedge C\}_{m-n}$$
for any subcomplexes $L \subset K$ of X, natural in K and L, including boundary maps.

<u>4.4 Theorem</u> [Dold] Suppose given $i:\Sigma^{0} \to A$, and a spectrum B with A-action (see 2.9) $\mu:A\wedge B \to B$. Suppose the virtual vector bundle α over X is A-oriented. Then
$$\Phi^{\alpha}:\{X^{\beta},\ B\}^{m} \cong \{X^{\beta+\alpha},\ B\}^{m+n}$$
$$\Phi_{\alpha}:\{\Sigma^{0},\ X^{\beta+\alpha}\wedge B\}_{m} \cong \{\Sigma^{0},\ X^{\beta}\wedge B\}_{m-n}$$
are isomorphisms, for any virtual vector bundle β over X. We also have Thom isomorphisms 4.3 for any subcomplexes $L \subset K$ of X.

<u>Proof</u> By induction on cells. Suppose first that $L \subset K \subset X$, and $K = L \cup e^{k}$, i.e. K is obtained from L by adjoining one k-cell. Let $\chi:D^{k} \to K$ be its characteristic map; then to prove 4.4 for K/L we need only prove 4.4 for $D^{k}/\partial D^{k}$ for the virtual bundles $\chi^{*}\alpha$ and $\chi^{*}\beta$, which are trivial. For either Thom homomorphism, this result follows from 4.1 and the hypothesis on μ, by suspending.

Let X_r be the r-skeleton of X. Then by additivity, from the previous case, we have 4.4 for X_r/X_{r-1}. Suppose, by induction on r, that 4.4 holds for X_{r-1}. Then by naturality of 4.3 and the five lemma, we deduce the isomorphism for X_r. Hence we have the Thom isomorphism for X_r for all r, by induction. By Milnor's lemma (IV), we therefore have the isomorphisms for general X.

In particular, we have isomorphisms for the restricted bundles $\alpha|K$ and $\alpha|L$, whenever $L \subset K \subset X$ are subcomplexes. The theorem for K/L now follows from the five lemma.]]]

<u>Remark</u> Again, when the vector bundles are genuine, this theorem does not make essential use of spectra, and could be expressed in terms of half-exact functors. Theorem 4.4 is in some sense 'dual' to 2.10.

Because we have allowed for the possibility $\beta \neq 0$, Thom isomorphisms can evidently be composed.

<u>4.5 Lemma</u> Suppose given spectra $i:\Sigma^0 \to A$, $i:\Sigma^0 \to B$, and a spectrum C with A- and B-actions $\mu:A \wedge C \to C$, $\mu:B \wedge C \to C$ that commute, in the sense that they yield only one $(A \wedge B)$-action $\mu:A \wedge B \wedge C \to C$ on C. Suppose given virtual vector bundles α, β over X, such that α is A-oriented, and β is B-oriented. Then we can $(A \wedge B)$-orient $\alpha + \beta$ canonically. With these orientations, we have $\Phi^{\alpha+\beta} = \Phi^{\alpha}\Phi^{\beta}$, $\Phi_{\alpha+\beta} = \Phi_{\alpha}\Phi_{\beta}$.

<u>Proof</u> Let $u:X^{\alpha} \to A$ and $v:X^{\beta} \to B$ be the given orientations of α and β. We orient $\alpha + \beta$ by

$$X^{\alpha+\beta} \xrightarrow[\Delta]{} X^{\alpha} \wedge X^{\beta} \xrightarrow[u \wedge v]{} A \wedge B.$$

The commutativity and associativity of Δ yield the composition laws.]]]

We always orient the zero bundle by means of the obvious projection $X^0 \longrightarrow \Sigma^0 \xrightarrow[i]{} A$.

<u>4.6 Lemma</u> Then Φ^0 and Φ_0 are identity homomorphisms.]]]

These two results yield a simpler proof of the Thom isomorphism, in the common cases. Given an A-action on A, $\mu: A \wedge A \to A$, commutative and associative, one deduces from 4.5 and 4.6 that $\Phi^{-\alpha}$ is inverse to Φ^{α}, and that $\Phi_{-\alpha}$ is inverse to Φ_{α}.

Let X be a compact smooth n-manifold with tangent bundle τ, and $i:\Sigma^0 \to A$ a map of spectra. We can consider A-orientability of X, or of τ.

<u>4.7 Theorem</u> With X and A as above, X is an A-orientable manifold if and only if $-\tau$ is an A-orientable bundle. The possible fundamental classes correspond, under the isomorphisms

$DX^0 = X^{-\tau}$ (see 3.1), and $\{DX^0, A\}^* \approx \{\Sigma^0, X^0 \wedge A\}_*$ (see IV).

The Thom isomorphisms $\Phi^{-\tau}:\{X^0, B\}^* \cong \{X^{-\tau}, B\}^*$ and $\Phi_{-\tau}:\{\Sigma^0, X^{-\tau} \wedge B\}_* \cong \{\Sigma^0, X^0 \wedge B\}_*$ agree with the isomorphisms 2.15, for any spectrum B with A-action.

<u>Proof</u> We have the stated isomorphisms. We must check that, if $u:X^{-\tau} \to A$ corresponds to $z:\Sigma^0 \to X^0 \wedge A$, the local conditions on u and z for these to be fundamental classes are equivalent.

We embed X smoothly in Σ^{n+k}, with normal bundle ν. Then we may conveniently take $u:X^{\nu} \to A$, of degree $-k$, instead of the given map $X^{-\tau} \to A$, by suspending, since $\tau + \nu = n + k$. The Thom construction gives a map $\Sigma^{n+k} \to X^{\nu}$. Then z is obtained from u as the composite

$$\Sigma^{n+k} \longrightarrow X^{\nu} \xrightarrow[\Delta]{} X^0 \wedge X^{\nu} \xrightarrow[1 \wedge u]{} X^0 \wedge A,$$

desuspended as necessary. The assertion about Thom isomorphisms follows.

Take any point x of X. Then we have the composite $v:\Sigma^k \to A$, defined by inclusion of a fibre, $\Sigma^k \cong X^{\nu|x} \subset X^{\nu} \xrightarrow[u]{} A$. Take a disk neighbourhood D^n of x in X, and let $q:X^0 \to \Sigma^n$ be the Thom construction applied to this disk neighbourhood D^n of x in X. Then our two local conditions are that for all $x \in X$, the maps $v:\Sigma^k \to A$ and $(q \wedge 1) \circ z:\Sigma^{n+k} \to \Sigma^n \wedge A$ are each $\pm i$, apart from suspensions. But it is immediate from the diagram below that these conditions are equivalent. This diagram is made up of Thom constructions, and commutes up to homotopy (compare 1.11).

$$\begin{tikzcd}
\Sigma^{n+k} \arrow[r] \arrow[d, "\simeq"] & X^{\nu} \arrow[r, "\Delta"'] & X^0 \wedge X^{\nu} \arrow[r, "1\wedge u"'] \arrow[d, "q\wedge 1"] & X^0 \wedge A \arrow[d, "q\wedge 1"] \\
\Sigma^n \wedge \Sigma^k \arrow[rr] & & \Sigma^n \wedge X^{\nu} \arrow[r, "1\wedge u"'] & \Sigma^n \wedge A.
\end{tikzcd}$$]]]

<u>Multiplicative structure</u>

Take $i:\Sigma^0 \to A$, and let B and C be spectra with A-action, such that $B \wedge C$ inherits a well-defined A-action $\mu:A \wedge B \wedge C \to B \wedge C$. Let ξ be an A-oriented virtual vector bundle of rank r over X.

Then we have seen that the diagonal $X^{\xi} \xrightarrow[\Delta]{} X^0 \wedge X^{\xi} \longrightarrow X^0 \wedge A$ induces Thom isomorphisms. Also, the diagonals $\Delta : X^{\xi} \to X^0 \wedge X^{\xi}$ and $\Delta : X^0 \to X^0 \wedge X^0$ induce cup and cap products. Then by commutativity and associativity of cup products and diagonals Δ (see 1.8), and the mixed rule for cup and cap products, we deduce the multiplication formulae

<u>4.8</u> $\qquad \Phi^{\xi}(\alpha \cup \beta) = \Phi^{\xi}\alpha \cup \beta = (-)^{mr}\,\alpha \cup \Phi^{\xi}\beta \quad (\alpha \in \{X^0, B\}^m,\ \beta \in \{X^0, C\}^n)$

<u>4.9</u> $\qquad \Phi_{\xi}(x \cap \alpha) = (-)^{mr}\, x \cap \Phi^{\xi}\alpha = \Phi_{\xi}x \cap \alpha$

$$(x \in \{\Sigma^0, X^{\xi} \wedge B\}_m, \quad \alpha \in \{X^0, C\}^n).$$

<u>Naturality</u>

Consider the Thom maps $T(f) : Y^{\alpha} \to X^{\mu + f^*\alpha}$ induced by a smooth embedding $f : X \subset Y \times \underset{\sim}{R}^k$ of smooth manifolds, as in 3.2, where α may be any virtual vector bundle over Y.

<u>4.10 Lemma</u> Let β be an A-oriented virtual vector bundle over Y. Then the Thom maps T(f) and Thom isomorphisms Φ^{β} yield commutative diagrams, for any spectrum B with A-action,

$$\begin{array}{ccc}
\{X^{\mu+f_1^*\alpha}, B\}^* & \xrightarrow{T(f)^*} & \{Y^{\alpha}, B\}^* \\
\cong \big\downarrow \Phi^{f_1^*\beta} & & \cong \big\downarrow \Phi^{\beta} \\
\{X^{\mu+f_1^*\alpha+f_1^*\beta}, B\}^* & \xrightarrow{T(f)^*} & \{Y^{\alpha+\beta}, B\}^*
\end{array}$$

and

$$\begin{array}{ccc}
\{\Sigma^0, Y^{\alpha+\beta} \wedge B\}_* & \xrightarrow[T(f)_*]{} & \{\Sigma^0, X^{\mu+f_1^*\alpha+f_1^*\beta} \wedge B\}_* \\
\cong \big\downarrow \Phi_{\beta} & & \cong \big\downarrow \Phi_{f_1^*\beta} \\
\{\Sigma^0, Y^{\alpha} \wedge B\}_* & \xrightarrow[T(f)_*]{} & \{\Sigma^0, X^{\mu+f_1^*\alpha} \wedge B\}_*
\end{array}$$

<u>Proof</u> Both parts are immediate from 3.5.]]]

§5. Bordism and cobordism theories

Thom's fundamental lemma [T1] relating cobordism classes to homotopy classes shows that the use of Thom spectra as coefficient spectra gives rise to geometrically interesting homology and cohomology theories.

Let ξ be any vector bundle, rank k, over a CW-complex B. Let M be any smooth (n + k)-manifold, and M_c its one-point compactification.

5.1 Definition A ξ-submanifold, or submanifold with ξ-structure, is a smooth compact submanifold V^n of M, with a bundle map from its normal bundle ν to ξ. Two ξ-submanifolds V_1 and V_2 are cobordant if there exists a ξ-submanifold W of $M \times I$, with boundary $V_i = W \cap (M \times i)$ $(i = 0, 1)$, where W meets $M \times 0$ and $M \times 1$ transversely, and the ξ-structure on W extends that on V_0 and V_1, under the natural identifications $\nu | V_i = \nu_i$ $(i = 0, 1)$, where ν, ν_0, ν_1, are the normal bundles of W in $M \times I$, V_0 in M, V_1 in M.

In particular, if the same submanifold V is given two homotopic structure maps $\nu \to \xi$, the two resulting ξ-submanifolds are cobordant. Cobordism is an equivalence relation.

The Thom construction applied to the ξ-submanifold V of M yields a map $M \to B^\xi$ with compact support, and therefore a map $M_c \to B^\xi$. Uniqueness of tubular neighbourhoods and the definition of cobordism show that this construction yields a well defined map from the set of cobordism classes of ξ-submanifolds of M, to $[M_c, B^\xi]$.

5.2 Lemma [after Thom] The Thom construction induces an isomorphism

$$L(M;\ \xi) \cong [M_c, B^\xi],$$

where $L(M;\ \xi)$ denotes the set of cobordism classes of ξ-submanifolds of M.

Proof The method of proof in [T1] is valid for the case when ξ is a smooth vector bundle over a smooth manifold B. We reduce the general case to this case.

Since any ξ-submanifold V of M is compact, its structure map $\nu \to \xi$ factors through $\xi|C$, for some finite subcomplex C of B. Similarly for the structure map of a cobordism manifold between two ξ-submanifolds. Hence

$$L(M;\ \xi) = \varinjlim L(M;\ \xi|C);\ \text{and}\ [M_c, B^\xi] = \varinjlim\, [M_c, C^{\xi|C}];$$

as C runs through finite subcomplexes of B. Thus we need consider only the case when B is a finite CW-complex.

We may clearly replace B by any space B' of the same homotopy type, and ξ by the induced bundle ξ' over B'. We can choose B' to be a smooth manifold (e.g. an open neighbourhood of B in $\underset{\sim}{R}^m$ for some suitable m), and give ξ' a smooth structure.]]]

Remark The condition on B can be removed.

We now stabilize. We shall be concerned only with the case $M = \underset{\sim}{R}^{n+k}$; then $M_c \cong \Sigma^{n+k}$, a sphere. We may replace B by its $(n+1)$-skeleton without loss of generality; then if $k > n$ any virtual vector bundle ξ over B of rank k is isomorphic to a genuine vector bundle. Also, if $k > n+1$ the particular embedding of V in $\underset{\sim}{R}^{n+k}$ becomes irrelevant, and we ignore it.

We next give the stable version of 5.1. Let ξ be a virtual vector bundle with base B (i.e. a map $\xi : B \to B\underset{\sim}{O}$), of rank 0.

5.3 Definition The smooth n-manifold V^n is a ξ-manifold if we are given a map $-\tau \to \xi - n$ of virtual vector bundles, where τ is the tangent bundle of V. Two compact ξ-manifolds without boundary V_0^n and V_1^n are said to be cobordant if there is a compact ξ-manifold W^{n+1} with boundary $V_0 \cup V_1$, whose ξ-structure extends those of V_0 and V_1. We define $L_n(\xi)$ as the set of cobordism classes of compact smooth ξ-manifolds without boundary.

The word 'extends' needs amplification. In comparing the structures of W and V_i over V_i $(i = 0, 1)$, we need to make use of bundle isomorphisms $\tau | V_i \cong \tau_i \oplus 1$, where τ, τ_0, τ_1, are the tangent bundles of W, V_0, V_1, respectively, and the extra trivial bundle 1 represents the inward normal bundle of V_0 in W or the outward normal bundle of V_1 in W.

With the help of the remarks preceding 5.3, we apply 5.2 with $M = \underset{\sim}{R}^{n+k}$, $k \geq n+2$, to $\xi + k$, which bundle may be assumed honest.

5.4 Theorem Let ξ be a virtual vector bundle over B, and $L_n(\xi)$ the set of cobordism classes of ξ-manifolds of dimension n. Then the Thom construction induces the isomorphism

$$L_n(\xi) \cong \{\Sigma^0, B^\xi\}_n. \quad]]]$$

This theorem leads to the computation of $L_n(\xi)$ in various well known cases [T1, M1, A6]. As examples, we have:

ξ	ξ-manifolds	$L_*(\xi)$
zero bundle over point	stably framed manifolds	framed cobordism groups
identity virtual bundle over $B\underset{\sim}{O}$	unoriented manifolds (i.e. no extra structure)	$\underline{\underline{N}}_*$
universal virtual bundle over $B\underset{\sim}{SO}$	oriented manifolds	Ω_*
universal virtual bundle over $B\underset{\sim}{U}$	'unitary' manifolds	$\underline{\underline{U}}_*$
universal virtual bundle over B Spin	spin manifolds	spin cobordism groups

The 'unitary' manifolds are commonly called 'weakly almost complex' manifolds. Spin cobordism appears not to have been properly defined until [M6].

Track addition in $L_n(\xi)$ in 5.4 is clearly expressed geometrically by disjoint union of ξ-manifolds. If we are given a bundle map $\xi \times \xi \to \xi$, we can introduce multiplication into $L_*(\xi)$ in the obvious way, which corresponds under 5.4 to that induced by the map of Thom spectra $B^\xi \wedge B^\xi \cong (B \times B)^{\xi\times\xi} \to B^\xi$ (from 1.7).

Singular manifolds

Suppose we are given a space X and a virtual vector bundle ξ over B of rank 0.

5.5 Definition A singular ξ-manifold of X is a pair (V, f), where V is a ξ-manifold and $f: V \to X$ is an (unbased) map. Two singular manifolds (V, f) and (V', f') are bordant if there is a cobordism ξ-manifold W between V and V', and a map $g: W \to X$ extending f and f'. Denote by $B_n(X; \xi)$ the set of bordism classes of singular ξ-manifolds of dimension n. (Compare [C5].)

Thus the structure of a singular ξ-manifold (V, f) of X consists of a bundle map $-\tau \to \xi - n$ and a map $V \to X$. We may combine these into a single virtual bundle map $-\tau \to \eta - n$, where η is the virtual bundle over $X \times B$ induced from ξ by projection. So singular ξ-manifolds of X correspond to η-manifolds, and cobordism classes correspond. We have, therefore, $B_n(X; \xi) \cong L_n(\eta)$. By 1.7, $(X \times B)^\eta \approx X^0 \wedge B^\xi$.

5.6 Theorem The Thom construction induces an isomorphism, natural in X and ξ,

$$B_*(X;\ \xi) \cong \{\Sigma^0,\ X^0 \wedge B^\xi\}_*,$$

between the graded group $B_*(X;\ \xi)$ of bordism classes of singular ξ-manifolds of X and the stable homotopy groups of $X^0 \wedge B^\xi$.]]]

Thus $B_*(\ ;\xi)$, apart from reduction, is the standard homology theory (see IV) with B^ξ as coefficient spectrum. It is therefore prudent to introduce the associated cohomology theory. In accordance with general policy, we define all homology and cohomology theories in the <u>reduced</u> form.

<u>5.7 Definition</u>

$$\begin{aligned} \underline{\underline{N}}_n(X) &= \{\Sigma^0,\ X \wedge \underset{\sim}{MO}\}_n, & \underline{\underline{N}}^n(X) &= \{X,\ \underset{\sim}{MO}\}^n \\ \underline{\underline{U}}_n(X) &= \{\Sigma^0,\ X \wedge \underset{\sim}{MU}\}_n, & \underline{\underline{U}}^n(X) &= \{X,\ \underset{\sim}{MU}\}^n \\ \Omega_n(X) &= \{\Sigma^0,\ X \wedge \underset{\sim}{MSO}\}_n, & \Omega^n(X) &= \{X,\ \underset{\sim}{MSO}\}^n \end{aligned}$$

(see 1.9 for $\underset{\sim}{MU}$ and $\underset{\sim}{MSO}$).

Then $\underline{\underline{N}}_*(X^0)$ are the bordism groups of X; but $\underline{\underline{N}}_*$ and $\underline{\underline{N}}^*$ are now defined on all spectra. The products $\underset{\sim}{MO} \wedge \underset{\sim}{MO} \to \underset{\sim}{MO}$, etc. from 1.8. induce commutative and associative products in all the above pairs of theories. In particular, these are modules over the coefficient rings $\underline{\underline{N}} = \{\Sigma^0,\ \underset{\sim}{MO}\}_*$, etc.

Conner and Floyd show in [C5] that when A is a subspace of X, the relative bordism group $\underline{\underline{N}}_n(X,\ A) = \{\Sigma^0,\ (X/A) \wedge \underset{\sim}{MO}\}_n$, etc., can also be given a geometric interpretation. Elements are represented as equivalence classes of singular manifolds with boundary, $f:(V,\ \partial V) \to (X,\ A)$, under a rather artificial equivalence relation. This has some uses. For purely homological considerations, relative groups are superfluous: instead a Mayer-Vietoris boundary homomorphism is all that is required. Berstein has given an elegant method of doing this. Let $X = A \cup B$, where A and B are open subsets, and let $f:M \to X$ be a singular manifold of X. Then $f^{-1}(X - A)$ and $f^{-1}(X - B)$ are disjoint closed subsets of M. Take a smooth Urysohn function $\varphi:M \to \underset{\sim}{R}$, 0 on $f^{-1}(X - A)$, 1 on $f^{-1}(X - B)$, and transverse to $\frac{1}{2}$. Then $N = \varphi^{-1}(\frac{1}{2})$ is a smooth $(n-1)$-manifold, and the class of the singular manifold $g = f|N:N \to A \cap B$ is the required boundary.

Trivially, any virtual bundle over any CW-complex X is $\underset{\sim}{MO}$-oriented (4.1) by means of its classifying map $X \to \underset{\sim}{MO}$ (assuming it has constant rank), where $i:\Sigma^0 \to \underset{\sim}{MO}$ is the classifying map of the zero bundle over a point. Hence we always have Thom isomorphisms for the $\underset{\sim}{MO}$ theories. Let us give the geometric interpretation in terms of singular manifolds.

<u>5.8 Lemma</u> Let ξ be a smooth vector bundle over the manifold X. Let $f:(M, \partial M) \to (X^\xi, o)$ be a singular manifold of (X^ξ, o), smooth near and transverse to $X \subset X^\xi$. Put $N = f^{-1}(X)$, and $g = f|N$, so that $g:N \to X$ is a singular manifold of X. Then the Thom isomorphism $\Phi_\xi:\underline{\underline{N}}_*(X^\xi) \to \underline{\underline{N}}_*(X^0)$ is given by $\Phi_\xi[M, f] = [N, g]$.

<u>Proof</u> In effect, Φ_ξ is induced by the map of Thom spectra

$$X^{\xi} \xrightarrow[\Delta]{} X^0 \wedge X^{\xi} \to X^0 \wedge M\underset{\sim}{O}$$

over the map of base spaces $X \to X \times B\underset{\sim}{O}$. This, composed with f, is homotopic to the map $M \to X^0 \wedge M\underset{\sim}{O}$ obtained by applying the Thom construction to N in M.]]]

In particular, let X be a smooth submanifold of the smooth manifold Y with normal bundle ν, $f:M \to Y$ a singular manifold of Y transverse to X, $N = f^{-1}(X)$, and $g = f|N$, so that $g:N \to X \subset Y$ is a singular manifold of Y. We recall that the Thom isomorphism is a cap product. Let $\alpha:X \to M\underset{\sim}{O}$ be the classifying map of ν. Then naturality of Thom spectra yields the geometric interpretation of cap products:

<u>5.9 Lemma</u> We have $[M, f] \cap \alpha = [N, g]$.]]]

Again, any smooth manifold is canonically $M\underset{\sim}{O}$-oriented by means of the identity singular manifold. One can deduce that in this case Poincaré duality is given by the Thom map (see 3.2) : given $f:M \subset X \times \underset{\sim}{R}^n$, we use the map $X^n \xrightarrow[T(f)]{} M^{\nu} \to M\underset{\sim}{O}$.

Evidently, everything we have done for the $M\underset{\sim}{O}$ theories carries over to the other theories, provided the bundles and manifolds have suitable structures.

It is well known when bundles are orientable for ordinary cohomology.

<u>5.10 Definition</u>

We have the fundamental classes

$\sigma_{\underset{\sim}{O}}:M\underset{\sim}{O} \to K(\underset{\sim}{Z}_2)$, $\sigma_{\underset{\sim}{SO}}:M\underset{\sim}{SO} \to K(\underset{\sim}{Z})$, hence $\sigma_{\underset{\sim}{U}}:M\underset{\sim}{U} \to K(\underset{\sim}{Z})$,

defined in the usual way (e.g. [T1]).

§6. Transfer homomorphisms

There are certain well-known important homomorphisms in algebraic topology which do not quite fit into the usual functorial framework. We propose to call them *transfer* homomorphisms, by analogy with the representation theory of groups. There is one for each homology and cohomology theory.

The transfer homomorphism is defined like a function on a manifold - we define it locally, on various domains of definition, and show that it is well defined on the intersection of any two of these domains. We shall content ourselves with eight types of transfer homomorphism; there are many more in common use.

Since our applications are to smooth manifolds, we shall often restrict attention to this easy case, even though it is well known, and sometimes obvious, that the definitions work much more generally. We also say nothing about manifolds with boundary. We give bordism transfer homomorphisms only for the theory $\underline{\underline{N}}_*$; but again the definitions hold for other bordism theories, under the obvious orientation conditions.

We shall use the same symbol $\natural$ for all transfer homomorphisms, however defined, to distinguish them from ordinary induced homomorphisms (this differs from some current practice).

Axiomatic description

We recall that $X^0 = X/\emptyset$. For *certain* maps $f:X \to Y$ of spaces with additional structure, in addition to the ordinary induced homomorphisms

$$f^*:\{Y^0, B\}^* \to \{X^0, B\}^* \qquad f_*:\{\Sigma^0, X^0 \wedge B\}_* \to \{\Sigma^0, Y^0 \wedge B\}_*$$

we have <u>transfer</u> homomorphisms, both of the same generally non-zero degree, m say,

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^* \qquad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*.$$

These are functorial to the extent that $1_\natural$ and $1^\natural$ are identities, and that if we have $g_\natural$ and $g^\natural$, where $g:Y \to Z$, then

<u>6.1</u> $$(g \circ f)_\natural = g_\natural f_\natural \qquad (g \circ f)^\natural = (-)^{mn} f^\natural g^\natural,$$

where $f_\natural$, $g_\natural$, and $(g \circ f)_\natural$ have degrees m, n, $m+n$. Under the conditions favourable to cup and cap products, we have

<u>6.2</u> (a) $f_\natural(\alpha \cup f^*\beta) = f_\natural\alpha \cup \beta \qquad (\alpha:X^0 \to B,\ \beta:Y^0 \to C)$

(b) $f_\natural(f^*\alpha \cup \beta) = (-)^{m|\alpha|}\ \alpha \cup f_\natural\beta \qquad (\alpha:Y^0 \to B,\ \beta:X^0 \to C)$

(c) $f^\natural(x \cap \alpha) = f^\natural x \cap f^*\alpha \qquad (x:\Sigma^0 \to Y^0 \wedge B,\ \alpha:X^0 \to C)$

(d) $f_*(f^\natural x \cap \alpha) = (-)^{m|x|}\ x \cap f_\natural\alpha \qquad (x:\Sigma^0 \to Y^0 \wedge B,\ \alpha:X^0 \to C)$;

in particular, for Kronecker products,

(e) $\langle f^\natural x, \alpha\rangle = (-)^{m|x|}\ \langle x, f_\natural\alpha\rangle \qquad (x:\Sigma^0 \to Y^0 \wedge B,\ \alpha:X^0 \to C)$.

If we are also given $C = B = A$ and a multiplication $\mu:A \wedge A \to A$, then $f_\natural$ and $f^\natural$ are homomorphisms of $\{\Sigma^0, A\}_*$-modules. We shall give the products in the simplest forms available; they may all be embellished with suitable multiplication maps $B \wedge C \to D$, etc.

<u>Various transfer homomorphisms</u>

We shall always need a spectrum A with a map $i:\Sigma^0 \to A$, and a spectrum B with A-action $\mu:A \wedge B \to B$ (see 2.9).

(a) Poincaré duality transfer

Let X and Y be A-oriented combinatorial homology manifolds, of dimensions m and n, with fundamental classes $z_X:\Sigma^0 \to X^0 \wedge A$, and $z_Y:\Sigma^0 \to Y^0 \wedge A$. Then $|z_X| = m$, $|z_Y| = n$. Take $f:X \to Y$.

6.3 Definition We define the Poincaré duality transfers

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^* \qquad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

by the formulae

$$z_Y \cap f_\natural\alpha = f_*(z_X \cap \alpha) \qquad (\alpha:X^0 \to B)$$

$$f^\natural(z_Y \cap \beta) = (-)^{(m-n)n}\, z_X \cap f^*\beta \quad (\beta:Y^0 \to B).$$

Then $f_\natural$ and $f^\natural$ have degree m - n, and the diagrams

$$\begin{array}{ccc} \{X^0, B\}^* & \xrightarrow{f_\natural} & \{Y^0, B\}^* \\ \cong\downarrow z_X\cap & & \cong\downarrow z_Y\cap \\ \{\Sigma^0, X^0 \wedge B\}_* & \xrightarrow{f_*} & \{\Sigma^0, Y^0 \wedge B\}_* \end{array} \qquad \begin{array}{ccc} \{Y^0, B\}^* & \xrightarrow{f^*} & \{X^0, B\}^* \\ \cong\downarrow z_Y\cap & & \cong\downarrow z_X\cap \\ \{\Sigma^0, Y^0 \wedge B\}_* & \xrightarrow{f^\natural} & \{\Sigma^0, X^0 \wedge B\}_* \end{array}$$

commute up to sign. The definition works in virtue of the duality isomorphisms 2.11.

For these transfers, 6.1 is trivial.

Suppose we have also a spectrum C with A-action $A \wedge C \to C$, such that the two resulting A-actions on $B \wedge C$ coincide. Then the multiplicative formulae 6.2 follow from the standard formulae (IV) for associativity, commutativity, and induced homomorphisms of cup and cap products, by algebraic manipulation or commutative diagrams, according to taste. (To prove (a) and (b), apply $z_Y\cap$ to each side. To prove (c) and (d), express x in the form $z_Y \cap \beta$.)

Next, assume that $B = C = A$, and put $\beta = 1 \in \{Y^0, B\}$, the 'unit element', in 6.3. Then

<u>6.4</u> $$f^\natural z_Y = (-)^{(m-n)n} z_X.$$

If we put $x = z_Y$ in 6.2 (d) and (c) and substitute from 6.4, we recover the formulae of 6.3. Thus

<u>6.5 Lemma</u> In this case, 6.4 and the multiplicative formulae 6.2 characterize $f^\natural$ and $f_\natural$.]]]

<u>Remark</u> The Principle of Signs breaks down in 6.3, because z_X and z_Y have non-zero degree. In any case, we cannot regard $f^\natural$ or $f_\natural$ as being obtained from f by a unary operation. We have chosen the signs to make 6.1 hold.

<u>(b) Spanier-Whitehead duality transfer</u>

Let X and Y be A-oriented combinatorial homology manifolds, of dimensions m and n, with fundamental classes $z_X:\Sigma^0 \to X^0 \wedge A$, $z_Y:\Sigma^0 \to Y^0 \wedge A$. Let $f:X \to Y$ be a map.

<u>6.6 Definition</u> We define the <u>Spanier-Whitehead duality transfers</u>

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^*, \quad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

by requiring the diagrams

$$\begin{tikzcd}
\{X^0, B\}^p \arrow[r, "f_\natural"'] \arrow[d, "\cong"'] & \{Y^0, B\}^{p-m+n} \arrow[d, "\cong"'] \\
\{DX^0, B\}^{p-m} \arrow[r, "(Df)^*"'] & \{DY^0, B\}^{p-m}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{\Sigma^0, Y^0 \wedge B\}_p \arrow[r, "f^\natural"'] \arrow[d, "\cong"'] & \{\Sigma^0, X^0 \wedge B\}_{p+m-n} \arrow[d, "\cong"'] \\
\{\Sigma^0, DY^0 \wedge B\}_{p-n} \arrow[r, "(Df)_*"'] & \{\Sigma^0, DX^0 \wedge B\}_{p-}
\end{tikzcd}$$

to commute up to the signs $+ 1$ and $(-)^{(m-n)n}$ respectively, in which D denotes dual as in IV, and the vertical isomorphisms are provided by 2.15.

6.7 Lemma The Spanier-Whitehead and Poincaré duality transfers agree.

Proof We recall from 2.15 the definition of the vertical isomorphisms. This shows that the first diagram of 6.6 can be expanded to give the commutative diagram

$$\begin{tikzcd}
\{X^0, B\}^p \arrow[r, "f_\natural"'] \arrow[d, "\cong"', "z_X\cap"] & \{Y^0, B\}^{p-m+n} \arrow[d, "\cong"', "z_Y\cap"] \\
\{\Sigma^0, X^0\wedge B\}_{m-p} \arrow[r, "f_*"'] \arrow[d, "\cong"'] & \{\Sigma^0, Y^0\wedge B\}_{m-p} \arrow[d, "\cong"'] \\
\{DX^0, B\}^{p-m} \arrow[r, "(Df)^*"'] & \{DY^0, B\}^{p-m},
\end{tikzcd}$$

which shows the result for $f_\natural$. Similarly for $f^\natural$.]]]

(c) The transfer of an oriented map

Let $f:X \to Y$ be a map of CW-complexes, which need not now be finite.

6.8 Definition We say f is an A-oriented map if we are given a map of spectra $\hat{f}:Y^0 \to X^0 \wedge A$, of degree n, say, such that

$$\begin{tikzcd}
Y^0 \arrow[rr, "\Delta"'] \arrow[d, "\hat{f}"] & & Y^0 \wedge Y^0 \arrow[d, "1\wedge\hat{f}"] \\
X^0 \wedge A \arrow[r, "\Delta\wedge 1"'] & X^0 \wedge X^0 \wedge A \arrow[r, "f\wedge 1\wedge 1"'] & Y^0 \wedge X^0 \wedge A
\end{tikzcd}$$

commutes (up to homotopy). It induces transfers

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^*, \quad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

as follows: Given $\alpha : X^0 \to B$, we have $Y^0 \xrightarrow{\hat{f}} X^0 \wedge A \xrightarrow{\alpha \wedge 1} B \wedge A \xrightarrow{\mu} B$, and we define $f_\natural \alpha = (-)^{n|\alpha|} \mu \circ (\alpha \wedge 1) \circ \hat{f}$. Given $x : \Sigma^0 \to Y^0 \wedge B$, we have $\Sigma^0 \xrightarrow{x} Y^0 \wedge B \xrightarrow{\hat{f} \wedge 1} X^0 \wedge A \wedge B \xrightarrow{1 \wedge \mu} X^0 \wedge B$, and we define $f^\natural x = (1 \wedge \mu) \circ (\hat{f} \wedge 1) \circ x$. (One could always take $\hat{f} = 0$, which would make $f_\natural$ and $f^\natural$ zero.)

Suppose that we are also given a spectrum C with A-action, such that the two resulting A-actions on $B \wedge C$ agree. Then the deduction of the multiplicative formulae 6.2 from the commutative diagram of 6.8 is another exercise in manipulating commutative diagrams or algebraic formulae, according to taste.

The multiplicative properties of the Thom isomorphisms (see §4) are included as a special case.

Now assume that we are in the simplified multiplicative situation with $A = B = C$, and we have a commutative and associative map $\mu : A \wedge A \to A$. Then given maps $f : X \to Y$, $g : Y \to Z$, A-oriented by $\hat{f} : Y^0 \to X^0 \wedge A$, $\hat{g} : Z^0 \to Y^0 \wedge A$, we can A-orient $g \circ f : X \to Z$ by putting

<u>6.9</u> $\quad \widehat{g \circ f} = (-)^{mn} (1 \wedge \mu) \circ (\hat{f} \wedge 1) \circ \hat{g} : Z^0 \to Y^0 \wedge A \to X^0 \wedge A \wedge A \to X^0 \wedge A$,

where m and n are the degrees of $\hat{f}$ and $\hat{g}$. The commutative diagram of 6.8 for $\widehat{g \circ f}$ follows immediately.

We next compare this transfer with the Poincaré duality transfer. Let X and Y be A-oriented manifolds, of dimensions m and n.

<u>6.10 Definition</u> We say the A-orientation of $f:X \to Y$ is <u>compatible</u> with the A-orientations of X and Y if f has degree $m-n$, and the transfer induced by $\hat{f}$ gives $f^{\natural} z_Y = (-)^{(m-n)n} z_X$, where z_X and z_Y are the fundamental classes of X and Y.

<u>6.11 Lemma</u> Suppose the map $f:X \to Y$ of A-oriented manifolds is A-oriented compatibly, by $\hat{f}:Y^0 \to X^0 \wedge A$, where $\mu:A \wedge A \to A$ is commutative and associative. Then the transfer induced by $\hat{f}$ agrees with the Poincaré duality transfer. If also Z is an A-oriented manifold, and $g:Y \to Z$ is oriented compatibly, then the A-orientation of $g \circ f$ is compatible with the A-orientations of X and Z.

<u>Proof</u> It is immediate from 6.5 that the two transfers agree, for they agree on z_Y by 6.4 and 6.10, and are both multiplicative.]]]

<u>(d) The Grothendieck transfer</u>

Let $f:X \to Y$ be a map of compact smooth manifolds, whose tangent bundles are $\tau(X)$ and $\tau(Y)$. We suppose that the virtual vector bundle $f^*\tau(Y) - \tau(X)$ over X is A-oriented, and deduce a transfer.

We lift f, up to homotopy, to a smooth embedding $f_1:X \subset Y \times \underset{\sim}{R}^k$. Let ν be the normal bundle. Then $\nu = k + f^*\tau(Y) - \tau(X)$ is A-oriented. The Thom construction (3.2) yields a map $T(f):Y^0 \to X^\nu$, of degree k, and hence, by using diagonals (1.8), a map of spectra $\hat{f}:Y^0 \to X^\nu \xrightarrow{\Delta} X^0 \wedge X^\nu \to X^0 \wedge A$.

This A-orients the map f (the commutative diagram of 6.8 is immediate), and hence induces transfer homomorphisms $f_\natural$ and $f^\natural$. However, we can express these transfers slightly differently, as being induced by the composites of the ordinary homomorphisms induced by T(f) with Thom isomorphisms.

<u>6.12 Definition</u> In this situation, the <u>Grothendieck transfers</u>

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^*, \quad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

are the composite homomorphisms

$$\{X^0, B\}^* \xrightarrow[\Phi^\nu]{} \{X^\nu, B\}^* \xrightarrow[T(f)^*]{} \{Y^0, B\}^*$$

and

$$\{\Sigma^0, Y^0 \wedge B\}_* \xrightarrow[T(f)_*]{} \{\Sigma^0, X^\nu \wedge B\}_* \xrightarrow[\Phi_\nu]{} \{\Sigma^0, X^0 \wedge B\}_*.$$

We see from 3.4, or by direct geometric construction, that these transfers are well defined. We deduce the multiplicative formulae 6.2 from those for Thom isomorphisms (4.8 and 4.9), and the composition law 6.1 from 3.6.

We have already observed that the Grothendieck transfers are special cases of transfers induced by an oriented map.

<u>6.13 Lemma</u> Suppose that X and Y are also A-oriented manifolds, and that we are given $\mu:A \wedge A \to A$, commutative and associative. Then there is a canonical A-orientation for $f^*\tau(Y) - \tau(X)$, and with this orientation the Grothendieck and Spanier-Whitehead duality transfers agree.

<u>Proof</u> Write $\nu = f^*\tau(Y) - \tau(X)$. By 4.7, we have A-orientations $v:Y^{-\tau(Y)} \to A$ and $w:X^{-\tau(X)} \to A$. We choose $u:X^\nu \to A$ corresponding

to w under the Thom isomorphism

$$\Phi^{-f^*\tau(Y)}:\{X^\nu, A\}^* \cong \{X^{-\tau(X)}, A\}^* ;$$

naturality of Thom isomorphisms for the inclusion in X of any point shows that u is an A-orientation of the virtual bundle ν. Further, by means of 4.5 and μ, the orientations u and v give back the orientation w, and $\Phi^{-\tau(X)} = \Phi^{-f^*\tau(Y)}\,\Phi^\nu$. We consider the diagram

$$\begin{tikzcd}
\{X^0, B\}^* \arrow[r, "\Phi^\nu"'] & \{X^\nu, B\}^* \arrow[r, "\Phi^{-f^*\tau(Y)}"] \arrow[d, "T(f)^*"] & \{X^{-\tau(X)}, B\}^* \arrow[d, "T(f)^*"] \\
 & \{Y^0, B\}^* \arrow[r, "\Phi^{-\tau(Y)}"'] & \{Y^{-\tau(Y)}, B\}^*,
\end{tikzcd}$$

which commutes by 4.10. From 4.7, the Thom isomorphisms of $-\tau(X)$ and $-\tau(Y)$ are the Spanier-Whitehead duality isomorphisms 2.15 for X and Y, apart from putting $DX^0 = X^{-\tau(X)}$ and $DY^0 = Y^{-\tau(Y)}$, and from 3.3 $T(f)^* = (Df)^*$; we are back to 6.6.

Similarly for homology.]]]

(e) Integration over the fibre

We now consider a fibre bundle $\pi:E \to B$ whose fibre F is a compact n-manifold, with fundamental classes $z_F \in H_n(F; G)$ and $u_F \in H^n(F; G)$, where $G = \mathbb{Z}$ or $\mathbb{Z}_2$. In the case $G = \mathbb{Z}$ we also require the fundamental group of B to act trivially on $H_n(F; G)$. Then we have transfer homomorphisms (see e.g. [B3]) known as 'integration over the fibre'. The picturesque name arises from the case when B, E, and F are smooth manifolds and π is a smooth bundle, in which the cohomology transfer can be expressed in terms of integrating differential forms.

We shall also call these transfers the spectral sequence transfers, and use the definition [B3] in terms of the Leray-Serre spectral sequences of π.

<u>6.14 Definition</u> We define the <u>spectral sequence transfers</u>, or <u>integration over the fibre</u>, $\pi_\natural : H^i(E) \to H^{i-n}(B)$, $\pi^\natural : H_i(B) \to H_{i+n}(E)$, in terms of the spectral sequences of π as follows:

$\pi_\natural$ is the composite $H^i(E) \to E_\infty^{i-n,n} \subset E_2^{i-n,n} \cong H^{i-n}(B)$,

$\pi^\natural$ is the composite $H_i(B) \cong E^2_{i,n} \to E^\infty_{i,n} \subset H_{i+n}(E)$,

where the isomorphisms are $\otimes\, u$ and $\otimes\, z$.

We know (from IV) that we can put cup and cap products into these spectral sequences. It is easy to deduce from this fact the multiplicative formulae 6.2.

It is clear that $\pi_\natural$ and $\pi^\natural$ are natural for maps of bundles with fibre F, because the spectral sequences are natural.

<u>6.15 Lemma</u> (Chern [C2]) Suppose B is a manifold. Then E is also a manifold, and the spectral sequence transfers agree with the Poincaré duality transfers.

<u>Proof</u> Both pairs of transfers are multiplicative, and hence, by 6.5, we need only check $\pi^\natural z_B = \pm z_E$. This is evident from the definition 6.14, assuming we choose the correct orientation z_E for E.]]]

(f) The pullback transfer

One would expect that for a geometrically defined homology theory such as bordism theory various transfers could be defined geometrically. This is indeed the case, although we shall restrict attention to the theory $\underline{\underline{N}}_*$ for simplicity.

Suppose we are given a smooth map $f:X \to Y$ of compact smooth manifolds, of dimensions m and n. Given a singular manifold $h:N \to Y$ of Y, we can construct the pullback space M and a map $g:M \to X$. Under a suitable transversality condition (viz. $f \times h:X \times N \to Y \times Y$ transverse to the diagonal of $Y \times Y$) $g:M \to X$ is a singular manifold of X.

6.16 Definition The pullback transfer $f^\natural:\underline{\underline{N}}_i(Y^0) \to \underline{\underline{N}}_{i+m-n}(X^0)$ is defined by taking the class of $h:N \to Y$ to the class of $g:M \to X$.

One can show directly that $f^\natural$ is well defined.

6.17 Lemma The pullback transfer agrees with the Grothendieck transfer.

Proof We lift f to a smooth embedding $f':X \subset Y \times \underset{\sim}{R}^k$. Then our assertion is evident from two applications of 5.8.]]]

(g) The bundle transfer

There is another case, very similar to the previous, in which a geometric definition can be given. Suppose $\pi:E \to B$ is a fibre bundle whose fibre F is a smooth compact n-manifold, and whose structure group is a Lie group G acting smoothly on F.

Then given a singular manifold $f:M \to B$ of B, we construct the induced bundle $\tilde{M} \to M$ over M and a map $\tilde{f}:\tilde{M} \to E$. We may give this induced bundle a smooth structure, which makes $\tilde{f}:\tilde{M} \to E$ a singular manifold of E.

<u>6.18 Definition</u> The <u>bundle transfer</u> $\pi^{\natural}:\underline{\underline{N}}_i(B^0) \to \underline{\underline{N}}_{i+n}(E^0)$ is defined by taking the class of $f:M \to B$ to the class of $\tilde{f}:\tilde{M} \to E$.

Again one can show that $\pi^{\natural}$ is well defined. It obviously agrees with the pullback transfer when B is a smooth manifold. It is also trivial that $\pi^{\natural}$ is natural for maps of bundles with fibre F.

<u>(h) The Grothendieck bundle transfer</u>

A serious disadvantage of the two previous transfers is that there is no obvious way to define the corresponding cohomology transfer, because $\underline{\underline{N}}^*$ is not a geometric theory. We should like to have multiplicative transfers. Again, integration over the fibre has only been defined for ordinary homology and cohomology. We fill this gap by constructing another transfer, available for general cohomology and homology theories.

Let $\pi:E \to B$ be a fibre bundle whose fibre F is a smooth compact n-manifold, and whose structure group is a compact Lie group acting smoothly on F. We shall need the bundle τ of tangents along the fibre (see e.g. [B3]); this is a vector bundle over E whose restriction to a typical fibre F is the tangent bundle of F.

6.19 Lemma Let F be a smooth compact manifold, and G a compact Lie group acting smoothly on F. Then there exists a finite-dimensional representation space V for G, and a smooth G-equivariant embedding $F \subset V$.

Proof We use a useful lemma of Mostow [M7], based on the Peter-Weyl theorem. Let S be the algebra of smooth real functions on F; then G acts on S. Take a finite set of elements $\{h_i\}$ of S which separate points of F. By [M7] we can approximate these by $\{h_i'\}$, still separating, such that for all i Gh_i' is contained in a finite-dimensional subspace of S. Let W be the subspace of S spanned by all the sets Gh_i'; it is finite-dimensional. Put $V = \mathrm{Hom}(W, \mathbb{R})$. Then evaluation of W at each point of F yields the required equivariant embedding $F \subset V$.]]]

Given a representation space V as in 6.19, let η be the vector bundle over B with fibre V associated to π [S5]. Then 6.19 yields an embedding of E in the total space of η. Choose an equivariant metric on V, and let U be a metric tubular neighbourhood of F in V. Then U gives rise to an associated subbundle of η having fibre U, total space N, say. We have a tubular neighbourhood disk bundle N of E in η, with normal bundle ν, say. Without loss of generality N is contained in the unit disk bundle of η. The

Thom construction now gives a map $B^{\eta} \to E^{\nu}$, and hence a map of spectra $B^{\eta+\xi} \to E^{\eta+\pi^*\xi}$ by 1.12 for any virtual vector bundle ξ over B. In particular, we have a map of spectra

<u>6.20</u> $$T(\pi): B^0 \to E^{-\tau},$$

since $\pi^*\eta \cong \tau \oplus \nu$. The Thom map 6.20 is well defined (for if ζ is another bundle containing π, adding ζ to η does not affect $T(\pi)$, and we then have to compare two isotopic embeddings of E in $\eta \oplus \zeta$).

Now we suppose that $-\tau$ is A-oriented, and that C is a spectrum with A-action.

<u>6.21 Definition</u> We define the <u>Grothendieck bundle transfers</u>

$$\pi_\natural: \{E^0, C\}^* \to \{B^0, C\}^*, \qquad \pi^\natural: \{\Sigma^0, B^0 \wedge C\}_* \to \{\Sigma^0, E^0 \wedge C\}_*$$

as the composite homomorphisms

$$\{E^0, C\}^* \xrightarrow[\Phi^{-\tau}]{} \{E^{-\tau}, C\}^* \xrightarrow[T(\pi)^*]{} \{B^0, C\}^*$$

$$\{\Sigma^0, B^0 \wedge C\}_* \xrightarrow[T(\pi)_*]{} \{\Sigma^0, E^{-\tau} \wedge C\}_* \xrightarrow[\Phi_{-\tau}]{} \{\Sigma^0, E^0 \wedge C\}_*$$

Formally we have exactly the same situation as for the Grothendieck transfers, and we shall not trouble to repeat the details. These transfers are multiplicative, i.e. satisfy 6.2. It is clear that they are natural for maps of fibre bundles with fibre F. As before, we have here a particular case of an oriented map.

If B is in fact a smooth manifold, we can choose η to be a trivial vector bundle (it does not <u>have</u> to come from a representation space). Then these transfers agree with the Grothendieck transfers 6.12.

Again, naturality and two applications of 5.8 show that our transfer includes the transfer 6.18 in bordism groups.

<u>6.22 Lemma</u> If the spectral sequence transfers 6.14 are also defined, they agree with the Grothendieck bundle transfers.

<u>Proof</u> We observe that we can relativize the transfers 6.21 as we did Thom isomorphisms, by constructing a map of spectra

<u>6.23</u> $$T(\pi):B_1/B_2 \to (E_1/E_2) \wedge E^{-\tau}$$

for subcomplexes $B_2 \subset B_1 \subset B$, where $E_i = \pi^{-1}(B_i)$. The spectral sequences of π can also be relativized. By using naturality of both pairs of transfers, we quickly reduce to the case of a trivial bundle over $(D^r, \partial D^r)$, which is clear.]]]

<u>Summary</u>

Let us gather together what we have. For the map $f:X \to Y$ of CW-complexes, under the respective orientation conditions, we have various transfer homomorphisms $f_\natural$ and $f^\natural$:

(a) Poincaré duality - X and Y A-oriented manifolds.

(b) Spanier-Whitehead duality - X and Y A-oriented manifolds.

(c) Oriented map - A-orientation $\hat{f}:Y^0 \to X^0 \wedge A$.

(d) Grothendieck - X, Y smooth manifolds, $f^*\tau(Y)-\tau(X)$ A-oriented.

(e) Integration over the fibre - f a bundle, fibre a manifold.

(f) Pullback - X and Y smooth manifolds. (Only in bordism.)

(g) Bundle - f a bundle, fibre a smooth manifold, group a Lie group acting smoothly. (Only in bordism.)

(h) Grothendieck bundle - f a bundle, fibre a smooth manifold, group a compact Lie group acting smoothly, $-\tau$ A-oriented.

<u>6.24 Theorem</u> Under suitable conditions, these transfers are multiplicative. If two transfers are defined for f, and the appropriate compatibility conditions on the orientations hold, the two transfers agree.]]]

§7. Riemann-Roch theorems

We give here the formal theory (compare [D2]) of Riemann-Roch theorems for smooth manifolds (e.g. [A8]) and other situations in which transfer homomorphisms are available.

We shall suppose throughout this section that we are given two coefficient spectra A and C, with maps $i:\Sigma^0 \to A$, $i:\Sigma^0 \to C$, and commutative and associative multiplication maps $\mu: A \wedge A \to A$, $\mu: C \wedge C \to C$, such that $A \approx \Sigma^0 \wedge A \xrightarrow{i \wedge 1} A \wedge A \xrightarrow{\mu} A$ and similarly $C \to C$ are identity maps of spectra. We suppose also that we have a 'homomorphism' $\theta: A \to C$, such that $\theta \circ \mu = \mu \circ (\theta \wedge \theta): A \wedge A \to C$, and $\theta \circ i = i: \Sigma^0 \to C$.

If ξ is a virtual vector bundle over X, with A- and C-orientations, there is no reason for expecting the diagram

$$\begin{array}{ccc} \{X^0, A\}^* & \xrightarrow{\theta_*} & \{X^0, C\}^* \\ \cong\downarrow\Phi^\xi & & \cong\downarrow\Phi^\xi \\ \{X^\xi, A\}^* & \xrightarrow{\theta_*} & \{X^\xi, C\}^* \end{array}$$

to commute. Indeed, we use this diagram to define a new homomorphism.

<u>7.1 Definition</u> We define a homomorphism

$$\theta_\xi: \{X^0, A\}^* \to \{X^0, C\}^*$$

by putting $\theta_\xi \alpha = (\Phi^\xi)^{-1}\theta_*\Phi^\xi \alpha$.

Associativity of cup products yields the formula

<u>7.2</u> $$\theta_\xi(\alpha \cup \beta) = \theta_\xi \alpha \cup \theta_* \beta. \qquad (\alpha, \beta \in \{X^0, A\}^*).$$

Next, we suppose that X and Y are smooth manifolds, each A- and B- oriented, and $f:X \to Y$ a map. Then the transfer $f_{\natural}^{A}:\{X, A\}^* \to \{Y, A\}^*$ may be defined (6.6 and 4.7) by Thom isomorphisms and the Thom map $Y^{-\tau(Y)} \to X^{-\tau(X)}$ (essentially the dual of f by 3.3), where $\tau(X)$ and $\tau(Y)$ are the tangent bundles of X and Y. Then naturality and 7.1 yields the formula

<u>7.3</u> $$f_{\natural}^{C}\theta_{-\tau(X)} = \theta_{-\tau(Y)}\, f_{\natural}^{A}:\{X^0, A\}^* \to \{Y^0, C\}^*.$$

Take $\alpha:Y^0 \to A$, $\beta:X^0 \to A$. Then 6.2, 7.2, and 7.3 give

$$\begin{aligned} f_{\natural}^{C}(\theta_{-\tau(X)}f^*\alpha \cup \theta_*\beta) &= f_{\natural}^{C}\theta_{-\tau(X)}(f^*\alpha \cup \beta) = \theta_{-\tau(Y)}f_{\natural}^{A}(f^*\alpha \cup \beta) \\ &= \theta_{-\tau(Y)}(\alpha \cup f_{\natural}^{A}\beta) = \theta_{-\tau(Y)}\alpha \cup \theta_* f_{\natural}^{A}\beta, \end{aligned}$$

i.e.

<u>7.4</u> $$f_{\natural}^{C}(\theta_{-\tau(X)}f^*\alpha \cup \theta_*\beta) = \theta_{-\tau(Y)}\alpha \cup \theta_* f_{\natural}^{A}\beta \quad (\alpha:Y^0 \to A,\ \beta:X^0 \to A).$$

In the cohomology ring $\{X^0, A\}^*$ of X we have the identity element 1, given by $X^0 \to \Sigma^0 \xrightarrow[i]{} A$, where the first map is induced by projecting X to a point.

<u>7.5 Definition</u> We put $\hat{a}(X) = \theta_{-\tau(X)}1 \in \{X^0, C\}$.

<u>7.6 Theorem</u> $f_{\natural}^{C}(\hat{a}(X) \cup \theta_*\beta) = \hat{a}(Y) \cup \theta_* f_{\natural}^{A}\beta \quad (\beta:X^0 \to A)$.

<u>Proof</u> We put $\alpha = 1$ in 7.4, and use 7.5.]]]

The proof of this 'Riemann-Roch' theorem is trivial. It is the verification of orientability and computation of $\hat{a}(X)$ and $\hat{a}(Y)$ that are liable to cause difficulties (e.g. [A8]).

More generally, we need only the difference bundle $f^*\tau(Y) - \tau(X)$ to be oriented, if we use the Grothendieck transfer 6.12.

<u>7.7 Definition</u> If $\nu = f^*\tau(Y) - \tau(X)$ is A- and C- oriented, we put

$$\hat{a}(f) = \theta_\nu 1 \in \{X^0, C\}.$$

If also X and Y are oriented as before, and the difference bundle ν is oriented according to 6.13, it is not difficult to prove

<u>7.8</u> $$\hat{a}(X) = f^*\hat{a}(Y) \cup \hat{a}(f),$$

and that $\hat{a}(Y)$ is invertible in $\{Y^0, C\}$, so that in this case $\hat{a}(f)$ can be determined from this formula.

In the same way as for 7.6, we obtain

<u>7.9 Theorem</u> $$f_\natural^C(\hat{a}(f) \cup \theta_*\beta) = \theta_* f_\natural^A \beta \qquad (\beta : X^0 \to A). \quad]]]$$

Again, suppose we have a fibre bundle $\pi : E \to B$ as in the context of the Grothendieck bundle transfer 6.21, and let τ be the bundle of tangents along the fibres. Formally, the situation is exactly that of 7.9. Suppose $-\tau$ is A- and C- oriented.

<u>7.10 Definition</u> We put $\hat{a}(\pi) = \theta_{-\tau} 1 \in \{E^0, C\}$.

<u>7.11 Theorem</u> $$\pi_\natural^C(\hat{a}(\pi) \cup \theta_*\alpha) = \theta_* \pi_\natural^A \alpha \qquad (\alpha : E^0 \to A). \quad]]]$$

Thus if θ_* is mono, and we know $\pi_\natural^C$ and $\hat{a}(\pi)$, we can compute $\pi_\natural^A$. It is this case that will concern us.

One could, of course, derive Riemann-Roch-type theorems for homology and cap products, along the same lines. There are obvious advantages, however, in arranging the computations so that they only involve cohomology and cup products.

§8. Characteristic cobordism classes

Given a complex vector bundle ξ over a CW-complex B, we shall define natural characteristic classes of ξ, called the Chern cobordism classes of ξ, taking values in $\underline{\underline{U}}^*(B^0)$. (Recall that $B^0 = B/\emptyset$, and that all cohomology is taken reduced.) Similarly we obtain Whitney classes in the real case, and an Euler class. For the Chern cobordism classes, this has been done by Conner and Floyd, under the restriction that B is finite. Now that we have $\underline{\underline{U}}^*(\ )$ defined satisfactorily for arbitrary CW-complexes, this restriction is irrelevant; also we may parallel Borel's approach [B2] and work only with the universal bundle over $B\underset{\sim}{U}(n)$, thus guaranteeing naturality. Finally, we show that our definition agrees with that of Conner and Floyd.

The importance of these classes in $\underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0)$ and $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$ is that they pick out canonical elements, which makes more precise investigations possible, as we shall see in VI. Further, we shall find in §9 some geometric properties of these classes.

For any honest vector bundle ξ over B, there is a canonical inclusion of B^0 in the Thom complex B^ξ, as the zero section (apart from base point). In particular, we have $B\underset{\sim}{U}(1)^0 \subset M\underset{\sim}{U}(1)$, etc.

<u>8.1 Definition</u> The <u>first universal Chern cobordism class</u> $C_1 \in \underline{\underline{U}}^2(B\underset{\sim}{U}(1)^0)$ is the composite $B\underset{\sim}{U}(1)^0 \subset M\underset{\sim}{U}(1) \to M\underset{\sim}{U}$.
The <u>first universal Stiefel-Whitney cobordism class</u> $W_1 \in \underline{\underline{N}}^1(B\underset{\sim}{O}(1)^0)$

is the composite $BO(1)^0 \subset MO(1) \to MO$.
The *n th universal Euler cobordism class* $X_n \in \{BSO(n)^0, MSO\}^n$ is the composite $BSO(n)^0 \subset MSO(n) \to MSO$.

In each case, the second map of spectra is the classifying map of a Thom spectrum.

These are our initial characteristic classes, from which we shall construct the others. We do this by using the fundamental classes (see 5.10) $\sigma_O : MO \to K(Z_2)$ and $\sigma_U : MU \to K(Z)$, observing that they induce ring homomorphisms from cobordism to ordinary cohomology, and using the results of Borel [B2].

Denote by $T(n)$ the usual maximal torus of diagonal matrices in $U(n)$, and $Q(n)$ the diagonal subgroup of $O(n)$. Then $T(n) \cong T(1) \times T(1) \times \ldots \times T(1)$, and we may therefore take $BT(n) = BT(1) \times BT(1) \times \ldots \times BT(1)$, and similarly for $Q(n)$. Define the cobordism classes $S_i \in U^2(BT(n)^0)$ $(1 \leq i \leq n)$ induced from $C_1 \in U^2(BT(1)^0) \cong U^2(BU(1)^0)$ by projection $BT(n) \to BT(1)$ to the i th factor; similarly we obtain $T_i \in N^1(BQ(n)^0)$.

In cohomology we have the corresponding cohomology classes s_i and t_i, and by Borel we have

8.2 $$H^*(BT(n)^0; Z) = Z[s_1, s_2, \ldots, s_n],$$
$$H^*(BQ(n)^0; Z_2) = Z_2[t_1, t_2, \ldots, t_n],$$

graded polynomial rings.

8.3 Lemma (a) $U^*(BT(n)^0) = U[S_1, S_2, \ldots, S_n]^\wedge$, and $\sigma_U \circ S_i = s_i$.

(b) $\underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0) = \underline{\underline{N}}[T_1, T_2, \ldots, T_n]^\wedge$, and $\sigma_{\underset{\sim}{Q}} \circ T_i = t_i$.
In each case, completion $\wedge$ is with respect to the skeleton topology (see IV), which here is the augmentation ideal generated by the S_i or the T_i, and its powers.

Proof We see from 8.1 that $\sigma_{\underset{\sim}{U}} \circ C_1 = c_1$, the first cohomology Chern class, from [B2] or [H1], and hence $\sigma_{\underset{\sim}{U}} \circ S_i = s_i$. (We may take the inclusion $B\underset{\sim}{U}(1) \subset M\underset{\sim}{U}(1)$ as inclusion of a hyperplane in $P_\infty(\underset{\sim}{C})$.) By IV we have a spectral sequence with E_2 term $E_2^{p,q} = H^p(B\underset{\sim}{T}(n)^0; \underline{\underline{U}}^q)$, where we write $\underline{\underline{U}}^q = \underline{\underline{U}}_{-q}$. Milnor has shown that $\underline{\underline{U}}$ is a graded polynomial ring over $\underset{\sim}{Z}$, with one generator in each even negative codegree, and in particular is free abelian [M1]; hence by 8.2 all the differentials vanish. Therefore the derived term RE_ω vanishes, and the spectral sequence converges (see IV; this is a fourth quadrant spectral sequence) to E_∞, associated to a complete Hausdorff filtration of $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0)$. The homomorphism induced by $\sigma_{\underset{\sim}{U}}$ appears here as an edge homomorphism $\underline{\underline{U}}^p(B\underset{\sim}{T}(n)^0) \to E_2^{p,0}$. Since $\sigma_{\underset{\sim}{U}} \circ S_i = s_i$, and the spectral sequence has products and $\underline{\underline{U}}$-module structure, $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0)$ must be as stated.

Similarly for $B\underset{\sim}{Q}(n)$, except that we have to invoke the fact (compare [C5]) that again the differentials vanish, because as we shall see in VI, $M\underset{\sim}{O}$ is a graded Eilenberg-MacLane spectrum.]]]

Still following Borel, we consider the map $\rho : B\underset{\sim}{T}(n) \to B\underset{\sim}{U}(n)$ induced by inclusion $\underset{\sim}{T}(n) \subset \underset{\sim}{U}(n)$.

<u>3.4 Lemma</u> (a) Inclusion induces the monomorphism $\rho^* : \underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0) \to \underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0)$, whose image is the symmetric subalgebra of $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0) = \underline{\underline{U}}[S_1, S_2, \ldots, S_n]^\wedge$.

(b) Inclusion $\underset{\sim}{Q}(n) \subset \underset{\sim}{O}(n)$ induces the monomorphism $\rho^* : \underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0) \to \underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0)$, whose image is the symmetric subalgebra of $\underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0) = \underline{\underline{N}}[T_1, T_2, \ldots, T_n]^\wedge$.

<u>Proof</u> The symmetric group G of permutations of n objects acts on $\underset{\sim}{T}(n)$ by permuting the factors, and hence also on $B\underset{\sim}{T}(n)$. However, each permutation can be expressed as conjugation by an element of $\underset{\sim}{U}(n)$, which is path connected. It follows that G acts on $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0) = \underline{\underline{U}}[S_1, S_2, \ldots, S_n]^\wedge$ by permuting the S_i, and that the image of ρ^* is contained in the symmetric subalgebra of $\underline{\underline{U}}[S_1, \ldots, S_n]^\wedge$. Consideration of the spectral sequences for $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0)$ and $\underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0)$ and of the map between them induced by ρ shows that ρ^* must be mono and its image the whole of the symmetric subalgebra, since $H^*(B\underset{\sim}{U}(n)^0) = \underset{\sim}{Z}[c_1, c_2, \ldots, c_n]$ and $\rho^* H^*(B\underset{\sim}{U}(n)^0)$ is the symmetric subalgebra of $\underset{\sim}{Z}[s_1, s_2, \ldots, s_n]$.

Similarly for $B\underset{\sim}{O}(n)$.]]]

The situation is therefore exactly as we would expect from that in cohomology, apart from the need for completion. We can therefore proceed.

8.5 Definition We define the <u>universal Chern cobordism classes</u> $C_i \in \underline{\underline{U}}^{2i}(B\underset{\sim}{U}(n)^0)$ $(1 \leqslant i \leqslant n)$ so that $\rho^* C_i$ is the i th elementary symmetric function of the S_j.

We define the <u>universal Stiefel-Whitney cobordism classes</u> $W_i \in \underline{\underline{N}}^i(B\underset{\sim}{O}(n)^0)$ so that $\rho^* W_i$ is the i th elementary symmetric function of the T_i.

This definition is permitted by 8.4.

Remark We can now complement 8.1.

8.6 $C_n \in \underline{\underline{U}}^{2n}(B\underset{\sim}{U}(n)^0)$ is the composite $B\underset{\sim}{U}(n)^0 \subset M\underset{\sim}{U}(n) \to M\underset{\sim}{U}$.

$W_n \in \underline{\underline{N}}^n(B\underset{\sim}{O}(n)^0)$ is the composite $B\underset{\sim}{O}(n)^0 \subset M\underset{\sim}{O}(n) \to M\underset{\sim}{O}$.

We are ready for the main theorem.

8.7 Theorem

(a) $\underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0) = \underline{\underline{U}}[C_1, C_2, \ldots, C_n]^\wedge$; $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0) = \underline{\underline{N}}[W_1, W_2, \ldots, W_n]^\wedge$.

(b) $\sigma_{\underset{\sim}{U}} \circ C_i = c_i$; $\sigma_{\underset{\sim}{O}} \circ W_i = w_i$.

(c) Inclusion $\underset{\sim}{U}(n) \subset \underset{\sim}{U}(n+1)$ induces the homomorphism

$$\underline{\underline{U}}^*(B\underset{\sim}{U}(n+1)^0) = \underline{\underline{U}}[C_1, \ldots, C_{n+1}]^\wedge \to \underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0) = \underline{\underline{U}}[C_1, \ldots, C_n]^\wedge,$$

which takes C_i to C_i for $1 \leqslant i \leqslant n$, and C_{n+1} to 0. Similarly for $\underset{\sim}{O}(n) \subset \underset{\sim}{O}(n+1)$.

(d) By (c), we may define $C_i \in \underline{\underline{U}}^{2i}(B\underset{\sim}{U}^0)$ as the inverse limit of the elements $C_i \in \underline{\underline{U}}^{2i}(B\underset{\sim}{U}(r)^0)$ $(r \geqslant i)$. Similarly $W_i \in \underline{\underline{N}}^i(B\underset{\sim}{O}^0)$.

(e) $\underline{\underline{U}}^*(B\underset{\sim}{U}^0) = \underline{\underline{U}}[C_1, C_2, \ldots.]^\wedge$; $\underline{\underline{N}}^*(B\underset{\sim}{O}^0) = \underline{\underline{N}}[W_1, W_2, \ldots]^\wedge$.

(f) Inclusion $\underset{\sim}{U}(m) \times \underset{\sim}{U}(n) \subset \underset{\sim}{U}(m+n)$ induces the homomorphism

$$\begin{array}{ccc} \underline{\underline{U}}^*(B\underset{\sim}{U}(m+n)^0) & \to & \underline{\underline{U}}^*(\{B\underset{\sim}{U}(m) \times B\underset{\sim}{U}(n)\}^0) \\ \| & & \| \\ \underline{\underline{U}}[C_1, C_2, \ldots, C_{m+n}]^\wedge & \to & \underline{\underline{U}}[C_1 \otimes 1, C_2 \otimes 1, \ldots, C_m \otimes 1, 1 \otimes C_1, \ldots, 1 \otimes C_n]^\wedge \end{array}$$

in which

$$C_i \rightsquigarrow C_i \otimes 1 + C_{i-1} \otimes C_1 + C_{i-2} \otimes C_2 + \dots + 1 \otimes C_i,$$

with the convention that $C_j = 0$ in $\underline{\underline{U}}^*(B\underset{\sim}{U}(r)^0)$ if $j > r$.

Similarly for $\underset{\sim}{O}(m) \times \underset{\sim}{O}(n) \subset \underset{\sim}{O}(m+n)$.

Proof We give only the unitary proofs, as the orthogonal proofs are similar. By 8.4, $\underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0)$ is isomorphic by ρ^* to the symmetric subalgebra of $\underline{\underline{U}}[S_1, S_2, \dots, S_n]^\wedge$, which is known (Newton?) to be a completed graded polynomial algebra on the elements $\rho^* C_i$. We have (a). Since C_i and c_i are both defined in terms of elementary symmetric functions, (b) follows from 8.3. The inclusion $\underset{\sim}{T}(n) \subset \underset{\sim}{T}(n+1)$ induces a homomorphism taking S_i to S_i $(1 \leqslant i \leqslant n)$ and S_{n+1} to 0, clearly. Hence (c), by 8.4 and 8.5. By Milnor's lemma (see IV) $\underline{\underline{U}}^*(B\underset{\sim}{U}^0) = \varprojlim \underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0)$, since we have here a sequence of epimorphisms; hence (d) and (e). In (f), we need only work with maximal tori, $\underset{\sim}{T}(m) \times \underset{\sim}{T}(n) = \underset{\sim}{T}(m+n)$, by 8.4. For these, we have $S_i \rightsquigarrow S_i \otimes 1$ (if $1 \leqslant i \leqslant m$) or $S_i \rightsquigarrow 1 \otimes S_{i-m}$ (if $m < i \leqslant m+n$). The result follows from 8.5.]]]

8.8 Corollary The comultiplications in $\underline{\underline{U}}^*(B\underset{\sim}{U}^0)$ and $\underline{\underline{N}}^*(B\underset{\sim}{O}^0)$ are given by

$$C_i \rightsquigarrow C_i \otimes 1 + C_{i-1} \otimes C_1 + C_{i-2} \otimes C_2 + \dots + 1 \otimes C_i;$$

$$W_i \rightsquigarrow W_i \otimes 1 + W_{i-1} \otimes W_1 + W_{i-2} \otimes W_2 + \dots + 1 \otimes W_i. \quad]]]$$

8.9 Definition Given a virtual vector bundle ξ over the CW-complex X, its Stiefel-Whitney cobordism characteristic classes

$W_i(\xi) \in \underline{\underline{N}}^i(X^0)$ are defined by $W_i(\xi) = \xi^* W_i$ (recall that by definition 1.3 we have $\xi : X \to B\underset{\sim}{O}$). Similarly, $C_i(\xi) \in \underline{\underline{U}}^{2i}(X^0)$ is defined, if ξ is factored through $B\underset{\sim}{U}$.

<u>8.10 Corollary</u> Let ξ and η be virtual vector bundles over X. Then $W_k(\xi + \eta) = \Sigma_{i+j=k} W_i(\xi).W_j(\eta)$.]]]

Finally we give an alternative description of the Chern (and equally of the Stiefel-Whitney) cobordism classes, which is the version adopted by Conner and Floyd.

Suppose ξ is an honest complex vector bundle over the finite-dimensional CW-complex X, with complex fibre dimension n. Let Y be the total space of the associated projective bundle with fibre $P_{n-1}(\underset{\sim}{C})$, projection $\pi : Y \to X$, and let Z be the unit sphere bundle in ξ. Then the map $Z \to Y$ is a principal $\underset{\sim}{U}(1)$-bundle, with Chern class $C \in \underline{\underline{U}}^2(Y^0)$, say.

<u>8.11 Theorem</u> By means of $\pi^* : \underline{\underline{U}}^*(X^0) \to \underline{\underline{U}}^*(Y^0)$, $\underline{\underline{U}}^*(Y^0)$ is a free $\{\underline{\underline{U}}^*(X^0)\}$-module with base $\{1, C, C^2, \ldots, C^{n-1}\}$. Multiplicatively, there is one relation

<u>8.12</u> $$C^n - C^{n-1}.\pi^* C_1(\xi) + C^{n-2}.\pi^* C_2(\xi) - \ldots + (-)^n \pi^* C_n(\xi) = 0.$$

<u>Proof</u> We first consider the universal example $\pi : E \to B$. When we have unravelled the various definitions, we find we are to investigate the Borel bundle [B2]

<u>8.13</u> $$\underset{\sim}{U}(n)//\{\underset{\sim}{U}(n-1) \times \underset{\sim}{U}(1)\} \to B\underset{\sim}{U}(n-1) \times B\underset{\sim}{U}(1) = E \underset{\pi}{\to} B\underset{\sim}{U}(n) = B$$

induced by $\underset{\sim}{U}(n-1) \times \underset{\sim}{U}(1) \subset \underset{\sim}{U}(n)$. The class C is induced by projection from $C_1 \in \underline{\underline{U}}^2(B\underset{\sim}{U}(1)^0)$. Write $\underline{\underline{U}}^*(B\underset{\sim}{U}(n-1)^0) = \underline{\underline{U}}[C_1', C_2', \ldots, C_{n-1}']^\wedge$, by 8.7. Then we know (from the usual spectral sequence) that $\underline{\underline{U}}^*(E^0) = \underline{\underline{U}}[C, C_1', C_2', \ldots, C_{n-1}']^\wedge$. From 8.7, the homomorphism π^* is given by

$$\pi^* C_1 = C_1' + C, \quad \pi^* C_i = C_i' + CC_{i-1}' \ (1 < i < n), \quad \pi^* C_n = CC_{n-1}'.$$

Eliminating the C_i' yields the relation 8.12. We also note that for the fibre $F = P_{n-1}(\underset{\sim}{C}) \cong \underset{\sim}{U}(n)//\{\underset{\sim}{U}(n-1) \times \underset{\sim}{U}(1)\}$, $\underline{\underline{U}}^*(F^0)$ is $\underline{\underline{U}}$-free with base $\{1, C, C^2, \ldots, C^{n-1}\}$, and therefore a free abelian group.

Let us now return to $\pi: Y \to X$. Certainly 8.12 holds, by naturality. We must show there are no new relations.

Consider the Leray-Serre spectral sequences of $\pi: Y \to X$ and $1: X = X$, with $\underset{\sim}{MU}$ as coefficient spectrum. Let us write them as $(E_r(Y))$ and $(E_r(X))$ respectively, and $\pi^*: E_r(X) \to E_r(Y)$ for the map induced by π. Now $\pi_1(X)$ acts trivially on $\underline{\underline{U}}^*(F^0)$, and

$$E_2(X) = H^*(X; \underset{\sim}{Z}) \otimes \underline{\underline{U}}^*; \qquad E_2(Y) = H^*(X; \underset{\sim}{Z}) \otimes \underline{\underline{U}}^*(F^0),$$

since these facts are true of 8.13. Now these are graded rings, and by means of π^*, $E_2(Y)$ is a free $E_2(X)$-module, with base $\{1, C, C^2, \ldots, C^{n-1}\}$. Moreover, the differentials are all derivations, and vanish on C since they do for 8.13. It follows, by induction on r, that $E_r(Y)$ is a free $E_r(X)$-module with base $\{1, C, C^2, \ldots, C^{n-1}\}$. The spectral sequences converge without difficulty, to show that $\underline{\underline{U}}^*(Y^0)$ is a free $\underline{\underline{U}}^*(X^0)$-module with base $\{1, C, C^2, \ldots, C^{n-1}\}$.]]]

The dimensional restriction on X can be removed.

§9. Some geometric homomorphisms

We consider here two geometrically defined homomorphisms in cobordism theory discussed by Conner and Floyd [C5], and called by them the Smith homomorphism and J. The second is of crucial importance in the study of fixed points sets of involutions on manifolds, as we shall see in VI. We show here that both homomorphisms are special cases of homomorphisms already considered.

We know [C5] that $\underline{\underline{N}}_*(BG^0)$ classifies equivariant cobordism classes of manifolds with free smooth G-action, where G is a Lie group (by considering the orbit spaces). Take a manifold

$\tilde{M}$ with free involution, representing $x \in \underline{\underline{N}}_i(B\underset{\sim}{O}(1)^0)$. Its orbit space is a singular manifold $f: M \to B\underset{\sim}{O}(1)$. We take $B\underset{\sim}{O}(1) = P_\infty(\underset{\sim}{R})$. Since M is compact, we may assume $fM \subset P_q(\underset{\sim}{R}) \subset P_\infty(\underset{\sim}{R})$, for some large q, and that f is smooth and transverse to P_{q-1} in P_q. Put $N = f^{-1}(P_{q-1})$, a submanifold of M, and $g = f|N$. Let $\tilde{N} \subset \tilde{M}$ be the double covering of N.

<u>9.1 Definition</u> The <u>Smith homomorphism</u>

$$\Delta: \underline{\underline{N}}_i(B\underset{\sim}{O}(1)^0) \to \underline{\underline{N}}_{i-1}(B\underset{\sim}{O}(1)^0)$$

is defined by taking the class of $f: M \to B\underset{\sim}{O}(1)$ to the class of $g: N \to B\underset{\sim}{O}(1)$.

The importance of N is that the involution on $\tilde{M}$ is trivial on $\tilde{M} - \tilde{N}$.

By 5.9, we are here simply taking the cap product with the class of the Thom map of the normal bundle of P_{q-1} in P_q. Write ξ for the canonical line bundle over P_r, for any r. Then clearly this normal bundle is ξ, and its classifying map $P_{q-1} \to B\underset{\sim}{O}(1)$ is simply inclusion $P_{q-1} \subset P_\infty$.

Let us be more general. The normal bundle η of P_{m-1} in P_{m+n-1} is the Whitney sum of n copies of ξ. Clearly η extends over P_{m+n-1} in the obvious way. We therefore obtain <u>two</u> maps from P_{m+n-1} to P^η_{m+n-1}; the first is inclusion of the base of the Thom complex, and the second is the composite

$$P_{m+n-1} \to P^\eta_{m-1} \subset P^\eta_{m+n-1}.$$

9.2 Lemma These two maps are homotopic. Moreover, they are *equivariantly* homotopic with respect to the obvious action of $\underset{\sim}{O}(m) \times \underset{\sim}{O}(n)$ on the pair (P_{m+n-1}, P_{m-1}), if we also make $\underset{\sim}{O}(n)$ transform the n copies of ξ in η.

Proof This becomes clear if we note that $P^{\eta}_{m+n-1} = P_{m+n+n-1}/P_{n-1}$, and work with the two obvious inclusion maps $\underset{\sim}{R}^m \times \underset{\sim}{R}^n \subset \underset{\sim}{R}^m \times \underset{\sim}{R}^n \times \underset{\sim}{R}^n$ omitting either factor $\underset{\sim}{R}^n$; these maps are plainly equivariantly homotopic with respect to the actions of $\underset{\sim}{O}(m) \times \underset{\sim}{O}(n)$.]]]

In our case, the required map $P_q \to M\underset{\sim}{O}(1)$ is homotopic to the composite $P_q \subset P_\infty = B\underset{\sim}{O}(1) \subset M\underset{\sim}{O}(1)$. The inclusion $B\underset{\sim}{O}(1) \subset M\underset{\sim}{O}(1)$ gives W_1, by definition 8.1.

9.3 Lemma The Smith homomorphism Δ is given by $\Delta x = x \cap W_1$.]]]

The bordism J-homomorphism

Take a smooth vector bundle ξ over a manifold X^i with fibre dimension n; such are classified up to bordism by $\underset{=}{N}_i(B\underset{\sim}{O}(n)^0)$. Its unit sphere bundle Y^{i+n-1} when equipped with the antipodal involution represents an element $x \in \underset{=}{N}_{i+n-1}(B\underset{\sim}{O}(1)^0)$.

9.4 Definition The bordism J-homomorphism

$$J_n : \underset{=}{N}_i(B\underset{\sim}{O}(n)^0) \to \underset{=}{N}_{i+n-1}(B\underset{\sim}{O}(1)^0)$$

is defined by taking the class of the bundle ξ to x.

Now consider X as a singular manifold of $B\underset{\sim}{O}(n)$. We see that Y is the covering singular manifold obtained by the construction 6.18 of the bundle transfer for the universal case, over $B\underset{\sim}{O}(n)$. Let E be a universal $\underset{\sim}{O}(n)$-space, and put $B\underset{\sim}{O}(n) = E//\underset{\sim}{O}(n)$. Then the universal case is the Borel fibre bundle [B2]

$$P_{n-1}(\underset{\sim}{R}) = \underset{\sim}{O}(n)//\{\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1)\} \to E//\{\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1)\} \to E//\underset{\sim}{O}(n)$$

induced by the inclusion $\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1) \subset \underset{\sim}{O}(n)$. Now $E//\{\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1)\} \simeq B\underset{\sim}{O}(n-1) \times B\underset{\sim}{O}(1)$, and the antipodal involution on the sphere bundle $E//\underset{\sim}{O}(n-1)$ is classified by the projection $B\underset{\sim}{O}(n-1) \times B\underset{\sim}{O}(1) \to B\underset{\sim}{O}(1)$.

Let us write this Borel bundle as

<u>9.5</u> $$P_{n-1}(\underset{\sim}{R}) = \underset{\sim}{O}(n)//\{\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1)\} \to B\underset{\sim}{O}(n-1) \times B\underset{\sim}{O}(1) \xrightarrow[\pi]{} B\underset{\sim}{O}(n).$$

Then we have proved

<u>9.6 Theorem</u> $J_n : \underline{\underline{N}}_*(B\underset{\sim}{O}(n)^0) \to \underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$ is the composite of the transfer homomorphism $\pi^\natural$ of 9.5 with the homomorphism $\underline{\underline{N}}_*(\{B\underset{\sim}{O}(n-1) \times B\underset{\sim}{O}(1)\}^0) \to \underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$ induced by projection.]]]

For the usual reasons, we would prefer to have multiplicative structure available, by means of a similar homomorphism in $\underline{\underline{N}}^*$. For transfer homomorphisms, this can be defined, if we use the Grothendieck form 6.21 of the bundle transfer.

<u>9.7 Definition</u> The cobordism J-homomorphism

$$J^n : \underline{\underline{N}}^i(B\underset{\sim}{O}(1)^0) \to \underline{\underline{N}}^{i-n+1}(B\underset{\sim}{O}(n)^0)$$

is defined as the composite of the transfer homomorphism $\pi_\natural$ with the homomorphism induced by the projection $B\underset{\sim}{O}(n-1) \times B\underset{\sim}{O}(1) \to B\underset{\sim}{O}(1)$.

Then we have the multiplicative properties 6.2.

We shall need to compare the homomorphisms J_n for different values of n.

9.8 Lemma Write $i: B\underset{\sim}{O}(n) \subset B\underset{\sim}{O}(n+1)$ for the map induced by inclusion. Then $J_n x = \Delta J_{n+1} i_* x$ for $x \in \underline{\underline{N}}_*(B\underset{\sim}{O}(n)^0)$.

Proof This can be seen directly from the geometric definitions. It appears as Theorem 26.4 of [C5].]]]

9.9 Corollary $J^n \alpha = i^* J^{n+1}(\alpha \cup W_1)$ for $\alpha \in \underline{\underline{N}}^*(B\underset{\sim}{O}(1)^0)$.

Proof 9.8, with 9.3, 9.7, and 6.2, shows that $\langle x, J^n\alpha\rangle = \langle x, i^* J^{n+1}(\alpha \cup W_1)\rangle$ for all $x \in \underline{\underline{N}}_*(B\underset{\sim}{O}(n)^0)$. The result follows, since we know the structure of $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$, and (by e.g. [C5]) $\underline{\underline{N}}_*(B\underset{\sim}{O}(n)^0)$.

Still using cap products, we can obtain a very precise relation between J_m and J_{m+n}. Let π_1 be the bundle induced from 9.5 for m + n as follows:

$$\begin{array}{ccc} E_1 & \longrightarrow & B\underset{\sim}{O}(m+n-1) \times B\underset{\sim}{O}(1) \\ \downarrow \pi_1 & & \downarrow \pi \\ B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n) & \longrightarrow & B\underset{\sim}{O}(m+n). \end{array}$$

Write q for the composite $E_1 \to B\underset{\sim}{O}(m+n-1) \times B\underset{\sim}{O}(1) \to B\underset{\sim}{O}(1)$, and $p: B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(m)$ for the projection.

9.10 Lemma $J_m p_* x = q_* (\pi_1^! x \cap \alpha)$ for $x \in \underline{\underline{N}}_*(\{B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)\}^0)$, where the element $\alpha \in \underline{\underline{N}}^*(E_1^0)$ is induced from the Stiefel-Whitney cobordism class $W_n \in \underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$ by means of the maps

$$E_1 \to B\underset{\sim}{O}(n) \times B\underset{\sim}{O}(1) \to B\underset{\sim}{O}(n),$$

where the first is obtained from π_1 and q, and the second is induced by $\otimes : \underset{\sim}{O}(n) \times \underset{\sim}{O}(1) \to \underset{\sim}{O}(n)$.

Proof Geometrically, E_1 is a bundle over $B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)$ with fibre $P_{m+n-1}(\underset{\sim}{R})$, containing a subbundle E_2 with fibre $P_{m-1}(\underset{\sim}{R})$ and projection π_2, say. Given a singular manifold $X \to B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)$ representing x, the construction 6.18 of the bundle transfers gives singular manifolds of E_1 and E_2 which yield in $\underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$ representatives for $q_*\pi_1^{\natural}x$ and $J_m p_* x$ respectively.

By 5.9 applied over X, we obtain the required formula, where $\alpha : E_1 \to M\underset{\sim}{O}(n) \to M\underset{\sim}{O}$ is the Thom map of the normal bundle of E_2 in E_1. Write ξ and η for the universal line and vector bundles over $B\underset{\sim}{O}(1)$ and $B\underset{\sim}{O}(n)$. Then the normal bundle of E_2 in E_1 is $(\pi_1^*\eta \otimes q^*\xi)|E_2$. This time making strong use of 9.2, we see that the Thom map we require is homotopic to the classifying map $E_1 \to B\underset{\sim}{O}(n)$ of $\pi_1^*\eta \otimes q^*\xi$, followed by $B\underset{\sim}{O}(n) \subset M\underset{\sim}{O}(n)$. The latter gives W_n, by 8.6.]]]

9.11 Corollary $\pi_{1\natural}(q^*\beta \cup \alpha) = p^*J^m\beta$ for $\beta \in \underline{\underline{N}}^*(B\underset{\sim}{O}(1)^0)$.

Proof This dual result is obtained in the same way as 9.9.]]]

These results will enable us to carry out computations in VI.

STABLE HOMOTOPY THEORY

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CHAPTER VI - UNORIENTED BORDISM AND COBORDISM

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CHAPTER VI - UNORIENTED BORDISM AND COBORDISM

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Our methods so far have been applicable to general bordism and cobordism theories. In this Chapter we specialize to the case of the unoriented theories, with the Thom spectrum $\underset{\sim}{MO}$ as coefficient spectrum.

In principle these theories have been reduced to the study of ordinary homology and cohomology [C5]. Everything depends on the main structure theorem, that $\underset{\sim}{MO}$ is a graded Eilenberg-MacLane spectrum. (We have by now given this assertion a precise meaning.) Therefore to obtain better results, we must be more precise about this structure theorem, and we must develop more efficient methods of carrying out the necessary computations. Various devices for doing this are introduced in §1, §2, §3, and §4.

In §5 we deduce the structure of $\underset{\sim}{MO}$, and hence that of the theories $\underline{\underline{N}}^*(\ )$ and $\underline{\underline{N}}_*(\ )$. This is expressed in §6 in a different form. In §7 we obtain some information on the primary operations in cobordism theory.

In §8 we consider the relations between determinants and tensor products on one hand, and the Stiefel-Whitney cobordism characteristic classes W_i defined in V.8 on the other. These are quite complicated, and yield interesting results. For example we discover a canonically defined set of polynomial generators for the unoriented cobordism ring $\underline{\underline{N}}$. We also determine the behaviour of two maps $d_1, d_2 : \underline{\underline{N}} \to \underline{\underline{N}}$ defined by Rokhlin [R1] on products, a problem posed by Wall in [W2].

In §10 we study manifolds with smooth involutions, as in [C4],[C5], etc. This contains our main theorem, that a smooth involution on a non-bounding n-manifold must have a fixed-point set of dimension at least 2n/5. To prove this we need to know about the bordism J-homomorphism [C5]. We compute it in §9.

Throughout this Chapter we shall work over the groundfield $\underset{\sim}{Z}_2$. This will be the coefficient group for ordinary homology and cohomology unless otherwise stated. As always, we take all homology and cohomology theories <u>reduced</u>. We recall that X^0 is the disjoint union of X and a base point $_o$.

This Chapter comprises the sections:

1. Universal elements
2. The giant Steifel-Whitney class
3. The giant Steenrod square
4. Some Hopf algebras
5. The structure of the Thom spectrum $M\underset{\sim}{O}$
6. From cobordism to cohomology

§1. Universal elements

In this section we start to develop the intensive algebraic machinery we require.

Let V be a finite-dimensional vector space over the field K, and V' the dual vector space. Then it is a commonplace that we have the evaluation map

$$V'\otimes V \to K.$$

From the point of view of categories this is unsymmetrical: why not work with the dual map

1.1 $$K \to V\otimes V' \qquad ?$$

1.2 Definition We call the image of $1 \in K$ under the linear map 1.1 the universal element $u \in V\otimes V'$ of V or V'.

1.3 Lemma Let $\alpha\colon V \to K$ be any linear functional on V. Then the composite $V\otimes V' \to K\otimes V' \cong V'$ takes u to $\alpha \in V'$.

Proof. This is immediate when we dualize back.]]]

This is the universal property of the element u.

If $\{e_i\}$ is a K-base for V, we have the dual base $\{e_i'\}$ of V' defined by $e_i'e_j = \delta_{ij}$, and then

1.4 $$u = e_1 \otimes e'_1 + e_2 \otimes e'_2 + \dots + e_n \otimes e'_n.$$

It is often convenient to embed V and V' in larger vector spaces A and B say; but from the knowledge of $u \in A \otimes B$ we can recover the subspaces V and V' from 1.3 and its dual. Alternatively the choice of any element $u \in A \otimes B$ defines a duality between some subspace V of A and some subspace V' of B. The notion of universal element will be useful because it is easier to specify than a homomorphism, in customary notation.

All this applies equally well to graded vector spaces of finite type (each component vector space finite-dimensional), except that we must replace the ordinary graded tensor product $A \otimes B$ by the complete tensor product $A \hat{\otimes} B$, as below.

1.5 Definition Given graded modules A and B, their tensor product and complete tensor product are again graded modules, defined respectively by $(A \otimes B)_k = \Sigma_{i+j=k} A_i \otimes B_j$ and $(A \hat{\otimes} B)_k = \Pi_{i+j=k} A_i \otimes B_j$.

Let us topologize $(A \otimes B)_k$ by taking the submodules $\Sigma_{i<m} A_i \otimes B_{k-i}$ and $\Sigma_{i>n} A_i \otimes B_{k-i}$ as subbasic neighbourhoods of 0. Then the completion of $(A \otimes B)_k$ in this topology is the module $(A \hat{\otimes} B)_k$, with the obvious topology.

Let us give an important example of a universal element. Let $A = K[t_1, t_2, \dots, t_n]$ be a graded polynomial ring, in which each t_i has degree -1, and $B = K[a_1, a_2, a_3, \dots]$

a graded polynomial ring in which a_i has degree i.

__1.6 Theorem__ The element $u \in A\hat{\otimes}B$ given by

$$u = \Pi_{i=1}^{i=n} (1 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3 + a_4 t_i^4 + \ldots)$$

is the universal element of the symmetric subalgebra of A.

__Proof__. Let $C = K[u_1, u_2, \ldots, u_m]$, where each u_i has degree 1, and define the algebra homomorphism $B \to C$ by sending a_i to the i th elementary symmetric function of the u_j if $i \leqslant m$, or to 0 if $i > m$. This takes u to the element $v \in A\hat{\otimes}C$ given by

$$\begin{aligned} v &= \Pi_{i=1}^{i=n} \Pi_{j=1}^{j=m} (1 + u_j t_i) \\ &= \Pi_{j=1}^{j=m} (1 + s_1 u_j + s_2 u_j^2 + \ldots + s_n u_j^n), \end{aligned}$$

where s_i is the i th elementary symmetric sum of the t_k. Hence v is the universal element of the submodule of $K[s_1, s_2, \ldots, s_n]$ consisting of elements of polynomial degree $\leqslant m$ in the s_i. Now $B \to C$ is mono in degrees $\leqslant m$. It follows that u is the universal element of $K[s_1, s_2, \ldots, s_n] \subset A$, which is well known to be the symmetric subalgebra of A.]]]

One can juggle with universal elements in various ways. For example, suppose the module A has a multiplication $\varphi\colon A\otimes A \to A$. Then the dual A' has a comultiplication $\varphi'\colon A' \to A'\otimes A'$. The image of $x \in A'$ under φ' is the image of $u\otimes u$ under the composite

$$A\otimes A'\otimes A\otimes A' \cong A\otimes A\otimes A'\otimes A' \xrightarrow[\varphi\otimes 1\otimes 1]{} A\otimes A'\otimes A' \xrightarrow[\chi\otimes 1\otimes 1]{} K\otimes A'\otimes A' \cong A'\otimes A'.$$

A homomorphism $A \to B$ corresponds to an element of $A'\otimes B$. In this way we shall be able to make use of universal elements.

§2. The giant Stiefel-Whitney class

Denote by $B\underset{\sim}{O}(n)$, as usual, the classifying space for the orthogonal group $\underset{\sim}{O}(n)$, etc., and by $\underset{\sim}{Q}(n)$ the diagonal subgroup of $\underset{\sim}{O}(n)$. The inclusion $\underset{\sim}{Q}(n) \subset \underset{\sim}{O}(n)$ induces $\rho\colon B\underset{\sim}{Q}(n) \to B\underset{\sim}{O}(n)$. We have by Borel [B2], as in V.8,

$$H^*(B\underset{\sim}{Q}(n)^O) = \underset{\sim}{Z}_2[t_1, t_2, \ldots, t_n],$$

$$H^*(B\underset{\sim}{O}(n)^O) = \underset{\sim}{Z}_2[w_1, w_2, \ldots, w_n],$$

graded polynomial rings, where w_i is the i th universal Stiefel-Whitney class. Also, $\rho^* w_i$ is the i th elementary symmetric function of the t_i.

It is often inconvenient to have to give a list of elements when specifying the Stiefel-Whitney classes of a vector bundle. If so, it is customary to introduce formally the <u>total</u> Stiefel-Whitney class $w = 1 + w_1 + w_2 + \ldots$. This is multiplicative: $w(\xi \oplus \eta) = w(\xi).w(\eta)$.

In cobordism computations it is useful to have readily available all the Stiefel-Whitney numbers of a manifold, and therefore all the monomials in the Stiefel-Whitney classes w_i. To display these, let $\mathbb{Z}_2[a_1, a_2, \ldots]$ be the graded polynomial ring in which a_i has degree i, which we write formally as $\mathbb{Z}_2[a]$. Then by 1.6 the element

2.1 $\Pi_{i=1}^{i=n} (1 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3 + \ldots) \in H^*(BQ(n)^O) \hat{\otimes} \mathbb{Z}_2[a]$

yields the universal element of $H^*(BQ(n)^O)$, with the help of ρ^*. (We frequently write simply a_i instead of $1 \otimes a_i$, etc.) As n increases to ∞, we obtain a well-defined element

2.2 $$\underline{w} \in H^*(BQ^O) \hat{\otimes} \mathbb{Z}_2[a].$$

Here the dual space to $H^*(BQ^O)$ fills up the whole of $\mathbb{Z}_2[a]$ and in effect we have identified $\mathbb{Z}_2[a]$ with $H_*(BQ^O)$.

2.3 Definition We call $\underline{w} \in H^*(BQ^O) \hat{\otimes} \mathbb{Z}_2[a]$ as in 2.2 the universal giant Stiefel-Whitney class. If ξ is a vector bundle over the space X, its giant Stiefel-Whitney class $\underline{w}(\xi) \in H^*(X^O) \hat{\otimes} \mathbb{Z}_2[a]$ is induced from $\underline{w}$ by the classifying map $X \to BQ$.

Consider next the product map $\varphi: BQ(m) \times BQ(n) \to BQ(m+n)$ induced by the usual inclusion $Q(m) \times Q(n) \subset Q(m+n)$. In cohomology put $H^*(BQ(m+n)^O) = \mathbb{Z}_2[t_1, t_2, \ldots, t_{m+n}]$, $H^*(BQ(m)^O) = \mathbb{Z}_2[t'_1, \ldots, t'_m]$, and $H^*(BQ(n)^O) = \mathbb{Z}_2[t''_1, \ldots, t''_n]$. Then φ induces $t_i \to t'_i$ (for $i \leq m$), $t_i \to t''_{i-m}$ (for $i > m$), and hence, in obvious notation, $\underline{w} \to \underline{w}'.\underline{w}''$, from the form of 2.1 We deduce:

2.4 Theorem Let ξ and η be vector bundles over X. Then their giant Stiefel-Whitney classes multiply: $\underline{w}(\xi\oplus\eta) = \underline{w}(\xi)\underline{w}(\eta)$.]]]

We recover the total Stiefel-Whitney class $w(\xi)$ from $\underline{w}(\xi)$ by means of the ring homomorphism $\mathbb{Z}_2[a] \to \mathbb{Z}_2$ sending a_1 to 1 and a_i to 0 for $i \geq 2$. Conversely, the giant class $\underline{w}(\xi)$ is determined by the total class $w(\xi)$. Let $\psi: \mathbb{Z}_2[a] \to \mathbb{Z}_2[a'] \otimes \mathbb{Z}_2[a'']$ be the ring homomorphism defined by

2.5 $$a_i \to a_i' + a_{i-1}'a_1'' + a_{i-2}'a_2'' + \cdots + a_1'a_{i-1}'' + a_i'' ,$$

where $\mathbb{Z}_2[a']$ and $\mathbb{Z}_2[a'']$ are copies of $\mathbb{Z}_2[a]$. We see from 2.1 that this comultiplication in $\mathbb{Z}_2[a]$ is dual to the cup product multiplication in $H^*(BO^0)$, under the duality determined by $\underline{w} \in H^*(BO^0) \hat{\otimes} \mathbb{Z}_2[a]$. Given a vector bundle ξ, let us write $\underline{w}'(\xi)$ and $\underline{w}''(\xi)$ for the copies of $\underline{w}(\xi)$ in $H^*(BO^0) \hat{\otimes} \mathbb{Z}_2[a']$ and $H^*(BO^0) \hat{\otimes} \mathbb{Z}_2[a'']$.

2.6 Lemma Given a vector bundle ξ over X, its giant Stiefel-Whitney class $\underline{w}(\xi)$ is characterized in terms of its total Stiefel-Whitney class $w(\xi)$ by the properties:

(a) $\underline{w}(\xi)$ reduces to $w(\xi)$ when we put
$$a_1 = 1,\ a_i = 0\ (i \geq 2),$$

(b) $(1\otimes\psi)\underline{w}(\xi) = \underline{w}'(\xi)\ \underline{w}''(\xi)$ in $H^*(X^0) \hat{\otimes} \mathbb{Z}_2[a']\otimes\mathbb{Z}_2[a'']$.

Proof. Dualizing $\underline{w}(\xi)$ yields a linear map $H^*(BO^0) \to H^*(X^0)$. Then (b) asserts that this is a ring homomorphism, and (a) asserts that it takes the value

$w_i(\xi)$ on w_i and takes 1 to 1. But $H^*(B\underset{\sim}{O}^0)=\underset{\sim}{Z}_2[w_1,w_2,\ldots]$.]]]

2.7 Corollary Suppose that the total class of the bundle ξ over X has the form

$$w(\xi) = \Pi_{j=1}^{j=m} (1 + x_j)^{e(j)},$$

where $x_j \in H^{r(j)}(X^0)$, $r(j)$ is a power of 2, and $e(j)$ is any integer, positive or negative. Then the giant class of ξ is given by

$$\underset{\sim}{w}(\xi) = \Pi_{j=1}^{j=m} \{1 + a_1^{r(j)}x_j + a_2^{r(j)}x_j^2 + a_3^{r(j)}x_j^3 + \ldots\}^{e(j)}.$$

Proof. It is clear that this element satisfies (a) and (b) in 2.6, since we are working modulo 2.]]]

This result will suffice for our applications.

2.8 Definition For a smooth manifold V, we define $\underset{\sim}{w}(V) = \underset{\sim}{w}(\tau_V)$, where τ_V is its tangent bundle.

Then by 2.4, the giant Stiefel-Whitney class of its stable normal bundle is $\underset{\sim}{w}(V)^{-1}$.

Let us give some examples.

Projective spaces

Let $P_n(K)$ denote n-dimensional projective space over the skew field K, where $K = \underset{\sim}{R}$ (reals), $\underset{\sim}{C}$ (complex numbers), or $\underset{\sim}{H}$ (quaternions). Write $d = \dim_{\underset{\sim}{R}} K = 1,2,$ or 4 respectively.

It is well known (compare e.g. [H1]) that

$$w(P_n(K)) = (1 + \alpha)^{n+1}$$

where $H^*(P_n(K)^0) = \underset{\sim}{Z}_2[\alpha : \alpha^{n+1} = 0]$, in which α has codegree d.

Then 2.7 applies, and we find

2.9
$$\begin{cases} \underline{w}(P_n(\mathbb{R})) = (1 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots)^{n+1}, \\ \underline{w}(P_n(\mathbb{C})) = (1 + a_1^2\alpha + a_2^2\alpha^2 + a_3^2\alpha^3 + a_4^2\alpha^4 + \dots)^{n+1}, \\ \underline{w}(P_n(\mathbb{H})) = (1 + a_1^4\alpha + a_2^4\alpha^2 + a_3^4\alpha^3 + a_4^4\alpha^4 + \dots)^{n+1}. \end{cases}$$

Hypersurfaces

We shall need, for applications, the 'hypersurfaces of degree (1,1)' introduced into cobordism theory by Milnor. There are two reasons for introducing them: firstly they provide some useful generators for the varous cobordism rings, and secondly their characteristic classes are easily computed.

2.10 Definition. The hypersurface $H_{m,n}(K)$ in $P_m(K) \times P_n(K)$ is the subset defined by the equation

$$x_0y_0 + x_1y_1 + x_2y_2 + \dots + x_py_p = 0,$$

where $p = \min(m,n)$, and $(x_0, x_1, \dots, x_m)$ and $(y_0, y_1, \dots, y_n)$ are the standard homogeneous coordinates in $P_m(K)$ and $P_n(K)$ respectively.

It is a smooth submanifold of codimension d. It is easy to see that if $m \leq n$, the projection $P_m \times P_n \to P_m$ induces the fibre bundle

2.11
$$P_{n-1} \to H_{m,n} \to P_m.$$

(The other projection is not a bundle projection if $m < n$.)

Let us compute the giant class of $H_{m,n}(\mathbb{R})$. (The other cases are analogous.) We take $H^*(P_m^0) = \mathbb{Z}_2[\alpha:\alpha^{m+1} = 0]$ and $H^*(P_n^0) = \mathbb{Z}_2[\beta:\beta^{n+1} = 0]$. Then by the Künneth formula,

$$H^*((P_m \times P_n)^0) = \mathbb{Z}_2[\alpha,\beta:\ \alpha^{m+1} = 0,\ \beta^{n+1} = 0].$$

Let $j:H_{m,n} \subset P_m \times P_n$ be the embedding, and suppose $m \leqslant n$. Write α and β also for $\alpha\circ j$ and $\beta\circ j$. One can deduce from the spectral sequence of the fibration 2.11 that α and β generate $H^*(H^0_{m,n})$, though we do not need this fact. We have the Gysin transfer homomorphism V.6.12

$$j_\natural:\ H^*(H^0_{m,n}) \to H^*((P_m \times P_n)^0).$$

By V.6.2, with our identifications, this is a homomorphism of $H^*((P_m \times P_n)^0)$-modules, and must be in fact multiplication by $\alpha+\beta$, since this is the cohomology class in $P_m \times P_n$ represented by $H_{m,n}$. Moreover, $\alpha+\beta$ is the first characteristic class of the normal bundle of $H_{m,n}$ in $P_m \times P_n$ (compare [H1]). Then for the tangent bundle of $H_{m,n}$ (compare 9.2 in [H1]) we have

$$w(H_{m,n}) = (1 + \alpha)^{m+1}\ (1 + \beta)^{n+1}\ (1 + \alpha+ \beta)^{-1}.$$

We can apply 2.7 and write down the giant class

<u>2.12</u>

$$\underline{w}(H_{m,n}(\mathbb{R}))=\{1+a_1\alpha + a_2\alpha^2+a_3\alpha^3+ ..\}^{m+1}\{1+a_1\beta+a_2\beta^2+a_3\beta^3+..\}^{n+1}$$

$$.\{1 + a_1(\alpha+\beta) + a_2(\alpha+\beta)^2 + a_3(\alpha+\beta)^3 + ..\}^{-1}.$$

3. The giant Steenrod square

We first remark that the mod 2 Steenrod algebra A can be regarded exactly as the ring $\{K(\mathbb{Z}_2),K(\mathbb{Z}_2)\}^*$, with the multiplication induced by composition. (Indeed, this is how we would define A.) For if $n > k+1$, we have

$$\{K(\mathbb{Z}_2),K(\mathbb{Z}_2)\}^k \cong \{K(\mathbb{Z}_2,n),K(\mathbb{Z}_2)\}^{n+k} = H^{n+k}(K(\mathbb{Z}_2,n)),$$

by a trivial application of Milnor's lemma (H.4 in Summary).

One frequently economizes on notation by putting

$$Sq = 1 + Sq^1 + Sq^2 + Sq^3 + \dots ,$$

the total Steenrod square. In the same way that in §2 we introduced the giant Steifel-Whitney class $\underline{w}$, we find it useful to introduce the giant Steenrod square, which displays the action of the whole Steenrod algebra A. Iterated Steenrod squares and Adem relations are difficult to handle; therefore we shall not use them. Our approach avoids this difficulty.

We lean heavily on work of Milnor [M2]. Now A is a Hopf algebra, whose dual algebra A' is a polynomial ring $\mathbb{Z}_2[\lambda] \equiv \mathbb{Z}_2[\lambda_1,\lambda_2,\lambda_3, \dots]$, in which λ_i has degree 2^i-1.

3.1 Definition Given any spectrum X, the giant Steenrod square

$$\underline{Sq}\colon H^*(X) \to A' \hat{\otimes} H^*(X)$$

is the adjoint of the action $A \otimes H^*(X) \to H^*(X)$ of A on $H^*(X)$.

For each monomial $\lambda^\rho \equiv \lambda_1^{\rho_1}\lambda_2^{\rho_2}\dots$, where $\rho = (\rho_1,\rho_2, \dots)$ is a sequence of non-negative integers, all but finitely many zero, we have the dual base element $Sq^\rho \in A$. These form Milnor's base [M2] of A. We have

$\underline{3.2}$ $$\underline{Sq}\ \alpha = \Sigma_\rho \quad \lambda^\rho \quad Sq^\rho \alpha.$$

In particular $Sq^1 = Sq^{1,0,0,\dots}$.

$\underline{3.3\ \text{Theorem}}$ (a) The giant Steenrod square

$$\underline{Sq} : H^*(X) \to \mathbb{Z}_2[\lambda] \hat{\otimes} H^*(X)$$

is a natural transformation, defined for all spectra X.

(b) When X is a space, $\underline{Sq}$ is a ring homomorphism.

(c) If X is a space, and $\alpha \in H^1(X)$, we have

$$\underline{Sq}\ \alpha = \alpha + \lambda_1\alpha^2 + \lambda_2\alpha^4 + \lambda_3\alpha^8 + \lambda_4\alpha^{16} + \dots$$

Proof. (a) is trivial. (b) is immediate, because the comultiplication in A, and hence the multiplication in A', was defined to make $\underline{Sq}$ a ring homomorphism. As for (c), we know from [M2] that the only operations which do not vanish on α are 1, Sq^1, Sq^2Sq^1, $Sq^4Sq^2Sq^1$, etc., whose values are α, α^2, α^4, α^8, etc. These operations are dual to 1, λ_1, λ_2, λ_3, etc.]]]

$\underline{3.4\ \text{Corollary}}$ A operates effectively on the A-module $\Sigma_n\ H^*(B\mathbb{Q}(n)^0)$.

Proof. We have $H^*(B\mathbb{Q}(n)^0) = \mathbb{Z}_2[t_1, t_2, \dots, t_n]$. Since each t_i has codegree 1, we can use 3.3 (b) and (c) to evaluate $\underline{Sq}$; in particular

$$\underline{Sq}\ t_1 t_2 \dots t_n = \Pi_{i=1}^{i=n} (t_i + \lambda_1 t_i^2 + \lambda_2 t_i^4 + \lambda_3 t_i^8 + \dots).$$

The result follows, by letting n vary]]]

We see from 3.3 that $\underline{Sq}$ shares with Sq the property of being a ring homomorphism; which accounts for the comultiplication in A. Our approach also elucidates the multiplicative structure. Let $\underline{Sq}'$: $H^*(X) \to \underset{\sim}{Z}_2[\lambda'] \hat{\otimes} H^*(X)$, etc., be other copies of $\underline{Sq}$. We consider the composite ring homomorphism

$$\underline{Sq}'\ \underline{Sq}'' : H^*(X) \to \underset{\sim}{Z}_2[\lambda'] \hat{\otimes} \underset{\sim}{Z}_2[\lambda''] \hat{\otimes} H^*(X),$$

in which $\underline{Sq}'$ acts trivially on $\underset{\sim}{Z}_2[\lambda'']$. If X is a space and $t \in H^1(X)$, evaluation by (3.3) (b) and (c) gives

$$\underline{Sq}'\ \underline{Sq}''\ t = (t+\lambda_1't^2+\lambda_2't^4+\ldots)+\lambda_1''(t+\lambda_1't^2+\ldots)^2+\lambda_2''(t+\lambda_1't^2+\ldots)^4+\ldots$$

This agrees with

$$\underline{Sq}\ t = t + \lambda_1 t^2 + \lambda_2 t^4 + \lambda_3 t^8 + \ldots$$

if we define a ring homomorphism $\underset{\sim}{Z}_2[\lambda] \to \underset{\sim}{Z}_2[\lambda'] \otimes \underset{\sim}{Z}_2[\lambda'']$ by

<u>3.5</u>
$$\begin{cases} \lambda_1 \to \lambda_1' + \lambda_1'', \\ \lambda_2 \to \lambda_2' + \lambda_1'^2\lambda_1'' + \lambda_2'', \\ \lambda_3 \to \lambda_3' + \lambda_2'^2\lambda_1'' + \lambda_1'^4\lambda_2'' + \lambda_3'', \\ \cdots\cdots\cdots\cdots \end{cases}$$

By 3.3 and 3.4, we must have $\underline{Sq} = \underline{Sq}'\ \underline{Sq}''$ generally.

<u>3.6 Theorem</u> We have

$$\underline{Sq} = \underline{Sq}'\ \underline{Sq}'' : H^*(X) \to \underset{\sim}{Z}_2[\lambda'] \hat{\otimes} \underset{\sim}{Z}_2[\lambda''] \hat{\otimes} H^*(X)$$

if we use 3.5 to embed $\underset{\sim}{Z}_2[\lambda]$ in $\underset{\sim}{Z}_2[\lambda'] \otimes \underset{\sim}{Z}_2[\lambda'']$.]]]

Hence, in conjunction with 3.5 and 3.2, we can read off the multiplication table of the operations Sq^ρ, and derive Theorem 4B of [M2]. In particular, one can verify the Adem relations

$$Sq^i\, Sq^j = \Sigma_k \{i-2k,\ j+k-i-1\}\ Sq^{i+j-k}\, Sq^k \quad (0 < i < 2j),$$

where $\{p,q\}$ denotes the binomial coefficient, i.e. the coefficient of $t^p u^q$ in $(t+u)^{p+q}$. The necessary arithmetic is not trivial, and may be found in the Appendix to Steenrod and Epstein's book [S6].

We can also treat the canonical anti-automorphism c of the Hopf algebra A. Define the ring homomorphism

3.7 $$\theta\colon \mathbb{Z}_2[\lambda] \hat{\otimes} H^*(X) \to \mathbb{Z}_2[\lambda] \hat{\otimes} H^*(X)$$

by means of $\underline{Sq}$ on $H^*(X)$, and the inclusion on $\mathbb{Z}_2[\lambda]$.

3.8 Theorem The ring homomorphism θ is a ring isomorphism. For any element $\alpha \in H^*(X)$ we have

3.9 $$\theta^{-1}(1 \hat{\otimes} \alpha) = \Sigma_\rho\, \lambda^\rho\, (c(Sq^\rho))\alpha.$$

Proof. Because $Sq^0 = 1$, it is easy to show that θ is an isomorphism. In view of 3.6, the formula 3.9 is essentially the definition of c.]]]

Again, we can pick out coefficients and recover Theorem 5 of [M2].

A similar treatment can be given for the mod p Steenrod algebra, for any odd prime p.

§4. Some Hopf algebras

We work over the groundfield $\mathbb{Z}_2$ for simplicity. We refer to Milnor and Moore [M8] for the usual graded concepts, such as graded module, coalgebra, and comodule. In particular, a Hopf algebra A has, among other structure, a multiplication $A\otimes A \to A$ and a comultiplication $A \to A\otimes A$. However, it must be noted thatin our case the grading need not be non-negative. Indeed we have two names, degree and codegree, for the grading, each of which is minus the other.

Suppose that A is a Hopf algebra, and that B and C are A-modules, with actions L: $A\otimes B \to B$ and L: $A\otimes C \to C$. It is usual to make the $\mathbb{Z}_2$-module $B\otimes C$ into a A-module by means of the action

$$A\otimes B\otimes C \xrightarrow[\psi\otimes 1\otimes 1]{} A\otimes A\otimes B\otimes C \cong A\otimes B\otimes A\otimes C \xrightarrow[L\otimes L]{} B\otimes C,$$

where ψ: $A \to A\otimes A$ is the comultiplication in A. We say the Hopf algebra A is <u>connected</u> if it is zero in negative codegrees and has $\mathbb{Z}_2$ in codegree 0 (e.g. the Steenrod algebra).

<u>4.1 Lemma</u> Suppose A is a connected associative Hopf algebra, B is a free A-module, and C is any A-module. Then $B\otimes C$ is a free A-module, with A-base $\{b_\alpha\otimes c_i\}$, where $\{b_\alpha\}$ is a A-base for B, and $\{c_i\}$ is a $\mathbb{Z}_2$-base for C.

<u>Proof</u>. Let B_α be the submodule Ab_α of B generated by b_α; clearly $B\otimes C = \Sigma_\alpha\, B_\alpha\otimes C$, as A-modules. Thus we need consider only the case $B = A$. Take $a \in A$ and $c \in C$, where a has positive codegree. Suppose

$$a = a\otimes 1 + 1\otimes a + \Sigma_j\, a'_j\otimes a''_j.$$

Then $\quad a\otimes c = a(1\otimes c) - 1\otimes ac - \Sigma_j\, a'_j\otimes a''_j c.$

It follows by induction on the codegree of a that the elements $1 \otimes c_i$ span the A-module $A \otimes C$. A similar proof shows that there are no relations between the $1 \otimes c_i$.]]]

We remark that a module action $L: A \otimes B \to B$ can be dualized in various ways (e.g. *Sq*). In particular, $L': B' \to A' \otimes B'$ is a coaction of the dual coalgebra A' on B'.

4.2 Definition The *invariant part* of the comodule B' is the set of all elements $x \in B'$ such that $L'x = 1 \otimes x$.

The invariant part is dual to the quotient module $B \otimes_A \underset{\sim}{Z}_2$.

Left A-module coalgebras (see 4.2 of [M8]), where A is the Steenrod algebra, are important in cobordism theory. If M is a connected left A-module coalgebra, there are canonical homomorphisms of left A-module coalgebras $i: A \to M$ and $\pi: M \to M \otimes_A \underset{\sim}{Z}_2$.

We write $C = M \otimes_A \underset{\sim}{Z}_2$, a coalgebra with trivial A-module action. We have the following wierd structure theorem, part of which is well known (see 4.4 of [M8]).

4.3 Theorem Let M be a connected *left* A-module commutative associative coalgebra, *where* A *is the* *Steenrod algebra*. Suppose $i: A \to M$ is mono. Then the structure splits, in the sense that the exact sequence of coalgebras

<u>4.4</u> $$0 \to A \xrightarrow[i]{} M \xrightarrow[\pi]{} C \to 0$$

is isomorphic to the split-exact sequence of coalgebras

$$0 \to A \to A \otimes C \to C \to 0.$$

In particular we have an isomorphism $M \cong A \otimes C$ of left A-module coalgebras, and M is a free A-module.

<u>Proof</u> Let $\{c_j\}$ be a $\mathbb{Z}_2$-base for C, and let $f: C \to M$ be any $\mathbb{Z}_2$-module homomorphism such that $\pi f = 1$. We first show that $\{fc_j\}$ is a free A-base for M.

Let $\Sigma_j\ a_j fc_j = 0$ be any relation between the fc_j. We consider the homomorphism $(\pi\otimes 1)\psi: M \to M\otimes M \to C\otimes M$ of coalgebras. Then $(\pi\otimes 1)\psi(\Sigma_j a_j fc_j) = \Sigma_j\ c_j \otimes ia_j + \ldots.$ Let r be the highest codegree of any c_j such that $a_j \neq 0$; then picking out the terms in $C\otimes M$ having codegree r in C gives a contradiction. Hence there are no relations.

Let B be the quotient A-module of M by the submodule generated by fC; then $B \otimes_A \mathbb{Z}_2 = 0$, which implies that $B = 0$, since B is zero in negative codegrees. Hence we have the A-base $\{fc_j\}$ of M.

In particular, $1 \in M$ must be one of these base elements, and we therefore have a A-module homomorphism $g: M \to A$ such that $g1 = 1$. We assert that the composite

$$h = (\pi\otimes g)\psi : M \xrightarrow[\psi]{} M \otimes M \xrightarrow[\pi\otimes g]{} C\otimes A$$

is an isomorphism of A-modules. It follows from what we have already proved, applied to the coalgebra $C\otimes A$, that $\{hfc_j\}$ is a A-base for $C\otimes A$. Hence $h: M \cong C\otimes A$.

Further, h will be an isomorphism of A-module coalgebras if g is a homomorphism of coalgebras. Dually, $g'\colon A' \to M'$ is a homomorphism of A'-comodules, which is expressed by the commutativity of the diagram

4.5
$$\begin{tikzcd} A' \arrow[r, "g'"] \arrow[d, "\varphi'"'] & M' \arrow[d, "L'"] \\ A' \otimes A' \arrow[r, "1\otimes g'"'] & A' \otimes M'. \end{tikzcd}$$

We should like g' to be an algebra homomorphism. Now we have $A' = Z_2[\lambda_1, \lambda_2, \lambda_3, \ldots]$ as in §3. We define $g_1\colon M \to A$ by stipulating that $g_1'\colon A' \to M'$ is the algebra homomorphism satisfying $g_1'\lambda_i = g'\lambda_i$ for each i; it exists because M' was assumed commutative and associative. It remains to check that g_1 is a homomorphism of A-modules, or alternatively that 4.5 commutes with g_1' in place of g'. Since L', g_1', and φ' are algebra homomorphisms, it suffices to check commutativity on the generators $\lambda_i \in A'$. This follows from the commutativity of 4.5 for g', because $\varphi'\lambda_i$ has the form $\Sigma_j\, x_j \otimes \lambda_j$ (see the explicit formulae 3.5).]]]

Our proof of 4.3 breaks down for right module coalgebras over the Steenrod algebra.

It is easy to describe all the splittings of 4.4 in terms of one splitting $M = A \otimes C$.

4.6 Lemma The possible splittings $g\colon M \to A$ of 4.4 are described in terms of one splitting $M = A \otimes C$ as follows:

Choose any elements $c_1, c_2, c_3, \ldots$ in C' having codegrees $1, 3, 7, \ldots$, and define $g' : A' \to M'$ by

$$g'\lambda_i = \lambda_i + \lambda_{i-1}^2 c_1 + \lambda_{i-2}^4 c_2 + \ldots + \lambda_1^{2^{i-1}} c_{i-1} + c_i.$$

<u>Proof</u> One verifies that these choices do make 4.5 commute, and hence provide splittings. There are no more, for suppose $\lambda_i \to x_i$ is an arbitrary splitting, and $\lambda_i \to y_i$ is a splitting of the above form. Suppose we have $x_i = y_i$ for $i < k$; then commutativity of 4.5 shows that

$$L'(y_k - x_k) = 1 \otimes (y_k - x_k).$$

Thus $y_k - x_k$ lies in the invariant part (see 4.2) of M', which is C'. Hence we can change $c_k \in C'$ to make $y_k = x_k$. The induction proceeds.]]]

<u>Two useful Hopf algebras</u>

We shall need two Hopf algebra structures on the graded polynomial ring $\mathbb{Z}_2[a] = \mathbb{Z}_2[a_1, a_2, \ldots]$, in which a_j has dimension i. For any algebra containing

$$\mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[u],$$

where u is an element of degree -1, define

<u>4.7</u>
$$\begin{cases} x(u) \equiv 1 + a_1 u + a_2 u^2 + a_3 u^3 + \ldots \\ y(u) \equiv u + a_1 u^2 + a_2 u^3 + a_3 u^4 + \ldots, \end{cases}$$

and similarly $x'(u)$, $y''(u)$, etc., for copies $\mathbb{Z}_2[a']$ and $\mathbb{Z}_2[a'']$ of $\mathbb{Z}_2[a]$.

4.8 Definition The straight comultiplication $\psi: \mathbb{Z}_2[a] \to \mathbb{Z}_2[a'] \otimes \mathbb{Z}_2[a'']$ is defined as the unique algebra homomorphism such that

$$\psi\hat{\otimes}1: \mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[u] \to \mathbb{Z}_2[a'] \hat{\otimes} \mathbb{Z}_2[a''] \hat{\otimes} \mathbb{Z}_2[u]$$

takes $x(u)$ to $x'(u).x''(u)$.

The crooked comultiplication is defined as the unique algebra homomorphism $\psi: \mathbb{Z}_2[a] \to \mathbb{Z}_2[a'] \otimes \mathbb{Z}_2[a'']$ such that

$$\psi\hat{\otimes} 1: \mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[u] \to \mathbb{Z}_2[a'] \hat{\otimes} \mathbb{Z}_2[a''] \hat{\otimes} \mathbb{Z}_2[u]$$

takes $y(u)$ to $y''(y'(u))$.

It is easily verified that these comultiplications both induce Hopf algebra structures on $\mathbb{Z}_2[a]$. The first is easily given explicitly, by the familiar formula 2.5

$$a_i \to a'_i + a'_{i-1}a''_1 + a'_{i-2}a''_2 + \ldots + a'_1a''_{i-1} + a''_i.$$

For obvious reasons we shall not attempt to write down the crooked comultiplication explicitly.

4.9 Remark It is easy to see by comparing 4.8 with §3 that the dual $\mathbb{Z}_2[\lambda]$ of the Steenrod algebra is a quotient Hopf algebra of $\mathbb{Z}_2[a]$ with the crooked comultiplication, by means of the obvious projection

$$p: \mathbb{Z}_2[a] \to \mathbb{Z}_2[\lambda]$$

taking a_i to λ_k if $i = 2^k-1$, or to 0 otherwise.

In §2 we constructed the giant Stiefel-Whitney class $\underline{w} \in H^*(B\mathbb{O}^0) \hat{\otimes} \mathbb{Z}_2[a]$ as the universal element of $H^*(B\mathbb{O}^0)$. If we apply the Thom isomorphism $\Phi: H^*(B\mathbb{O}^0) \cong H^*(M\mathbb{O})$ we obtain the universal element

4.10 $$\Phi\underline{w} \in H^*(MQ) \hat{\otimes} Z_2[a]$$

of $H^*(MQ)$, and therefore a duality between $H^*(MQ)$ and $Z_2[a]$. This element will be very important in §6 and later sections. Meanwhile, we observe that the action of the Steenrod algebra on $H^*(MQ)$ gives rise to a comodule structure

4.11 $$L': Z_2[a] \to Z_2[\lambda] \otimes Z_2[a]$$

on $Z_2[a]$.

4.12 Lemma The coaction 4.11 may be expressed as the composite

$$Z_2[a] \xrightarrow{\psi} Z_2[a] \otimes Z_2[a] \xrightarrow{p \otimes 1} Z_2[\lambda] \otimes Z_2[a],$$

where ψ is the crooked comultiplication, and p is given in 4.9.

Proof We consider the restriction of $\Phi\underline{w}$ to $MQ(n) \subset MQ(n)$. The inclusion $BQ(n) \subset MQ(n)$ embeds $H^*(MQ(n))$ in $H^*(BQ(n)^0) = Z_2[t_1, t_2, .. , t_n]$ as the ideal generated by $t_1 t_2 .. t_n$, and $\Phi\underline{w}$ restricts to the element in $H^*(BQ(n)^0) \hat{\otimes} Z_2[a]$

4.13 $$\Pi_{i=1}^{i=n} (t_i + a_1 t_i^2 + a_2 t_i^3 + a_3 t_i^4 + ..) = \Pi_{i=1}^{i=n} y(t_i).$$

Applying the giant Steenrod square $\underline{Sq}$, with the help of 3.3, yields

$$\Pi_{i=1}^{i=n} y(\underline{Sq}\ t_i) = \Pi_{i=1}^{i=n} y(t_i + \lambda_1 t_i^2 + \lambda_2 t_i^4 + \lambda_3 t_i^8 + ..).$$

The result follows.]]]

5. The structure of the Thom spectrum MO

Let us write $M = H^*(\underset{\sim}{MO})$. Then M is a A-module, where A is the Steenrod algebra. Further, M is a left A-module coalgebra, by means of the map $\varphi: \underset{\sim}{MO}\wedge\underset{\sim}{MO} \to \underset{\sim}{MO}$ provided by V.1.7 and V.1.10. In §4 we gave the universal element $\Phi\underline{w} \in M \hat{\otimes} \underset{\sim}{Z}_2[a]$. The structure of M is well known:

5.1 Theorem M is a free A-module. Further, there exists an isomorphism $M \cong A \otimes C$ of left A-module coalgebras, where C has trivial A-structure, and the dual algebra C' of C is a graded polynomial algebra $\underset{\sim}{Z}_2[b_2, b_4, b_5, b_6, b_8, \ldots]$ with one generator b_i in each degree not of the form 2^k-1.

Proof. It is easy to show that 4.3 applies. Instead we shall give a high-speed version of Liulevicius's proof [L2]. The inclusion $\underset{\sim}{BO}(n) \subset \underset{\sim}{MO}(n)$ induces the inclusion

$$H^*(\underset{\sim}{MO}(n)) \subset H^*(\underset{\sim}{BO}(n)^0) = \underset{\sim}{Z}_2[t_1, t_2, \ldots t_n].$$

The restriction to $\underset{\sim}{MO}(n)$ of the universal element $\Phi\underline{w}$ of M yields the element (see 4.13)

$$u = \Pi_{i=1}^{i=n} (t_i + a_1 t_i^2 + a_2 t_i^3 + a_3 t_i^4 + \ldots) \in H^*(\underset{\sim}{BO}(n)^0) \hat{\otimes} \underset{\sim}{Z}_2[a].$$

Define the subgroup B of M by specifying its universal element $v \in M \hat{\otimes} \underset{\sim}{Z}_2[a]$, whose restriction to $\underset{\sim}{MO}(n)$ for each n gives

$$v_n = \Pi_{i=1}^{i=n} (t_i + b_2 t_i^3 + b_4 t_i^5 + b_5 t_i^6 + \ldots) \in H^*(\underset{\sim}{BO}(n)^0) \hat{\otimes} \underset{\sim}{Z}_2[b_2, b_4, \ldots]$$

Then by 3.3

$$\underline{Sq}\ v_n = \Pi_i\ \{t_i + \lambda_1 t_i^2 + \lambda_2 t_i^4 + \ldots + b_2(t_i+\lambda_1 t_i^2+\ldots)^3 + b_4(t_i+\ldots)^5 + \ldots\}$$

$$= \Pi_i\ (t_i + a_1 t_i^2 + a_2 t_i^3 + a_3 t_i^4 + a_4 t_i^5 + \ldots)$$

if we define the isomorphism $\underset{\sim}{Z}_2[a] \cong \underset{\sim}{Z}_2[\lambda] \otimes \underset{\sim}{Z}_2[b_2.b_4,b_5,\ldots]$ of algebras so that the formal identity holds:

$$5.2\quad \theta + a_1\theta^2 + a_2\theta^3 + a_3\theta^4 + \ldots = \theta + \lambda_1\theta^2 + \lambda_2\theta^4 + \lambda_3\theta^8 + \ldots$$
$$+ b_2(\theta+\lambda_1\theta^2+\lambda_2\theta^4+\ldots)^3$$
$$+ b_4(\theta+\lambda_1\theta^2+\lambda_2\theta^4+\ldots)^5$$
$$+ b_5(\theta+\lambda_1\theta^2+\ldots)^6 + \ldots$$

It follows that $\underline{Sq}\ v = u$. The interpretation of this equation in universal elements is that a $\underset{\sim}{Z}_2$-base of B is a A-base ofM, and that we have split the coalgebra structure of M. (We have used essentially the same A-base as Thom [T1].) Also

$$C' \cong B' = \underset{\sim}{Z}_2[b_2,b_4,b_5,\ldots\ ]. \qquad]]]$$

This is the main algebraic result on M. As in [T1] we deduce geometric properties of the spectrum $\underset{\sim}{MO}$.

<u>5.3 Theorem</u> We have $\underline{\underline{N}} \cong \pi_*(\underset{\sim}{MO}) \cong \underset{\sim}{Z}_2[b_2,b_4,b_5,\ldots\ ]$. The Thom spectrum $\underset{\sim}{MO}$ has the homotopy type of the graded Eilenberg-MacLane spectrum $K(\underline{\underline{N}})$. Moreover we can find a homotopy equivalence $\underset{\sim}{MO} \simeq K(\underline{\underline{N}})$ such that φ: $\underset{\sim}{MO}\wedge\underset{\sim}{MO} \to \underset{\sim}{MO}$ corresponds to the map $K(\underline{\underline{N}})\wedge K(\underline{\underline{N}}) \to K(\underline{\underline{N}})$ induced by the

multiplication $\underline{\underline{N}} \times \underline{\underline{N}} \to \underline{\underline{N}}$.

<u>Proof</u> By 5.1 choose a A-base $\{x_\alpha\}$ for M. For each α take a spectrum $K_\alpha = S^n K(\underset{\sim}{Z}_2)$, where n is the degree of α; then we have a map $x_\alpha : \underset{\sim}{MO} \to K_\alpha$. Their product $x : \underset{\sim}{MO} \to \Pi_\alpha K_\alpha \equiv K$ induces an isomorphism $M = H^*(\underset{\sim}{MO}) \cong H^*(K)$, since $\{x_\alpha\}$ is a A-base for M. As in [T1], there is no homology with odd torsion coefficients, and we can apply the Whitehead theorem to deduce that x is a homotopy equivalence (K and $\underset{\sim}{MO}$ are highly connected). If we chose the A-base compatible with the coalgebra structure of M (possible by 5.1), the map $\varphi : \underset{\sim}{MO} \wedge \underset{\sim}{MO} \to \underset{\sim}{MO}$ is induced by $\underline{\underline{N}} \times \underline{\underline{N}} \to \underline{\underline{N}}$. Also, 5.1 gives the structure of $\underline{\underline{N}} \cong \pi_*(\underset{\sim}{MO})$.]]]

What we are really after is statements about the bordism and cobordism theories $\underline{\underline{N}}_*$ and $\underline{\underline{N}}^*$.

<u>5.4 Theorem</u> There exist pairs of non-canonical natural equivalences

$$\underline{\underline{N}}_*(X) \approx H_*(X) \hat{\otimes} \underline{\underline{N}} \quad \text{and} \quad \underline{\underline{N}}^*(X) \approx H^*(X) \hat{\otimes} \underline{\underline{N}}$$

that respect all the product structures (namely cup, cap, slant, and Kronecker products, and $\underline{\underline{N}}$-module structures).

<u>Proof</u> We select by 5.3 a homotopy equivalence $\underset{\sim}{MO} \simeq K(\underline{\underline{N}})$ such that $\varphi : \underset{\sim}{MO} \wedge \underset{\sim}{MO} \to \underset{\sim}{MO}$ corresponds to the coefficient homomorphism $K(\underline{\underline{N}}) \wedge K(\underline{\underline{N}}) \to K(\underline{\underline{N}})$ induced by multiplication $\underline{\underline{N}} \times \underline{\underline{N}} \to \underline{\underline{N}}$. All the stated products are induced by φ.]]]

<u>N.B.</u> There is no <u>canonical</u> homotopy equivalence

$\underset{\sim}{MO} \simeq K(\underline{\underline{N}})$ in 5.3, and hence the equivalences in 5.4 are not canonical. The permissible variation of the choice of the homotopy equivalence $\underset{\sim}{MO} \simeq K(\underline{\underline{N}})$ so as to respect products is measured by 4.6. Thus there are many choices.

From the usual Künneth formula we deduce:

<u>5.5 Corollary</u> For any spectra X and Y we have <u>canonically</u>

$$\underline{\underline{N}}_*(X\wedge Y) \approx \underline{\underline{N}}_*(X) \otimes_{\underline{\underline{N}}} \underline{\underline{N}}_*(Y) \quad \text{and} \quad \underline{\underline{N}}^*(X\wedge Y) \approx \underline{\underline{N}}^*(X) \hat{\otimes}_{\underline{\underline{N}}} \underline{\underline{N}}^*(Y). \ ]]$$

Theorem 5.4 disposes of a large part of the theory of $\underline{\underline{N}}_*$ and $\underline{\underline{N}}^*$. However, quite apart from aesthetic considerations, the non-uniqueness of the equivalences in 5.4 is not acceptable. Transfer homomorphisms are not catered for, and neither are the cobordism Stiefel-Whitney classes we introduced in V.8.9. (We shall see in §8 that W_1 does not correspond to $w_1 \otimes 1$ under <u>any</u> of the equivalences in 5.4) We therefore formulate our results in an invariant manner.

<u>5.6 Theorem</u> We have canonically $\underline{\underline{N}} \cong \mathrm{Hom}_A(M,\underset{\sim}{Z}_2)$, and canonical natural equivalences

$$\underline{\underline{N}}_*(X) \approx \mathrm{Hom}_A(H^*(X)\otimes M,\ \underset{\sim}{Z}_2) \text{ and } \underline{\underline{N}}^*(X) \approx \mathrm{Hom}_A(M,\ H^*(X))$$

for any finite spectrum X. The second is valid for all spectra X. The product structures come from the coalgebra structure on M.

<u>Proof</u>. By definition and 5.3 we have

$$\underline{\underline{N}}^*(X) = \{X,\underset{\sim}{MO}\}^* \cong \{X,K(\underline{\underline{N}})\}^* \text{ and } \underline{\underline{N}}_*(X)=\{\Sigma^0,X\wedge \underset{\sim}{MO}\}_* \cong \{\Sigma^0,X\wedge K(\underline{\underline{N}})\}_*$$

We know that $X \wedge K(\underline{\underline{N}})$ is again a graded Eilenberg spectrum (see Summary, M.20, or we could use the algebraic counterpart 4.1).]]]

This theorem sheds more light on how the equivalences in 5.4 depend on the choice of the splitting of M.

We can remove the finiteness restriction in 5.6 if we agree to take only those homomorphisms $H^*(X) \otimes M \to \underset{\sim}{Z}_2$ that factor through $H^*(Y) \otimes M$, for some finite subspectrum Y of X. For we have $\underline{\underline{N}}_*(X) = \varinjlim_Y \underline{\underline{N}}_*(Y)$, in common with all homology theories.

§6. From cobordism to cohomology

In §5 we elucidated the structure of the cohomology theory $\underline{\underline{N}}^*$. In this section we express its structure in a form suitable for the computation of transfer homomorphisms. This will involve the machinery of universal elements, giant Stiefel-Whitney classes, giant Steenrod squares, etc., which we have developed in previous sections. (From now on we concentrate on $\underline{\underline{N}}^*$ rather than $\underline{\underline{N}}_*$, mainly because it has cup products.)

We fix attention on the canonical natural equivalence 5.6

<u>6.1</u> $$\underline{\underline{N}}^*(X) \approx \mathrm{Hom}_A(M, H^*(X)).$$

We have the universal element 4.10 $\Phi\underline{w} \in H^*(M\underset{\sim}{O}) \hat{\otimes} \underset{\sim}{Z}_2[a]$, where $\underset{\sim}{Z}_2[a]$, as usual, stands for the graded polynomial algebra $\underset{\sim}{Z}_2[a_1, a_2, a_3, \ldots]$ with a generator a_i in degree i for each $i > 0$. The element $\Phi\underline{w}$ establishes a duality between $H^*(M\underset{\sim}{O})$ and $\underset{\sim}{Z}_2[a]$. We use this duality to rewrite 6.1 in the form

<u>6.2</u> $$\underline{\underline{N}}^*(X) \subset H^*(X) \hat{\otimes} \underset{\sim}{Z}_2[a].$$

To decide which elements of $H^*(X) \hat{\otimes} \underset{\sim}{Z}_2[a]$ are in $\underline{\underline{N}}^*(X)$, we must express differently the A-module structures of $H^*(X)$ and M. To this end, we have the giant Steenrod square 3.1

$$\underline{Sq} : H^*(X) \longrightarrow \underset{\sim}{Z}_2[\lambda] \hat{\otimes} H^*(X),$$

and the dual coaction 4.11 on $\underset{\sim}{Z}_2[a]$ dual to the A-action on M

$$L' : \underset{\sim}{Z}_2[a] \longrightarrow \underset{\sim}{Z}_2[\lambda] \otimes \underset{\sim}{Z}_2[a].$$

We shall also write $H^*(X) \hat{\otimes} \underset{\sim}{Z}_2[a]$ more succinctly as $H^*(X;\underset{\sim}{Z}_2[a])$. Then what we are doing in 6.2 is to apply the map of coefficient spectra $\Phi\underline{w}: M\underset{\sim}{O} \to K(\underset{\sim}{Z}_2[a])$. We extend the giant Steenrod square $\underline{Sq}$ and the coaction L' to ring homomorphisms

$$H^*(X;\underset{\sim}{Z}_2[a]) \longrightarrow \underset{\sim}{Z}_2[\lambda] \hat{\otimes} H^*(X;\underset{\sim}{Z}_2[a])$$

by making $\underline{Sq}$ act trivially on $\underset{\sim}{Z}_2[a]$ and L' act trivially on $H^*(X)$.

<u>6.3 Theorem</u> The giant class $\Phi\underline{w}: M\underset{\sim}{O} \to K(\underset{\sim}{Z}_2[a])$ induces a multiplicative natural transformation

$$\Phi\underline{w}^\circ : \underline{\underline{N}}^*(X) \longrightarrow H^*(X;\underset{\sim}{Z}_2[a]),$$

which embeds $\underline{\underline{N}}^*X$ as the subalgebra of elements $x \in H^*(X;\mathbb{Z}_2[a])$ satisfying $\underline{Sq}\ x = L'\ x$.]]]

This is the description of $\underline{\underline{N}}^*(X)$ most convenient for our computations.

We have the canonically defined cobordism Stiefel-Whitney classes W_i.

6.4 Lemma In $H^*(BQ(1)^0;\mathbb{Z}_2[a])$ we have

$$\Phi\underline{w}\circ W_1 = w_1 + a_1 w_1^2 + a_2 w_1^3 + a_3 w_1^4 + a_4 w_1^5 + \cdots$$

Proof. By definition V.8.1. W_1 is the class of the inclusion $BQ(1)^0 \subset MQ(1)$. We can read off the answer from 4.13, by putting $n = 1$.]]]

6.5 Corollary In $H^*(BQ(n)^0;\mathbb{Z}_2[a])$ we have

$$\Phi\underline{w}\circ T_i = t_i + a_1 t_1^2 + a_2 t_i^3 + a_3 t_i^4 + \cdots \qquad]]]$$

By a formidable computation, one can now work out $\Phi\underline{w}\circ W_i$.

Transfer homomorphisms

A transfer homomorphism in ordinary cohomology extends trivially to one in $H^*(\ ;\mathbb{Z}_2[a])$. If we also have a corresponding transfer homomorphism in $\underline{\underline{N}}^*$, we can use $\Phi\underline{w}$ and the Riemann-Roch theorems of V.7 to compare them.

Any smooth manifold V has a canonical MQ-orientation, which corresponds by V.4.7 to an orientation of its stable normal bundle $-\tau$, where τ is the tangent bundle. Also, any vector bundle α over X is canonically oriented by its classifying map $X^\alpha \to \underset{\sim}{MO}$, defined in V.1.10.

We intend to apply the Riemann-Roch theorems of V.7 to the multiplicative natural transformation

$$\Phi\underline{w}\circ : \underline{\underline{N}}^* \to H^*(\ ;\underset{\sim}{Z}_2[a]).$$

To do this we need to compute $\hat{a}(V)$ for a manifold V, and $\hat{a}(\pi)$ for a fibre bundle of the type considered in V.6.20, etc. For any virtual vector bundle α over X, the classifying maps $X^0 \to \underset{\sim}{BO}$ and $X^\alpha \xrightarrow[u]{} \underset{\sim}{MO}$ induce the commutative diagram of Thom isomorphisms in ordinary cohomology:

$$\begin{array}{ccc} H^*(\underset{\sim}{MO};\underset{\sim}{Z}_2[a]) & \longrightarrow & H^*(X^\alpha;\underset{\sim}{Z}_2[a]) \\ \cong\uparrow\Phi & & \cong\uparrow\Phi \\ H^*(\underset{\sim}{BO}^0;\underset{\sim}{Z}_2[a]) & \longrightarrow & H^*(X^0;\underset{\sim}{Z}_2[a]). \end{array}$$

From this we deduce that $\Phi\underline{w}\circ u \in H^*(X^\alpha;\underset{\sim}{Z}_2[a])$ corresponds under Φ to the giant Stiefel-Whitney class 2.3 $\underline{w}(\alpha)$. If we now consider the definitions V.7.5 and V.7.7 of $\hat{a}(V)$ and $\hat{a}(\pi)$ we find:

<u>6.6 Lemma</u> With respect to the transformation

$$\Phi\underline{w}\circ : \underline{\underline{N}}^* \to H^*(\ ;\underset{\sim}{Z}_2[a])$$

(a) For any smooth manifold V we have $\hat{a}(V)=\underline{w}(V)^{-1}$,

(b) For any fibre bundle π as in V.6.20 we have $\hat{a}(\pi) = \underline{w}(\tau)^{-1}$, where τ is the bundle of tangents along the fibres.

(c) If $\Phi^{\underline{\underline{N}}}$ denotes the cobordism Thom isomorphism $\Phi^{\underline{\underline{N}}} : \underline{\underline{N}}^*(X^0) \cong N^*(X^\alpha)$, and $x \in \underline{\underline{N}}^*(X^0)$, we have $\Phi\underline{w}\circ\Phi^{\underline{\underline{N}}}x = \Phi((\Phi\underline{w}\circ x).\underline{w}(\alpha))$.]]]

Generators for $\underline{\underline{N}}$.

By 6.3, $\Phi\underline{w}^\circ : \underline{\underline{N}} \to \underset{\sim}{Z}_2[a]$ embeds $\underline{\underline{N}}$ as the invariant subalgebra of $\underset{\sim}{Z}_2[a]$ with respect to the coaction 4.11. In the proof of 5.1 we expressed $\underset{\sim}{Z}_2[a]$ as a tensor product algebra $\underset{\sim}{Z}_2[b] \otimes \underset{\sim}{Z}_2[\lambda]$ by means of the identity 5.2; in particular we observe that under this isomorphism $b_i = a_i$ + lower terms. The image of $\underset{\sim}{Z}_2[b]$ is just $\Phi\underline{w}^\circ\underline{\underline{N}}$. It is an algebraic triviality that if we are given an i-manifold M_i for each i not of the form 2^k-1, their classes $[M_i]$ in $\underline{\underline{N}}$ form a system of polynomial generators if and only if a_i appears in $\Phi\underline{w}^\circ[M_i]$ with non-zero coefficient for each i.

Let us compute $\Phi\underline{w} \circ [V]$ for a n-manifold V. This can be done by considering the transfer homomorphisms V.6.3 induced by the map $f:V \to P$, where P is a point. In $\underline{\underline{N}}_*$ we have $[V] = f_* f^! i \in \underline{\underline{N}}_*$, where $i \in \underline{\underline{N}}_*(P^0)$ is the fundamental class of P. By applying the Riemann-Roch theorem V.7.6 and V.6.2 (e), we obtain

$$\Phi\underline{w}^\circ\, [V] = \langle z, \hat{a}(V)\rangle,$$

where z is the homology fundamental class of V. Finally we substitute from 6.6.

<u>6.7 Lemma</u> For any n-manifold V, whose class in $\underline{\underline{N}}$ is $[V]$, we have

$$\Phi\underline{w} \circ [V] = \langle z, \underline{w}(V)^{-1}\rangle \in \underset{\sim}{Z}_2[a],$$

where z is the fundamental homology class of V. We may take $[V]$ as one of a system of polynomial generators of N if and only if a_n has non-zero coefficient in $\underline{w}(V)^{-1}$, or equivalently in $\underline{w}(V)$.]]] (Compare Thom [T1].)

In §2 we computed $\underline{w}(V)$ in various cases, to which we now apply 6.7.

<u>6.8 Lemma</u> $[P_n(\underset{\sim}{R})]$ may be taken as a generator for $\underline{\underline{N}}$ in degree n if n is even; $[P_n(\underset{\sim}{R})] = 0$ if n is odd.

<u>Proof</u>. In 2.9 we computed

$$\underline{w}(P_n(\underset{\sim}{R})) = (1 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots)^{n+1},$$

where α generates $H^1(P_n(\underset{\sim}{R})^0)$. The coefficient of a_n is $(n+1)\alpha^n$, which is non-zero if n is even. If n is odd, $n = 2k-1$ say, we have

$$\underline{w}(P_n) = (1 + a_1^2\alpha^2 + a_2^2\alpha^4 + a_3^2\alpha^6 + \dots)^k,$$

in which the coefficient of α^n is plainly zero.]]]

<u>6.9 Lemma</u> (a) Suppose $m \geqslant 2, n \geqslant 2$. Then $[H_{m,n}(\underset{\sim}{R})]$ may be taken as a generator for $\underline{\underline{N}}$ is degree $m + n - 1$ if and only if the binomial coefficient $\{m,n\}$ is non-zero (mod 2).

(b) $[H_{1,n}(\underset{\sim}{R})] = 0$ if $n \geqslant 1$.

<u>Proof</u> We computed $\underline{w}(H_{m,n})$ in 2.12. The fundamental class of $H_{m,n}$ is awkward to work with. Instead we use the embedding $j:H_{m,n} \subset P_m \times P_n$, and the transfer homomorphism $j_{\natural}$. We see that, in the notation of 2.12 and using information

there and in V.§6,

$$\langle H_{m,n}, \underline{w}(H_{m,n})^{-1}\rangle = \langle P_m \times P_n, (\alpha+\beta).\underline{w}(H_{m,n})^{-1}\rangle.$$

Now $(\alpha+\beta).\underline{w}(H_{m,n})^{-1} = \{(\alpha+\beta)+a_1(\alpha+\beta)^2+a_2(\alpha+\beta)^3 + \dots\}.$

$$\{1+a_1\alpha+a_2\alpha^2+ a_3\alpha^3+..\}^{-m-1}. \{1+a_1\beta+a_2\beta^2+a_3\beta^3+..\}^{-n-1}.$$

The coefficient of a_{m+n-1} in this expression is $(\alpha+\beta)^{m+n} = \{m,n\}\ \alpha^m\beta^n$, all the other terms being zero.

Hence we have (a), by 6.7.

If m=1 this computation is false. That $[H_{1,n}] = 0$ has been verified explicitly by Conner and Floyd as Lemma 2.2 of [C6]. In our notation their proof is as follows. In this case we have the relations $\alpha^2=0$ and $\beta^{n+1}=0$. Hence $(1+\alpha+\beta)^2 = 1 + \alpha^2 + \beta^2 = 1 + \beta^2 = (1+\beta)^2$, which enables us to rewrite $w = (1+\alpha)^2(1+\beta)^{n+1}(1+\alpha+\beta)^{-1}$ as $w = (1+\alpha+\beta)(1+\beta)^{n-1} = (1+\beta)^n + \alpha(1+\beta)^{n-1} = (1+\beta+ (\alpha/n))^n$. The last expression lies in $H^*(P_1 \times P_n)$, despite appearances. By the expansion lemma 2.6 the giant Stiefel-Whitney class is

$$\underline{w}(H_{m,n}) = \{1 + a_1(\beta + \alpha/n) + a_2(\beta + \alpha/n)^2 + \dots\}^n,$$

formally. We require, by the same device as before, the terms in $(\alpha+\beta).\underline{w}(H_{m,n})$ of polynomial degree n+1 in α and β. But these contain the factor

$$(\alpha+\beta).(\beta+\alpha/n)^n = \alpha\beta^n + \beta.n\beta^{n-1}.\alpha/n = 0.$$

(This argument requires more justification than we have given.)]]

<u>6.10 Corollary</u> (Milnor) As a set of polynomial generators for $\underline{\underline{N}}$ we may take:

(a) All $[P_n(\underset{\sim}{R})]$, for n even, and

(b) All $[H_{m,n}(\underset{\sim}{R})]$ for which $m=2^r$, $n=2^{r+1}s$, $r\geq 1$, $s\geq 1$.]]]

Let $x_n \in \underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$ be the class of the singular manifold $H_{1,n}(\underset{\sim}{R}) \subset P_1(\underset{\sim}{R}) \times P_n(\underset{\sim}{R}) \to P_n(\underset{\sim}{R}) \subset P_\infty(\underset{\sim}{R}) = B\underset{\sim}{O}(1)$. These elements were used in [C4] by Conner and Floyd to provide a good $\underline{\underline{N}}$-base of $\underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$. They will reappear in §8. It is easily seen [C4] that $\Delta x_n = x_{n-1}$, where Δ is the Smith homomorphism V.9.1, i.e. $x_{n-1} = x_n \cap W_1$ in view of V.9.3. We may write 6.9 (b) in the form $\langle x_n, 1\rangle = 0$ if $n \geq 1$. It follows that $\langle x_i, W_1^j\rangle$ is 1 if $i = j$, and 0 otherwise. This shows:

<u>6.11 Theorem</u> The monomials $\{W_1^i\}$ in $\underline{\underline{N}}^*(B\underset{\sim}{O}(1)^0)$ are the $\underline{\underline{N}}$-linear functionals on $\underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$ dual to the base $\{x_i\}$.]]]

<u>§7. Primary cobordism operations</u>

As with any represented functor, primary cobordism operations are in canonical 1-1 correspondence with $\{M\underset{\sim}{O},M\underset{\sim}{O}\}^*$, which is a graded ring under composition. In this section we make various remarks on its structure.

From 5.6 we see that $\{M\underset{\sim}{O},M\underset{\sim}{O}\}^* = \mathrm{Hom}_A(M,M)$, where $M = H^*(M\underset{\sim}{O})$ as a free A-module. Thus $\{M\underset{\sim}{O},M\underset{\sim}{O}\}^*$ is the graded algebra of matrices over A of a certain form. Unfortunately this form of the structure is not very useful, because it involves choosing a A-base for M.

A canonical approach is to make use of the Thom isomorphism, which is $\underline{\underline{N}}$-linear,

<u>7.1</u> $$\Phi : \underline{\underline{N}}^*(BQ^0) \cong \underline{\underline{N}}^*(MQ) = \{MQ, MQ\}^*;$$

this is more useful because we have from V.8.7

$$\underline{\underline{N}}^*(BQ^0) = \underline{\underline{N}} \hat{\otimes} Z_2[W_1, W_2, W_3, \dots],$$

where the W_i are the cobordism Stiefel-Whitney classes. By means of Φ, the composition product in $\{MQ,MQ\}^*$ induces a peculiar multiplication in $\underline{\underline{N}}^*(BQ^0)$, quite different from the cup product multiplication. We call it the crooked multiplication, on account of 7.5 below.

A particularly interesting subring of $\underline{\underline{N}}^*(BQ^0)$ is the graded polynomial algebra $Z_2[W_1, W_2, W_3, \dots]$. By V.8.8 the multiplication $\mu : BQ \times BQ \to BQ$ induces a coalgebra structure on $Z_2[W_1, W_2, \dots]$ and we have the same formal theory as for cohomology Stiefel-Whitney classes. In particular, just as in §2, we take the graded polynomial algebra $Z_2[b] \equiv Z_2[b_1, b_2, b_3, \dots]$ as the dual algebra.

<u>7.2 Definition</u> The <u>universal giant Stiefel-Whitney cobordism characteristic class</u> $\underline{W} \in \underline{\underline{N}}^*(BQ^0) \hat{\otimes} Z_2[b]$ is defined in terms of the W_i in the same way that the giant Stiefel-Whitney cohomology class $\underline{w}$ was defined in terms of the w_i. Thus $\underline{W}$ is the universal element of $Z_2[W_1, W_2, \dots]$.

Let us calculate $(\Phi\underline{w} \hat{\otimes} 1) \circ \underline{W} \in H^*(BQ^0) \hat{\otimes} Z_2[a] \hat{\otimes} Z_2[b]$.

7.3 Lemma We have

$$(\Phi\underline{w} \hat{\otimes} 1) \circ \underline{W} = (1 \hat{\otimes} \lambda)\underline{w} \in H^*(BO^0) \hat{\otimes} \mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[b],$$

where the algebra homomorphism $\lambda:\mathbb{Z}_2[a] \to \mathbb{Z}_2[a] \otimes \mathbb{Z}_2[b]$ is defined by the formal identity

7.4 $(\lambda\hat{\otimes}1)\, x_a(\theta) = x_b(y_a(\theta))$ in $\mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[b] \hat{\otimes} \mathbb{Z}_2[\theta]$.

Here we have written, as in 4.7

$$x_b(\theta) \equiv 1 + b_1\theta + b_2\theta^2 + b_3\theta^3 + \ldots$$
$$y_a(\theta) \equiv \theta + a_1\theta^2 + a_2\theta^3 + a_3\theta^4 + \ldots$$

Proof. It suffices, as usual, to consider the restrictions to $B\mathbb{Q}(n)$ for finite n. Then $\underline{W}$ restricts to $\Pi_{i=1}^{i=n}\, x_b(T_i)$, in the notation of V.8, and hence 6.5 and 6.3 show that the restriction of $(\Phi\underline{w} \hat{\otimes} 1) \circ \underline{W}$ is

$$\Pi_i\; x_b(y_a(t_i)) = \Pi_i(1 \hat{\otimes} \lambda)x_a(t_i).$$

But by 2.1 $\Pi_i\; x_a(t_i)$ is the restriction of $\underline{w}$.]]]

We note that we almost have here the formula for the crooked comultiplication.

7.5 Theorem Under 7.1, $\Phi(\mathbb{Z}_2[W_1, W_2, \ldots])$ is a subalgebra of the algebra $\{M\mathbb{Q}, M\mathbb{Q}\}^*$ equipped with the composition product. The dual comultiplication on the dual $\mathbb{Z}_2[b]$ is the crooked comultiplication 4.8. In cohomology we have

$$(\Phi\underline{w} \hat{\otimes} 1) \circ \Phi\underline{W} = (1 \hat{\otimes} \psi)\, \Phi\underline{w} \text{ in } H^*(M\mathbb{Q}) \hat{\otimes} \mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[b],$$

where $\psi:\mathbb{Z}_2[a] \to \mathbb{Z}_2[a] \hat{\otimes} \mathbb{Z}_2[b]$ is the crooked comultiplication.

Proof. We show $(\Phi\underline{w} \hat{\otimes} 1) \circ \Phi\underline{W} = (1 \hat{\otimes} \psi)\, \Phi\underline{w}$, from which the rest follows, by the associativity of the crooked comultiplication and the fidelity of $\Phi\underline{w}^{\circ}$. By 6.6 and 7.3, we have

$$(\Phi\underline{w} \hat{\otimes} 1) \circ \Phi\underline{W} = (\Phi \hat{\otimes} 1 \hat{\otimes} 1)((\Phi\underline{w} \circ \underline{W}).(\underline{w} \hat{\otimes} 1))$$
$$= (\Phi \hat{\otimes} 1 \hat{\otimes} 1)((1 \hat{\otimes} \lambda)\underline{w}.(\underline{w} \hat{\otimes} 1))$$
$$= (1 \otimes \psi)\ \Phi\underline{w}, \quad \text{comparing 7.4 and 4.8.} \ ]]$$

Remark This result does not determine the ring structure of $\{MO,MO\}^*$ completely, for the composition product is $\underline{\underline{N}}$-linear only in the first factor.

We now have, as foreshadowed in §4, two Hopf algebra structures on the subgroup $Z_2[W_1,W_2,\ldots]$ of $\underline{\underline{N}}^*(BO^0)$, having the same comultiplication. One multiplication is by cup products, and the other from composition in $\{MO,MO\}^*$ by 7.1. These are dual to the straight and crooked comultiplications on $Z_2[b]$ respectively, defined in 4.8.

There are various standard cobordism operations.

$\underline{\underline{N}}$-module multiplications For any spectrum X, $\underline{\underline{N}}^*(X)$ is a $\underline{\underline{N}}$-module. Since $\underline{\underline{N}}$ is commutative, multiplication by any element of $\underline{\underline{N}}$ is a $\underline{\underline{N}}$-linear cobordism operation, of degree $\geqslant 0$.

Steenrod operations By 5.3 there exist homotopy equivalences $MO \simeq K(\underline{\underline{N}})$ respecting the products $MO \wedge MO \to MO$ and $K(\underline{\underline{N}}) \wedge K(\underline{\underline{N}}) \to K(\underline{\underline{N}})$. The Steenrod algebra A acts canonically on $K(\underline{\underline{N}})$, and hence on $M\underset{\sim}{O}$, if we choose a homotopy equivalence $M\underset{\sim}{O} \simeq K(\underline{\underline{N}})$ as above. We call these cobordism operations the Steenrod operations. They have negative degree, and are obviously $\underline{\underline{N}}$-linear. Of course they depend decisively on the choice of $M\underset{\sim}{O} \simeq K(\underline{\underline{N}})$, e.g. Sq^8 definitely can vary, as we see from 4.6.

7.6 Theorem

(a) The only $\underline{\underline{N}}$- linear cobordism operations are the (infinite) $\underline{\underline{N}}$-linear combinations of the Steenrod operations. (These include the module multiplications.)

(b) Let $x \in \underline{\underline{N}}^*(B\underset{\sim}{O}^0)$; then by 7.1 Φx is a cobordism operation. This operation is a derivation if and only if x is primitive with respect to the straight coproduct; there are only the (infinite) $\underline{\underline{N}}$-linear combinations of those given formally as $\Phi(\Sigma\ T_i^r)$, for each integer $r \geq 1$.

(c) The only $\underline{\underline{N}}$-linear derivations are the (infinite) $\underline{\underline{N}}$-linear combinations of the Steenrod operations Sq^1, $Sq^{0,1}$, $Sq^{0,0,1}$,... .

Proof To prove (a) we may work in $K(\underline{\underline{N}})$, where the assertion is trivial.

Naturality of the Thom isomorphisms with respect to $B\underset{\sim}{O} \times B\underset{\sim}{O} \to B\underset{\sim}{O}$ yields the commutative diagram

$$\begin{tikzcd}
\underline{\underline{N}}^*(\underset{\sim}{MO}) \arrow[r, "\mu^*"'] & \underline{\underline{N}}^*(\underset{\sim}{MO}) \otimes_{\underline{\underline{N}}} \underline{\underline{N}}^*(\underset{\sim}{MO}) \\
\underline{\underline{N}}^*(\underset{\sim}{BO}^0) \arrow[r, "\mu^*"'] \arrow[u, "\cong"', "\Phi"] & \underline{\underline{N}}^*(\underset{\sim}{BO}^0) \otimes_{\underline{\underline{N}}} \underline{\underline{N}}^*(\underset{\sim}{BO}^0). \arrow[u, "\cong"', "\Phi \otimes \Phi"]
\end{tikzcd}$$

An element of $\underline{\underline{N}}^*(\underset{\sim}{MO})$ is a derivation if and only if it is primitive. The first assertion of (b) follows. The primitives of $\underline{\underline{N}}^*(\underset{\sim}{BO}^0)$ are easily to found from V.8.8 to be those given; they are dual to the indecomposable quotient of $\underline{\underline{N}} \otimes \underset{\sim}{Z}_2[b]$.

We deduce from (a) that to prove (c) all we have to do is to find the primitive elements of A. These can be read off from 3.2 (compare [M2]), as the elements dual to the generators of $\lambda_1, \lambda_2, ..$ of $\underset{\sim}{Z}_2[\lambda]$.]]]

Remark The previous caution notwithstanding, these operations Sq^1, $Sq^{0,1}$, ... are independent of the choice of $\underset{\sim}{MO} \simeq K(\underline{\underline{N}})$, as one can see from 4.6.

§8. Determinants

We introduced the cobordism Stiefel-Whitney classes W_i partly because they have geometric significance, partly because they enable us to write canonically

$$\underline{\underline{N}}^*(\underset{\sim}{BO}(n)^0) = \underline{\underline{N}} \mathbin{\hat{\otimes}} \underset{\sim}{Z}_2[W_1, W_2, .. , W_n].$$

In this section we determine their behaviour under the various standard maps involving $\underset{\sim}{BO}(n)$. Not all of them behave as one might expect from the situation in ordinary cohomology. Those involving determinants are particularly non-trivial, and give rise to interesting geometric operations.

We have various standard maps as follows:

Whitney sum maps

$\varphi: B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(m+n)$, and $\varphi: B\underset{\sim}{O} \times B\underset{\sim}{O} \to B\underset{\sim}{O}$

$\varphi: M\underset{\sim}{O}(m) \wedge M\underset{\sim}{O}(n) \to M\underset{\sim}{O}(m+n)$, and $\varphi: M\underset{\sim}{O} \wedge M\underset{\sim}{O} \to M\underset{\sim}{O}$.

(see V.1.8 and V.1.9)

Diagonal maps

$\Delta: B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(n) \times B\underset{\sim}{O}(n)$, hence $\Delta: B\underset{\sim}{O} \to B\underset{\sim}{O} \times B\underset{\sim}{O}$,

$\Delta: M\underset{\sim}{O}(n) \to B\underset{\sim}{O}(n)^0 \wedge M\underset{\sim}{O}(n)$, hence $\Delta: M\underset{\sim}{O} \to B\underset{\sim}{O}^0 \wedge M\underset{\sim}{O}$.

Determinant

$\det: B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(1)$, hence $\det: B\underset{\sim}{O} \to B\underset{\sim}{O}(1)$.

Tensor product

$\otimes: B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(mn)$, and $\otimes: B\underset{\sim}{O} \times B\underset{\sim}{O} \to B\underset{\sim}{O}$.

Multiplication

$\mu: B\underset{\sim}{O}(1) \times B\underset{\sim}{O}(1) \to B\underset{\sim}{O}(1)$ (defined since $\underset{\sim}{O}(1)$ is abelian).

and of course many maps induced by inclusion.

We have already dealt with most of these. All the diagonal maps do is to furnish $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$ etc. with cup products, and induce Thom isomorphisms (Chapter V). The Whitney sum maps were treated in V.8.7 and V.8.8, and yield no surprises. By restricting to $B\underset{\sim}{O}(n)$, etc., as in V.§8, we can reduce the study of det and $\otimes$ to that of the multiplication μ. This map contains all the difficulty.

The map $\mu: B\underset{\sim}{O}(1) \times B\underset{\sim}{O}(1) \to B\underset{\sim}{O}(1)$ and the diagonal induce a commutative and associative 'Hopf algebra' structure on $\underline{\underline{N}}^*(B\underset{\sim}{O}(1)^O)$ over the graded groundring $\underline{\underline{N}}$. Also the canonical antiautomorphism is the identity. All these facts follow from the easily verified assertion that $\underset{\sim}{O}(1)$ is an abelian group in which inversion is the identity. As algebra, we have $\underline{\underline{N}}^*(B\underset{\sim}{O}(1)^O) = \underline{\underline{N}} \hat{\otimes} \underset{\sim}{Z}_2[W_1]$. It remains to find the coproduct $W_1 \circ \mu$. We see from 6.4 that W_1 is <u>not</u> primitive.

<u>8.1 Theorem</u> There exist elements $z_2, z_4, z_5, z_6, z_8, \ldots$ in $\underline{\underline{N}}$, uniquely defined by the condition that

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_8 W_1^9 + \ldots$$

(omitting terms of the form $z_{k-1} W_1^k$ when k is a power of 2) is a primitive element in $\underline{\underline{N}}^*(B\underset{\sim}{O}(1)^O)$. Moreover, these elements z_i are a set of polynomial generators for $\underline{\underline{N}}$.

<u>Proof</u> We use $\Phi \underline{w} \circ : \underline{\underline{N}}^*(B\underset{\sim}{O}(1)^O) \to H^*(B\underset{\sim}{O}(1)^O) \hat{\otimes} \underset{\sim}{Z}_2[a] = \underset{\sim}{Z}_2[w_1] \hat{\otimes} \underset{\sim}{Z}_2[a]$, which is a homomorphism of Hopf algebras, and the fact that w_1 is primitive. In the proof of 5.1 we made use of an algebra isomorphism

$$\underset{\sim}{Z}_2[a] \cong \underset{\sim}{Z}_2[\lambda] \otimes \underset{\sim}{Z}_2[b]$$

defined by the formal identity 5.2

$$\begin{aligned}\theta + a_1\theta^2 + a_2\theta^3 + a_3\theta^4 + \ldots = {} & \theta + \lambda_1\theta^2 + \lambda_2\theta^4 + \lambda_3\theta^8 + \ldots \\ & + b_2(\theta + \lambda_1\theta^2 + \lambda_2\theta^4 + \ldots)^3 \\ & + b_4(\theta + \lambda_1\theta^2 + \lambda_2\theta^4 + \ldots)5 \\ & + b_5(\theta + \lambda_1\theta^2 + \ldots)^6 + \ldots,\end{aligned}$$

with the property that $\varphi \underline{w}^\circ : \underline{\underline{N}} \cong \mathbb{Z}_2[b]$. If we write $L = w_1 + \lambda_1 w_1^2 + \lambda_2 w_1^4 + \lambda_3 w_1^8 + \ldots \in \mathbb{Z}_2[w_1] \hat{\otimes} \mathbb{Z}_2[a]$ and use the above identity and 6.4 we find

$$\Phi \underline{w}^\circ W_1 = L + b_2 L^3 + b_4 L^5 + b_5 L^6 + \ldots .$$

We can solve this for L:

$$L = \Phi\underline{w}^\circ W_1 + c_1(\Phi\underline{w}^\circ W_1)^2 + c_2(\Phi\underline{w}^\circ W_1)^3 + c_3(\Phi\underline{w}^\circ W_1)^4 + \ldots,$$

where the c_i are certain complicated polynomials in the b_j <u>and therefore lie in</u> $\Phi\underline{w}^\circ\underline{\underline{N}}$. Hence we can write

$$L = \Phi\underline{w}\ (W_1 + y_1 W_1^2 + y_2 W_1^3 + y_3 W_1^4 + \ldots),$$

in which the y_i are in $\underline{\underline{N}}$. But L is primitive. It follows that the element

$$Q = W_1 + y_1 W_1^2 + y_2 W_1^3 + y_3 W_1^4 + \quad \in \underline{\underline{N}}^*(B\underline{O}(1)^0)$$

is primitive. This does not yet have the correct form, because $y_7 \neq 0$. However, Q^{2^k} is again primitive, and a suitable $\underline{\underline{N}}$-linear combination of these has the required form, Further, such combinations account for all the primitive elements, which is enough to show that P is unique.

Finally, if i does not have the form 2^k-1 we observe that z_i involves b_i with coefficient 1.]]]

<u>Remark</u> In unitary cobordism we have a corresponding 'Hopf algebra' $\underline{\underline{U}}^*(B\underline{U}(1)^0)$. The coproduct of C_1 is

$$C_1 \otimes 1 + 1 \otimes C_1 + a(C_1 \otimes C_1) + \text{higher terms},$$

where a generates $\underline{\underline{U}}_2$. It follows that this Hopf algebra has no non-zero primitive elements, so that the unitary analogue of 8.1 fails.

We study the coproduct of W_1 in more detail. This has the form

<u>8.2</u> $W_1\circ\mu = \Sigma_{i,j}\, y_{i,j}\, W_1^i \hat{\otimes}_{\underline{\underline{N}}} W_1^j$, $(y_{i,j} \in \underline{\underline{N}})$

<u>8.3 Definition</u> We define the <u>diagonal function</u> $D(u,v) \in \underline{\underline{N}} \hat{\otimes} \underset{\sim}{Z}_2[u,v]$ by $D(u,v) = \Sigma_{i,j}\, y_{i,j}\, u^i v^j$, so that $W_1\circ\mu = D(W_1 \hat{\otimes}_{\underline{\underline{N}}} 1,\ 1 \hat{\otimes}_{\underline{\underline{N}}} W_1)$.

Directly from the definition, by applying $\Phi\underline{w}\circ$ and using 6.4 we have:

<u>8.4</u> $D(u+a_1u^2 + a_2u^3 + a_3u^4 +\ldots,\ v+a_1v^2 + a_2v^3 + a_3v^4 +\ldots)$

$$= (u+v) + a_1(u+v)^2 + a_2(u+v)^3 + a_3(u+v)^4 + \ldots,$$

in which we have suppressed the inclusion $\Phi\underline{w}\circ : \underline{\underline{N}} \subset \underset{\sim}{Z}_2[a]$. Also, from the primitive element obtained in 8.1 we have:

<u>8.5</u> $D(u,v) + z_2D(u,v)^3 + z_4D(u,v)^5 + z_5D(u,v)^6 + \ldots$

$$= u + z_2u^3 + z_4u^5 + z_5u^6 + \ldots + v + z_2v^3 + z_4v^5 + z_5v^6 +\ldots$$

These formulae enable one to compute the elements $y_{i,j} \in \underline{\underline{N}}$ in terms of the z_i or the a_i, by a formidable algebraic computation. The first few terms are:

$$u+v+z_2(u^2v + uv^2) + z_4(u^4v + uv^4)+z_2^2(u^4v+u^3v^2+u^2v^3+uv^4)+ \ldots$$

Let us list the elementary properties of the diagonal function.

<u>8.6 Lemma</u> The <u>diagonal function</u> D has the properties:

(a) $D(u,v) = D(v,u)$,

(b) $D(u,v) = u + v +$ higher terms

(c) Every term in $D(u,v)$ except for u and v contains uv as a factor,

(d) $D(u,u) = 0$

(e) $u + v$ divides $D(u,v)$,

(f) $D(u,D(v,w)) = D(D(u,v),w)$.

Also, $y_{i,i} = 0$ for all i.

Proof (a), (d), and (f) follow from the corresponding properties of the comultiplication in

$$\underline{\underline{N}}^*(B\underset{\sim}{O}(1)^{O}).$$

(c) expresses the fact that this comultiplication has a counit. From (d), the coefficient $y_{i,i}$ of $u^i v^i$ in $D(u,v)$ is zero, which with (a) proves (e).]]]

It is possible to display explicit manifolds $Y_{i,j}$ whose cobordism classes are the coefficients $y_{i,j}$ appearing in 8.2. We recall the hypersurface

$$H_{m,n} \subset P_m \times P_n$$

from 2.10.

Some cobordism operations

8.8 Definition We define the class $W_{det} \in \underline{\underline{N}}^*(B\underset{\sim}{O}^{O})$ by

$$W_{det} = W_1 \circ det.$$

Then by 7.1 we have a cobordism operation $\Phi(W^r_{det})$ for each positive integer r. These are also bordism operations, since $\underset{=}{N}_*(X) = \{\Sigma^0, X\wedge \underset{\sim}{MO}\}_*$.

<u>8.9 Definition</u> We use the operation $\Phi(W^r_{det})$ to define the linear map $d_r : \underset{=}{N} \to \underset{=}{N}$, for each $r \geq 1$.

Geometrically we are considering the composites

$$\Sigma^0 \xrightarrow[x]{} \underset{\sim}{MO} \xrightarrow[\Delta]{} \underset{\sim}{BO}^0 \wedge \underset{\sim}{MO} \xrightarrow[\det\wedge 1]{} \underset{\sim}{BO}(1)^0 \wedge \underset{\sim}{MO} \xrightarrow[W_1^r \wedge 1]{} \underset{\sim}{MO}\wedge\underset{\sim}{MO} \xrightarrow[\varphi]{} \underset{\sim}{MO}.$$

We interpret d_r in terms of manifolds, as follows: We represent $x \in N$ by a manifold V, and let $V \to P_n \subset \underset{\sim}{BO}(1)$ be the classifying map of its orientation bundle, factored through P_n for some large n. Suppose V is transverse to $P_{n-r} \subset P_n$. Then $d_r x$ is represented by the inverse image in V of P_{n-r}.

Thus the operations d_1 and d_2 on $\underset{=}{N}$ are those introduced by Rokhlin [R1] and studied by Wall [W1], [W2]. As for the other operations, because the normal bundle of

P_{n-2} in P_n is orientable, and the normal bundle of P_{n-1} in P_n is classified by the inclusion $P_{n-1} \subset P_n \subset \underset{\sim}{BO}(1)$, we have the composition laws

<u>8.10</u> $$d_r d_2 = d_{r+2}, \text{ and } d_r d_1 = 0, \text{ for } r \geq 1.$$

A question posed in [W2] is the behaviour of d_1 and d_2 on products. We can now answer this.

<u>8.11 Theorem</u> d_1 and d_2 are not derivations on $\underline{\underline{N}}$; instead

$$d_1(uv) = \Sigma_{i,j}\, y_{i,j}\, d_i u.\, d_j v \quad \text{and}$$

$$d_2(uv) = \Sigma_{i,j}\, y_{i,j}^2\, d_{2i} u.\, d_{2j} v = \Sigma_{i,j}\, y_{i,j}^2\, d_2^i u.\, d_2^j v,$$

where the elements $y_{i,j} \in \underline{\underline{N}}$ are those defined in 8.2.

<u>Proof</u>. The naturality of the Thom isomorphisms yields the commutative diagram

$$\begin{tikzcd}
\underline{\underline{N}}^*(\underset{\sim}{MO}) \arrow[r, "\varphi^*"] & \underline{\underline{N}}^*(\underset{\sim}{MO}) \hat{\otimes}_{\underline{\underline{N}}} \underline{\underline{N}}^*(\underset{\sim}{MO}) \\
\underline{\underline{N}}^*(\underset{\sim}{BO}^0) \arrow[r, "\varphi^*"] \arrow[u, "\Phi"] & \underline{\underline{N}}^*(\underset{\sim}{BO}^0) \hat{\otimes}_{\underline{\underline{N}}} \underline{\underline{N}}^*(\underset{\sim}{BO}^0), \arrow[u, "\Phi \hat{\otimes} \Phi"']
\end{tikzcd}$$

from which we see that we have to find the coproduct under φ^* of W_{det}. From another commutative diagram

$$\begin{tikzcd}
\underset{\sim}{BO} \times \underset{\sim}{BO} \arrow[r, "\det \times \det"] \arrow[d, "\varphi"] & \underset{\sim}{BO}(1) \times \underset{\sim}{BO}(1) \arrow[d, "\mu"] \\
\underset{\sim}{BO} \arrow[r, "\det"] & \underset{\sim}{BO}(1)
\end{tikzcd}$$

we deduce that

$$\begin{aligned}\varphi^* W_{det} &= W_1 \circ \det \circ \varphi \\ &= W_1 \circ \mu \circ (\det \times \det) \\ &= (\Sigma_{i,j}\, y_{i,j}\, W_1^i \,\hat{\otimes}_{\underline{\underline{N}}}\, W_1^j) \circ (\det \times \det) \\ &= \Sigma_{i,j}\, y_{i,j}\, W^i_{det} \,\hat{\otimes}_{\underline{\underline{N}}}\, W^j_{det}.\end{aligned}$$

Hence, and by squaring, the required formulae.]]]

Both the formulae in 8.11 contain infinitely many potentially non-zero terms, since $d_2 : \underline{\underline{N}} \to \underline{\underline{N}}$ is epi (Theorem 1 in [W2]), and hence by 8.10 d_{2n} is epi for all n.

<u>Remark</u> The Wall subalgebra $\underline{\underline{W}}$ of $\underline{\underline{N}}$ [W1] [W2], is defined as the kernel of d_2. 8.11 shows that $\underline{\underline{W}}$ is a subalgebra. In the formula for $d_1(uv)$, all the terms except $d_1u.v$ and $u.d_1v$ vanish if u and v are in $\underline{\underline{W}}$ (since $y_{1,1} = 0$ by 8.6), so that we recover the result that d_1 is a derivation on $\underline{\underline{W}}$.

Some values of d_1 and d_2 are particularly easy to find. It is clear geometrically that

$$d_1[P_{2n}] = [P_{2n-1}] = 0, \text{ by } 6.8,$$
$$d_2[P_{2n}] = [P_{2n-2}].$$

Also that $H_{2m,2n}$ is orientable, which gives

$$d_1[H_{2m,2n}] = 0, \text{ and}$$
$$d_2[H_{2m,2n}] = 0.$$

Further, the orientation of $H_{2m,2n+1}$ is induced by the projection $H_{2m,2n+1} \to P_{2n+1} \subset P_\infty = B\underset{\sim}{O}(1)$, so that

$$d_1[H_{2m,2n+1}] = [H_{2m,2n}], \text{ and}$$
$$d_2[H_{2m,2n+1}] = [H_{2m,2n-1}] \text{ provided } m \neq n.$$

By 6.10 we have here enough manifolds to generate $\underline{\underline{N}}$.

§9. Computation of the bordism J-homomorphism

In V.§9 we introduced the bordism J-homomorphisms

$$J_n:\underline{\underline{N}}_i(B\underset{\sim}{Q}(n)^0) \to \underline{\underline{N}}_{i+n-1}(B\underset{\sim}{Q}(1)^0)$$

defined in [C4][C5] by Conner and Floyd.

We interpreted them in terms of transfer homomorphisms. For technical reasons we introduced the corresponding cobordism J-homomorphisms

$$J^n:\underline{\underline{N}}^i(B\underset{\sim}{Q}(1)^0) \to \underline{\underline{N}}^{i-n+1}(B\underset{\sim}{Q}(n)^0)$$

which we defined V.9.7 in terms of transfer homomorphisms. They are therefore dual to the homomorphisms J_n.

In this section we compute J^n, which of course determines J_n. Since it is $\underline{\underline{N}}$-linear, it will be enough to find $J^n\, W_1^r$. Further, we lose no information if we compose with $\rho:B\underset{\sim}{Q}(n) \to B\underset{\sim}{Q}(n)$, by V.8.4. In order to express the result, we need some more cobordism classes in $\underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0) = \underline{\underline{N}} \hat{\otimes} \underset{\sim}{Z}_2[T_1,T_2,\ldots,T_n]$. Define $T_{ij} = \mu_{ij}^* W_1$ for any distinct integers i and j $(1 \leq i,\ j \leq n)$, where μ_{ij} is the composite map

$$\mu_{ij}:B\underset{\sim}{Q}(n) = B\underset{\sim}{Q}(1) \times B\underset{\sim}{Q}(1) \times \ldots \times B\underset{\sim}{Q}(1) \to B\underset{\sim}{Q}(1)\times B\underset{\sim}{Q}(1) \xrightarrow[\mu]{} B\underset{\sim}{Q}(1),$$

where the centre map is the projection of the product to its i th and j th factors. Algebraically,

9.1 $$T_{ij} = D(T_i,T_j),$$

where D is the diagonal function defined in 8.3.

<u>9.2 Theorem</u> The homomorphism $\rho^* J^n : \underline{\underline{N}}^*(B\underset{\sim}{O}(1)^0) \to \underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$ is given by

$$\rho^* J^n \, W_1^r = \frac{T_1^r}{T_{12}T_{13}\cdots T_{1n}} + \frac{T_2^r}{T_{12}T_{23}\cdots T_{2n}} + \cdots + \frac{T_n^r}{T_{1n}T_{2n}\cdots T_{n-1,n}}$$

for any integer $r \geqslant 0$.

<u>Remark</u>. This formula demands some explanation; $J^n \, W_1^r$ is apparently in the field of fractions of $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$ rather than $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$ itself. We can put everything over the common denominator $\Pi_{i<j} \, T_{ij}$. By 8.6 (b) and (e) $(T_i+T_j)/T_{ij}$ is a respectable element of $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0)$. Then we need only show that the numerator of our fraction has $\Pi_{i<j} \, (T_i+T_j)$ as a factor. Symmetry shows this to be so.

According to V.9.7 we are considering the Borel fibration of the inclusion $\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1) \subset \underset{\sim}{O}(n)$, and that induced from it by $\rho : B\underset{\sim}{Q}(n) \to B\underset{\sim}{O}(n)$. These form the commutative diagram of bundles

$$\begin{tikzcd}
\underset{\sim}{O}(n)//\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1) \arrow[r] \arrow[d, equal] & E \arrow[r, "\pi'"] \arrow[d] & B\underset{\sim}{Q}(n) \arrow[d] \\
\underset{\sim}{O}(n)//\underset{\sim}{O}(n-1) \times \underset{\sim}{O}(1) \arrow[r] & B\underset{\sim}{O}(n-1) \times B\underset{\sim}{O}(1) \arrow[r, "\pi"] & B\underset{\sim}{O}(n).
\end{tikzcd} \tag{9.3}$$

We shall also write W_1, T_i, t_i, etc. for the induced classes in $\underline{\underline{N}}^*(E^0)$ and $H^*(E^0)$. Then by V.9.7 and naturality $J^n \, W_1^r = \pi'_4 \, W_1^r$.

To compute $\pi'_4 \, W_1^r$, we use the Riemann-Roch theorem V.7.11

with respect to the natural transformation $\Phi\underline{w}^\circ : \underline{N}^* \to H^*(\ ; Z_2[a])$. This gives

9.4 $$\Phi\underline{w}^\circ\ \pi'_! W_1^r = \pi'^H_!\{\hat{a}(\pi').\ \Phi\underline{w}^\circ\ W_1^r\},$$

where $\pi'^H_!$ denotes the transfer in ordinary cohomology. We have to find $\hat{a}(\pi')$ and the cohomology transfer $\pi'^H_!$; this will be enough because $\Phi\underline{w}^\circ$ is mono.

Now by 6.6 $\hat{a}(\pi') = \underline{w}(\tau)^{-1}$, where τ is the bundle over E of tangents along the fibre. Fortunately, Borel and Hirzebruch have already computed $w(\tau)$ in [B3]; the answer is $w(\tau) = (1+w_1)^n + v_1(1+w_1)^{n-1} + v_2(1+w_1)^{n-2} + \ldots + v_n$, where we write v_i for the cohomology class induced from the i th Stiefel-Whitney class over $BO(n)$. Expressed in terms of the t_i, this simplifies to

$$w(\tau) = \Pi_{i=1}^{i=n}\ (1 + w_1 + t_i).$$

The expansion lemma 2.6 therefore yields

9.5 $$\hat{a}(\pi') = \Pi_{i=1}^{i=n} \{1 + a_1(w_1+t_i) + a_2(w_1+t_i)^2 + .. \}^{-1}.$$

It remains to find $\pi'^H_!$.

9.6 Lemma $H^*(E^0)$ is generated by w_1 and the t_i, subject to the single relation

$$(w_1 + t_1)(w_1 + t_2) \ldots (w_1 + t_n) = 0.$$

All the differentials in the spectral sequence of π' vanish.

Proof This is proved by Borel [B2].]]]

For dimensional reasons, $\pi'^H_!\ w_1^r = 0$ if $r < n-1$. From the spectral sequence definition V.6.14 of the transfer $\pi'^H_!$, we see that $\pi_{\natural}^{!H}\, w_1^{n-1} = 1$. Since the w_1^i for $0 \leqslant i \leqslant n-1$ generate $H^*(E^0)$ as a $\underset{\sim}{Z}_2[t_1, t_2, \ldots, t_n]$-module, and with our identifications $\pi_{\natural}^{!H}$ is a module homomorphism, these values determine $\pi_{\natural}^{!H}$ completely.

Unfortunately direct substitution in 9.4 is not practicable. We must first express $\pi_{\natural}^{!H}$ in a different form.

<u>9.7 Lemma</u> The bundle map π' has canonical sections $\sigma_i : B\underset{\sim}{Q}(n) \to E$ $(1 \leqslant i \leqslant n)$, which induce $\sigma_i^* w_1 = t_i$. In terms of these, the transfer $\pi_{\natural}^{!H}$ is given by

$$\pi_{\natural}^{!H}\, \alpha = \Sigma_{i=1}^{i=n} \quad \frac{\sigma_i^*\, \alpha}{\Pi_{j \neq i}\ (t_i + t_j)} \qquad (\alpha \in H^*(E^0)).$$

<u>Proof</u>. The universal vector bundle over $B\underset{\sim}{Q}(n)$ splits as the Whitney sum of n line bundles. Each of these yields a section σ_i of the projective bundle π'. Clearly $\sigma_i^* w_1 = t_i$.

The rest is algebraic. Both $\pi_{\natural}^{!H}$ and $\Sigma_i\ \Pi_{j \neq i}\ (t_i + t_j)^{-1}\ \sigma_i^*$ are homomorphisms of $\underset{\sim}{Z}_2[t_1, t_2, \ldots, t_n]$-modules, taking values in the field of fractions of $\underset{\sim}{Z}_2[t_1, t_2, \ldots, t_n]$, which fortunately has no zero divisors. It is clear that they agree on the elements $\alpha_i = \Pi_{j \neq i}\ (w_1 + t_j)$. But for any $\alpha \in H^*(E^0)$ there exist elements $x, x_1, x_2, \ldots, x_n \in \underset{\sim}{Z}_2[t_1, t_2, \ldots, t_n]$ such that $x\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$, with $x \neq 0$. This shows that the two homomorphisms coincide.]]]

We can now substitute 9.7 and 9.5 in 9.4 to obtain the theorem.

<u>Proof of 9.2.</u> We recall from 4.7 the notation

$$y(\theta) \equiv \theta + a_1\theta^2 + a_2\theta^3 + a_3\theta^4 + \dots\dots$$

$$x(\theta) \equiv 1 + a_1\theta + a_2\theta^2 + a_3\theta^3 + \dots$$

We have from 6.4 and 9.5

$$\hat{a}(\pi').\ \Phi\underline{w} \circ W_1^r = y(w_1)^r.\ \Pi_j\ x(w_1+t_j)^{-1}.$$

Therefore from 9.4 and 9.7 we deduce (using $x(t_i+t_i) = 1$)

$$\begin{aligned}\Phi\underline{w} \circ \pi_4^! W_1^r &= \Sigma_i\ y(t_i)^r.\Pi_{j\neq i}\ x(t_i+t_j)^{-1}.\ (t_i+t_j)^{-1} \\ &= \Sigma_i\ y(t_i)^r.\Pi_{j\neq i}\ y(t_i+t_j)^{-1} \\ &= \Phi\underline{w} \circ \Sigma_i\ T_i^r.\ \Pi_{j\neq i}\ T_{ij}^{-1} \qquad \text{by 6.5 and 9.1.}\end{aligned}$$

Since $\Phi\underline{w}\ \circ$ is mono by 6.3, the theorem follows.]]]

The result 9.2 is still not convenient for calculation. We need to express it in another form. First, we note from V.9.9 that $J^nW_1^r = i^*J^{n+1}W_1^{r+1}$, where $i:B\underset{\sim}{Q}(n) \to B\underset{\sim}{Q}(n+1)$. Also, $\underline{\underline{N}}^*(B\underset{\sim}{Q}^0) = \varprojlim \underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0)$, under the homomorphisms i^*.

<u>9.8 Definition</u> For any integer i, positive or negative, we define $\eta_i \in \underline{\underline{N}}^{-i}(B\underset{\sim}{Q}^0)$ as the inverse limit of the elements $J^nW_1^{n-i-1} \in \underline{\underline{N}}^{-i}(B\underset{\sim}{Q}(n)^0)$ (which are defined for sufficiently large n). We define the Laurent series η by

$$\eta(\theta) = \Sigma_i\ \eta_i\theta^i.$$

This Laurent series η contains all the information about all the homomorphisms J^n.

We can also write the cobordism transfer homomorphism

$$\pi_4^!\ :\underline{\underline{N}}^*(E^0) \to \underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0)$$

in a form very similar to that in 9.7 for the cohomology transfer.

<u>9.9 Lemma</u> The sections σ_i of the bundle map π' induce $\sigma_i^* W_1 = T_i$. In terms of these, the cobordism transfer $\pi_\natural^!$ is given by

$$\pi_\natural^! \quad \beta = \Sigma_{i=1}^{i=n} \frac{\sigma_i^* \beta}{\Pi_{j \neq i} T_{ij}} \qquad (\beta \in \underline{\underline{N}}^*(E^O)).$$

<u>Proof</u>. This is exactly parallel to that of 9.7; we have two module homomorphisms that agree on the elements W_1^r by arrangement, and therefore generally.]]]

For any $m \leqslant n$, V.9.11 yields the commutative diagram

$$\begin{array}{ccccc}
\underline{\underline{N}}^*(B\underset{\sim}{Q}(1)^O) & \xrightarrow[\alpha .]{} & \underline{\underline{N}}^*(E^O) & & \\
\downarrow J^m & & \downarrow \pi_\natural^! & & \\
\underline{\underline{N}}^*(B\underset{\sim}{Q}(m)^O) & \xrightarrow[p^*]{} & \underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^O) & \xrightarrow[j^*]{} & \underline{\underline{N}}^*(B\underset{\sim}{Q}(k)^O),
\end{array}$$

where $n = m + k$, $p : B\underset{\sim}{Q}(n) \to B\underset{\sim}{Q}(m)$ denotes the projection to the <u>first</u> m factors, and $j : B\underset{\sim}{Q}(k) \subset B\underset{\sim}{Q}(n)$ denotes the inclusion of the <u>last</u> k factors. In view of §8, the class

$$\alpha \in \underline{\underline{N}}^*(E^O)$$

now reveals itself as

$$\alpha = D(T_{m+1}, W_1) D(T_{m+2}, W_1) \ \ldots \ D(T_n, W_1).$$

Let us write α as a power series $\Sigma\ \alpha_i\ W_1^i$, with coefficients in $\underline{\underline{N}} \hat{\otimes} \underset{\sim}{Z}_2[T_{m+1}, T_{m+2}, \ \ldots\ , T_n] = \underline{\underline{N}}^*(B\underset{\sim}{Q}(k)^O)$.

We evaluate the commutativity on the element W_1^r. Now

$$j^* \pi_\natural^! \ (\alpha . W_1^r) = j^* \pi_\natural^! \ (\Sigma_s\ \alpha_s\ W_1^{r+s}) = \Sigma_s \alpha_s\ \eta_{n-r-s+1}.$$

On the other hand, $pj: BQ(k) \to BQ(m)$ is the constant map, so that W_1^r goes to

$$\langle z, J^m W_1^r \rangle\, 1 \in \underline{\underline{N}}^*(BQ(k)^0),$$

where z is the bordism class of a point in $BQ(m)$. We can rewrite $\langle z, J^m W_1^r \rangle$ as $\langle J_m z,\ W_1^r \rangle$. By definition, $J_m z$ is the class of the singular manifold

$$P_{m-1}\ (R) \subset P\infty = BQ(1).$$

Hence by V.9.3

$$\langle J_m z,\ W_1^r \rangle = [P_{m-r-1}(R)] \in \underline{\underline{N}}.$$

We therefore have

9.10 $$\Sigma_s\ \alpha_s\ \eta_{n-r-s+1} = [P_{m-r-1}(R)].1 \text{ in } \underline{\underline{N}}^*(BQ(k)^0).$$

To express this result in concise form, we introduce two more power series. First, we note that it is possible to multiply any two homogeneous Laurent series over $\underline{\underline{N}}^*(BQ(k)^0)$ in an indeterminate θ of codegree 1, so they form a ring. In this ring $D(T_i, \theta)$ is invertible, by 8.6,

9.11 Definition We define the Laurent series ζ over $\underline{\underline{N}}^*(BQ^0)$ as the series whose restrictions to each $\underline{\underline{N}}^*\ (BQ(n)^0)$ is

$$\Pi_{i=1}^{i=n}\ \theta/D(T_i, \theta).$$

We also need the power series over $\underline{\underline{N}}$

9.12 $$p(\theta) = 1 + [P_2(R)]\theta^2 + [P_4(R)]\theta^4 + [P_6(R)]\theta^6 + \ldots.$$

(We do not need the odd terms, because they vanish by 6.8.)

9.13 Theorem We have

$$\eta(\theta) = p(\theta).\ \zeta(\theta),$$

an identity of Laurent series over $\underline{\underline{N}}^*(B\underset{\sim}{Q}^0)$.

Proof. It is sufficient to consider the restrictions to $\underline{\underline{N}}^*(B\underset{\sim}{Q}(k)^0)$. Write $\alpha(\theta) = \Sigma_s\ \alpha_s\ \theta^s$. Then we showed in 9.10 that over $\underline{\underline{N}}^*(B\underset{\sim}{Q}(k)^0)$ we have $\alpha(\theta).\eta(\theta) = p(\theta).\theta^k$. But by the definition of α,

$$\theta^k/\alpha(\theta) = \theta^k/D(T_{m+1},\theta)D(T_{m+2},\theta)\ldots D(T_n,\theta),$$

which is the restriction of $\zeta(\theta)$.]]]

In the remainder of this section we translate the results on the cobordism J-homomorphisms back into results on the bordism J-homomorphisms. At the same time we can use the precise information to obtain better theorems.

9.14 Definition We define

$$J:\underline{\underline{N}}_*(B\underset{\sim}{Q}^0) \to \text{the graded ring of Laurent series over } \underline{\underline{N}}$$

by setting $Jx = \left\langle x,\ \eta(\theta)\right\rangle$, where (θ) is the Laurent series defined in 9.8. More directly, we have

9.15 $$Jx = \Sigma_i \left\langle J_{n(i)}x,\ w_1^{n(i)-i-1}\right\rangle\theta^i$$

in which we choose for each i an integer n(i) such that $n(i) \geqslant i + 1$ and x lifts to $\underline{\underline{N}}_*(B\underset{\sim}{Q}(n(i))^0)$.

Let us recall that $\underline{\underline{N}}_*(B\underset{\sim}{Q}^0)$ has a multiplication induced by $\varphi:B\underset{\sim}{Q} \times B\underset{\sim}{Q} \to B\underset{\sim}{Q}$.

9.16 Lemma $$\underline{\underline{N}}_*(B\underset{\sim}{Q}^0) = \underline{\underline{N}}[b_1,b_2,b_3,\ \ldots\ ]$$

is a graded polynomial ring on generators b_i of degree i, such that the $\underline{\underline{N}}$-universal element $\in \underline{\underline{N}}_*(BQ^0) \hat{\otimes} \underline{\underline{N}}^*(BQ^0)$ restricts to

$$\Pi_{i=1}^{i=n} (1+b_1T_i+b_2T_i^2+b_3T_i^3+\ldots) \in \underline{\underline{N}}[b_1,b_2,b_3,\ldots] \hat{\otimes}\underline{\underline{N}}^*(BQ(n)^0).$$

Proof This follows from §7.

It is inconvenient to have Laurent series here (e.g. $Jb = \theta^{-1}$). We avoid them by forcing the image of J to have degree zero.

9.17 Definition We put $F = \Sigma_i \underline{\underline{N}}_i(BQ^0)$, so that F is an ordinary ungraded polynomial ring over Z_2. We filter F by the subgroups F_n defined by $F_n = \Sigma_{i=0}^{i=n} \underline{\underline{N}}_i(BQ^0)$. Then we define

$$J':F \to \underline{\underline{N}}[[\theta]]$$

by linearity, from $J'x = \theta^i .Jx$. whenever $x \in \underline{\underline{N}}_i(BQ^0)$.

We now find from 9.15 that

$$J'1 = 1 + [P_2]\theta^2 + [P_4]\theta^4 + [P_6]\theta^6 + \ldots = p(\theta).$$

9.18 Definition We normalize the homomorphism J' by putting

$$J''x = p(\theta)^{-1}.J'x.$$

Then we have arranged $J''1 = 1$. We can now state the main theorem of this section, in which $\underline{\underline{N}} = Z_2[z_2,z_4,z_5,\ldots]$ as in 8.1.

9.19 Theorem $J'':F \to \underline{\underline{N}}[[\theta]]$ is an injective ring homomorphism. Further, we can find polynomial generators e_i' for F (i not of the form 2^k-1) such that

(a) $F = \mathbb{Z}_2[e'_2, e'_4, e'_5, e'_6, e'_8, \ldots]$,

(b) e'_{2i} and e'_{2i+1} lie in F_i,

(c) $J''e'_i = z_i\theta^i$ + terms with higher powers of θ.

It follows from (b) that F_n consists of all polynomials in the e'_i with weight $\leqslant n$, where we assign weight i to e'_{2i} and e'_{2i+1}.

<u>9.20 Corollary</u> J'' induces $J'':F^\wedge \cong \underline{\underline{N}}[[\theta]]_0$, where $^\wedge$ denotes completion with respect to a suitable filtration of F, and we take the 0-dimensional subring of $\underline{\underline{N}}[[\theta]]$.]]]

A direct geometric proof that J'' is a ring homomorphism would be desirable. Also a direct description of the filtration on F determined by J''.

By definition 9.18, 9.13 and 9.14, we have

<u>9.21</u> $$J''x = \left\langle x\theta^i, \zeta(\theta)\right\rangle \qquad (x \in \underline{\underline{N}}_i(BO^0)).$$

It follows from the form of $\zeta(\theta)$ given in 9.11 that J'' is a ring homomorphism. Now $\zeta(\theta)$ is expressed in terms of the diagonal function $D(T,\theta) \in \underline{\underline{N}}[[T,\theta]]$; let us write, by 8.6,

<u>9.22</u> $$\theta/D(T,\theta) = 1 + B_1T/\theta + B_2T^2/\theta^2 + B_3T^3/\theta^3 + \ldots.,$$

where $B_i \in \underline{\underline{N}}[[\theta]]$.

<u>9.23 Lemma</u> $J''b_i = B_i$, $J''z_i = z_i\theta^i$.

<u>Proof</u> It is immediate from 9.16 that $J''b_i = B_i$. The $\underline{\underline{N}}$-linearity of J and $J''1 = 1$ yield $J''z_i = z_i\theta^i$.]]]

We must therefore study the formal power series B_i. Now the defining relation 8.5 of $D(T,\theta)$ suggests working with the dual polynomial generators of F. Let us write

9.24 $$D(T,\theta)/\theta = 1 + T/\theta + C_1T/\theta + C_2T^2/\theta^2 + C_3T^3/\theta^3 + \cdots$$
$$(1+b_1\theta+b_2\theta^2+b_3\theta^3+\cdots)^{-1} = 1 + \theta + c_1\theta + c_2\theta^2 + c_3\theta^3 + \cdots,$$
where $C_i \in \underline{\underline{N}}[[\theta]]$ and $c_i \in F$, and we have inserted extra terms T/θ and θ. It is immediate from 9.23 that $J''c_i = C_i$. The point of inserting the extra term T/θ is that now $C_i \in q$ for all i, where $q = \mathrm{Ker}(\underline{\underline{N}}[[\theta]] \to \mathbb{Z}_2[[\theta]])$ is the augmentation ideal generated by the z_i.

We now work in formal algebra, modulo q^2, and replace T in 9.24 by $\lambda\theta$, so that λ has degree 0. We have
$$D(\lambda\theta,\theta) = \theta.(1+\lambda + \lambda C_1 + \lambda^2C_2 + \lambda^3C_3 + \cdots).$$
The defining relation 8.5 for D, modulo q^2, simplifies to

9.25 $$\Sigma_{j>0}\ z_j\theta^j\ \{ (1+\lambda)^{j+1}+\lambda^{j+1}+1 \} +\lambda C_1 + \lambda^2C_2 +\lambda^3C_3 + \cdots \equiv 0.$$

Consider the quadratic equation $\lambda^2+\lambda = \rho$. If λ is one root, the other is $1+\lambda$, and the expression $(1+\lambda)^{i+1}+\lambda^{i+1}+1$, being symmetric, is a polynomial in ρ. Moreover, we can solve this equation in the ring $\mathbb{Z}_2[[\rho]]$, by setting
$$\lambda = \rho + \rho^2 + \rho^4 + \rho^8 + \cdots$$

9.26 Lemma Write $f_n(\rho) = (1+\lambda)^n + \lambda^n + 1$. Then

(a) $f_n(\rho)$ is polynomial in ρ, of degree $< \frac{1}{2}n$,

(b) $f_{2n+1}(\rho) = \rho^n$ + lower terms.

Proof We consider the generating function of the f_n.

$$\Sigma_{n\geq 0} \; [1+f_n(\rho)] \; \theta^n = \Sigma \quad (1+\lambda)^n\theta^n + \lambda^n\theta^n$$
$$= 1/(1+(1+\lambda)\theta) + 1/(1+\lambda\theta)$$
$$= \theta/(1+\theta+\rho\theta^2)$$
$$= \theta\{ \; 1+(\theta+\rho\theta^2) + (\theta+\rho\theta^2)^2+(\theta+\rho\theta^2)^3+..\} \; .$$

We obtain the result by picking out the coefficient of θ^n.]]]

We next work in the ring $\underset{\sim}{Z}_2[[\sigma]]$, and make the substitutions

9.27 $\rho = \sigma^2,$

$\mu = \sigma + \sigma^2 + \sigma^4 + \sigma^8 + \ldots,$

$\lambda = \mu^2 = \sigma^2 + \sigma^4 + \sigma^8 + \ldots$

9.28 Definition We define new elements $e_i \in F$ and $E_i \in \underline{\underline{N}}[[\theta]]$ for $i > 0$ by the identities

$$\Sigma_i \; e_i\sigma^{i+1} = (c_1\lambda+c_2\lambda^2+.. \;)\mu + \Sigma_j \; z_j\lambda^{j+1} \in F[[\sigma]],$$
$$\Sigma_i \; E_i\sigma^{i+1} = (C_1\lambda + C_2\lambda^2+.. \;)\mu + \Sigma_j \; z_j\theta^j \; \lambda^{j+1} \in \underline{\underline{N}}[[\theta]][[\sigma]]$$

where we sum over those j not of the form 2^k-1.

It is immediate from the similarity between these formulae that we still have $J''e_i = E_i$ for all i.

In the first formula of 9.28, the terms having odd powers of σ yield

$$\Sigma \; e_{2i}\rho^i = c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + \ldots.,$$

and the terms having even powers of θ yield

$$\begin{aligned}\Sigma \; e_{2i+1}\rho^{i+1} &= (c_1\lambda+c_2\lambda^2+c_3\lambda^3+..)(\sigma^2+\sigma^4+\sigma^8+..)+\Sigma \; z_j\lambda^{j+1} \\ &= (c_1\lambda+c_2\lambda^2+c_3\lambda^3+..)\lambda + \Sigma \; z_j\lambda^{j+1}.\end{aligned}$$

Thus $e_{2i} = c_i + \dots$, and $e_{2i+1} = z_i + \dots$. (if i is not of the form 2^k-1). We have

9.29 Lemma $F=\underset{\approx}{Z}_2[e_2,e_4,e_5,e_6,e_8,\dots]$, $J''e_i = E_i$, and F_n consists of the polynomials in the e_i of weight $\leqslant n$, if we assign weight i to e_{2i} and e_{2i+1}.]]]

These are almost the required generators of F. Let $m \subset \underset{=}{N}[[\theta]]$ be the augmentation ideal generated by θ.

9.30 Lemma For all i not of the form 2^k-1, we have

$$E_i = z_i\theta^i \qquad \text{mod } (q^2+m^{i+1}).$$

Proof We work mod q^2, to begin with. If we substitute 9.25 into 9.28 we find

$$\begin{aligned}\Sigma_i\ E_i\sigma^{i+1} &\equiv \Sigma_j\ z_j\theta^j\ \{(1+\lambda)^{i+1}+\bar{\lambda}^{i+1}+1\} + \Sigma\ z_j\theta^j\lambda^{j+1}\\ &\equiv \Sigma_j\ z_j\theta^j\ \{(1+\mu)^{2j+1}(1+\mu)\mu+\mu^{2j+1}.\mu^2+\mu+\mu^{2j+1}\}\\ &\equiv \Sigma_j\ z_j\theta^j\ \{[(1+\mu)^{2j+1}+\mu^{2j+1}+1](\mu^2+\mu)+\mu^2\}\\ &\equiv \Sigma_j\ z_j\theta^j\ \{f_{2j+1}(\sigma).\sigma + \mu^2\}\end{aligned}$$

If we now pick out the coefficient of σ^{j+1}, we find the result by 9.26.]]]

Proof of 9.19 Modulo q^2, we have 9.19 from 9.29 and 9.30. We need to alter each e_i (i not of the form 2^k-1) to e_i' by adding polynomials in the previous generators e_j' to cancel the unwanted terms in $J''e_i$, in such a way that

(a) $J''e_i' = z_i\theta^i$ modulo m^{i+1}

(b) e_{2i}' and e_{2i+1}' lie in F_i.

There is no difficulty here.]]]

§10. Manifolds with involution

In this section we apply our results on the bordism J-homomorphism to the study of smooth involutions on manifolds. We consider particularly non-bounding n-manifolds with a smooth involution whose fixed point sets have dimension $\leq k$. Conner and Floyd in Theorem 27.1 of [C5] gave a highly non-constructive proof that for k given, n could not be arbitrarily large. We show that $n \leq 5k/2$, the best possible result.

We recall from [C5] the elementary results on the bordism theory of manifolds with involution. Let M be a compact n-manifold without boundary with a smooth involution ω. Let F be the fixed-point set of ω. Then F is the disjoint union of submanifolds F_i of M, where F_i has dimension i. Each F_i has a tubular neighbourhood N_i on which ω is the antipodal map in each fibre; the N_i may be assumed disjoint. The classifying map of the normal bundle of F_i in M yields an element $\nu_i \in \underline{\underline{N}}_i(B\underset{\sim}{O}(n-i)^0)$. Let I_n be the cobordism group of manifolds with involution. The main result is that the elements ν_i are cobordism invariants, subject to the single relation

<u>10.1</u> $$\Sigma_i \; J_{n-i} \, \nu_i = 0 \quad \text{in } \underline{\underline{N}}_{n-1}(B\underset{\sim}{O}(1)^0),$$

and characterize the cobordism class of (M,ω). In other words, we have the short exact sequence (28.1) of [C5]

<u>10.2</u> $$0 \to I_n \to \Sigma_{i=0}^{i=n} \; \underline{\underline{N}}_i(B\underset{\sim}{O}(n-i)^0) \xrightarrow[J]{} \underline{\underline{N}}_{n-1}(B\underset{\sim}{O}(1)^0) \to 0,$$

where J is the sum of the homomorphisms J_{n-i}.

The cobordism class [M] is obviously a cobordism invariant, and is expressed in terms of the ν_i by the formula (24.2) of [C5]

<u>10.3</u> $$[M] = q_* \Sigma_{i=0}^{i=n} J_{n-i+1} (j_* \nu_i), \text{ in } \underline{\underline{N}},$$

where $j:B\underset{\sim}{O}(n-i) \to B\underset{\sim}{O}(n-i+1)$, and $q:B\underset{\sim}{O}(1) \to$ Point.

We combine these results and express them in terms of the homomorphism $J':F \to \underline{\underline{N}}[[\theta]]$, introduced in 9.17, where $F = \Sigma_i \underline{\underline{N}}_i(B\underset{\sim}{O}^0)$.

<u>10.4 Definition</u> We define $\nu_M \in F$ by

$$\nu_M = \nu_0 + \nu_1 + \nu_2 + \dots \nu_n,$$

where we include the normal invariants ν_i in $\underline{\underline{N}}_*(B\underset{\sim}{O}^0)$.

Then from 9.15 we have

<u>10.5 Theorem</u> The formulae 10.1 and 10.3 are combined in either of the formulae

$$J'\nu_M = [M]\theta^n + \text{terms with higher powers of } \theta,$$

$$J''\nu_M = [M]\theta^n + \text{terms with higher powers of } \theta.$$

<u>Proof</u>. The first follows from 9.15, 9.17, 10.1, and 10.3. The second is equivalent ot the first by the definition 9.18 of J".]]]

<u>10.6 Corollary</u> If the fixed-point sets of the involution on M have dimension $\leq k$, and M does not bound, then n cannot be arbitrarily large.

<u>Proof</u>. This is (27.1) of [C5]. For a given k, there are only finitely many relevant elements ν_M.]]]

<u>10.7 Definition</u> Given k, let $\varphi(k)$ be the maximum dimension of a non-bounding manifold carrying an involution whose fixed-point sets have dimension $\leqslant k$. This exists by 10.6.

The problem of determining $\varphi(k)$ is reduced by 10.5 to the computation of J'', which we carried out in §9, apart from the question of lifting elements of $\underline{\underline{N}}_*(B\underset{\sim}{O}^0)$ to $\underline{\underline{N}}_*(B\underset{\sim}{O}(n-i)^0)$ (which turns out to be irrelevant).

We next give some examples of manifolds with involution.

<u>Examples</u>

(a) We can give $V \times V$ the involution interchanging the factors. The fixed-point set is the diagonal. Thus $\varphi(n) \geqslant 2n$, except possibly when n=1 or n=3.

(b) On a complex algebraic variety with real coefficients, e.g. $H_{m,n}(\underset{\sim}{C})$ or $P_m(\underset{\sim}{C})$, we have the involution given by complex conjugation. The fixed-point sets are e.g. $H_{m,n}(\underset{\sim}{R})$ or $P_m(\underset{\sim}{R})$.

(c) Let V and W be manifolds with involution. We can give $V \times W$ the product involution, $\omega(x,y) = (\omega x, \omega y)$. If the fixed-point sets in V and W are F and G, the fixed-point set in $V \times W$ is $F \times G$. Hence $\varphi(m+n) \geqslant \varphi(m) + \varphi(n)$, since $\underline{\underline{N}}$ has no zero divisors.

(d) Take coordinates (x_0, x_1, x_1') on $P_2(\underset{\sim}{R})$, and involution given by $\omega(x_0, x_1, x_1') = (x_0, x_1, -x_1')$. The fixed-point sets are P_1 $(x_1' = 0)$ and P_0 $(x_0 = x_1 = 0)$. Hence $\varphi(1) \geqslant 2$, and by (c), $\varphi(n) \geqslant 2n$.

(e) Take coordinates (y_0,y_1,y_2,y_1',y_2') on $P_4(\underset{\sim}{R})$, and the involution given by

$$\omega(y_0,y_1,y_2,y_1',y_2') = (y_0,y_1,y_2,-y_1',-y_2').$$

The fixed-point sets are P_2 $(y_1' = y_2' = 0)$ and P_1 $(y_0 = y_1 = y_2 = 0)$.

(f) Consider the product involution (c) on $P_2 \times P_4$ obtained from the involutions (d) and (e). The hypersurface $H = H_{2,4} \subset P_2 \times P_4$ defined by $x_0y_0 + x_1y_1 + x_1'y_1' = 0$ is taken into itself by ω. The fixed-point sets of $\omega|H$ are $H_{1,2} \subset P_1 \times P_2$, $P_1 \times P_1$, $P_0 \times P_2$, and $H_{0,1} \subset P_0 \times P_1$. Thus $\varphi(2) \geqslant 5$, and we have, from (c) and (d),

<u>10.8</u> $$\varphi(2n) \geqslant 5n, \quad \varphi(2n+1) \geqslant 5n+2.$$

(g) More generally than (f), consider

$$H_{2i,2j} \subset P_{2i}(\underset{\sim}{R}) \times P_{2j}(\underset{\sim}{R}),$$

with coordinates $(x_0,x_1,x_2, \dots ,x_i,x_1',x_2', \dots ,x_i')$ on P_{2i}, and $(y_0,y_1, \dots y_j,y_1',\dots ,y_j')$ on P_{2j}. We suppose $i \leqslant j$, and that $H = H_{2i,2j}$ is defined by

$$x_0y_0 + x_1y_1 + \dots+x_iy_i + x_1'y_1' +\dots+x_i'y_i' = 0.$$

We condiser the involutions on P_{2i} and P_{2j} given by

$$\omega(x_0,x_1,\dots , x_i,x_1', \dots ,x_i') = (x_0,x_1,\dots , x_i.-x_1',\dots ,-x_i'),$$
$$\omega(y_0,y_1,\dots , y_j,y_1',\dots ,y_j') = (y_0,y_1,\dots ,y_j,-y_1',\dots ,-y_j').$$

The fixed-point sets in P_{2i} and P are P_i and P_{i-1}, and those in

P_{2j} are P_j and P_{j-1}. The product involution on $P_{2i} \times P_{2j}$ takes H into itself, and the fixed-point sets of $\omega|H$ are

$H_{i,j} \subset P_i \times P_j$, $P_i \times P_{j-1}$, $P_{i-1} \times P_j$, and

$H_{i-1,j-1} \subset P_{i-1} \times P_{j-1}$, which have dimensions $i + j - 1$, $i+j-1$, $i+j-1$, and $i+j-3$ respectively. H has dimension $2i+2j-1$. Its cobordism class [H] is indecomposable (6.10) if the binomial coefficient $\{2i,2j\} \neq 0$, i.e. $\{i,j\} \neq 0$. We can choose i and j such that $i+j = n$ and $\{i,j\} \neq 0$ whenever n is not a power of 2.

<u>10.9 Theorem</u> $\varphi(2k) = 5k$, and $\varphi(2k+1) = 5k+2$. Any smooth involution on a non-bounding n-manifold has a fixed-point set of dimension at least $2n/5$.

<u>Proof</u> Suppose the n-manifold M has an involution with fixed-point sets of dimension $\leqslant k$ only. The corresponding element ν_M of F lies in F_k. Let m be the ideal in $\underline{\underline{N}}[[\theta]]$ generated by θ. We proved in 9.19 that J'' embeds F_k in $\underline{\underline{N}}[[\theta]]/m^{r+1}$, where $r = 5k/2$ if k is even, or $r = (5k-1)/2$ if k is odd. If $n > r$, we must have $\nu_M = 0$ and $[M] = 0$ by 10.5. Our examples show that this is the best possible result.]]]

However, we can do better with an extra hypothesis.

<u>10.10 Theorem</u> Any smooth involution on a n-manifold whose unoriented cobordism class is indecomposable has a

fixed-point set of dimension at least $\frac{1}{2}(n-1)$.

Proof By 10.5 we have $J''\nu_M = z_n\theta^n$ apart from higher terms and terms involving more than one z_i. By 9.19 the generator e'_n must appear in ν_M, and hence ν_M lies in $F_{\frac{1}{2}n}$ or $F_{\frac{1}{2}(n-1)}$, and not in any smaller F_j. Therefore there is a fixed-point set with dimension $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ (whichever is an integer).]]]

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This is an expanding list of references. We abbreviate certain abbreviations as follows:

AM = Annals of Math., series 2

AJM= American J.Math.

Aarhus = Colloquium on algebraic topology, Aarhus, 1962

BAMS = Bulletin Amer. Math. Soc.

TAMS = Transactions Amer. Math. Soc.

CMH = Comment. Math. Helv.

PCPS = Proc. Cambridge Phil. Soc.

PLMS = Proc. London Math. Soc., series 3

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Further suggestions are welcome.