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Bimodules, Higher Relative Commutants and the Fusion Algebra Associated to a Subfactor

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Abstract. We prove in this paper that the tensor product of reduced bimodules associated to a subfactor can be recovered as a product of certain projections in the higher relative commutants associated to the subfactor. After giving an elementary introduction to bimodules of II_1 factors and their relative tensor product, we prove various formulas relating the representations of the Jones tower coming from different k-step basic constructions and show that the natural shift on the higher relative commutants, defined by two consecutive modular conjugations of the tower, can be computed in terms of orthonormal bases and the Jones projections e_i. We give a detailed account of how the principal graphs of a subfactor can be recovered by calculating dimensions of intertwiner spaces of certain (reduced) bimodules and show that each vertex of the principal graphs represents a unique reduced bimodule. Then we define the (full) fusion algebra associated to a subfactor and prove that this fusion algebra can be calculated by computing products of certain projections in the higher relative commutants of the subfactor. Explicit formulas for these products are given. Finally we discuss reduced subfactors and give a procedure to compute the fusion algebra of a subfactor in those situations, when the principal graphs are simple. We show the relation to reduced subfactors and discuss in detail the example of a subfactor with principal graph E_6 to illustrate the general algorithm.

Introduction

In this paper we prove various facts about bimodules associated to a subfactor, some of which are known to experts, but whose proofs are not readily available in the literature. We refer to (Connes [1994], Sauvageot [1983]), (see also Ocneanu [1988], Ocneanu [1991(a)], Popa [1986]) for some of the original papers on bimodules, and to (Anantharaman-Delaroche [1993], Denizeau and Havet [1993(a)] [1993(b)], Ocneanu [1991(b)], Sunder [1992], Yamagami [1993]), for additional material (see

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also Bisch [1994(b)], Bisch and Haagerup [1996], Evans and Kawahigashi [1996], Haagerup [1994], Goodman and Wenzl [1990], Izumi [1991], Jones and Sunder [1996], Longo [1989][1990], Popa [1994], Sunder and Vijayarajan [1993], Wassermann [1995], and Wenzl [1988]). The above list of citations is by no means intended to be complete.

Here is a detailed description of the sections below. We begin in Section 1 with an elementary introduction to bimodules associated to a pair of II_1 factors, define Connes' relative tensor product of two such bimodules and prove that this bimodule tensor product is associative (Proposition 1.12). We discuss bimodule intertwiners and state the Frobenius reciprocity theorem (Theorem 1.18). Finally, we show that the direct sum of bimodules is compatible with the bimodule tensor product. Most of the material in this section can be found in (Connes [1994], Sauvageot [1983], Ocneanu [1988] and [1991(a)], Popa [1986], see also Sunder [1992]).

Section 2 contains material that is needed in Sections 3 and 4 to establish the bimodule interpretation of the principal graphs associated to a subfactor and to identify the tensor product of reduced bimodules as a product of certain projections in the higher relative commutants of the subfactor. We discuss the representations of the tower of II_1 factors associated to a subfactor $N \subset M$ coming from the k-step basic constructions, i.e., the basic construction for the inclusion $N \subset M_k$ (resp. $M \subset M_k$) and prove various formulas relating them (Lemma 2.4, Propositions 2.2, 2.5). Next, we discuss the natural shift on the higher relative commutants. We show that the "spatial" definition using the modular conjugations is the same as the "abstract" one using the Jones projections e_i and orthonormal bases (Theorems 2.6 and 2.11). We do this for an inclusion of II_1 factors of the form $A \subset B \subset^{e_1} B_1$, which we apply then to $A = N$, $B = M_n$ (resp. $A = M$, $B = M_n$) in Section 4. We give explicit formulas of the $J \cdot J$-map and the shift in terms of the e_i's and orthonormal bases, which are useful for the computation of tensor products of reduced bimodules associated to a subfactor. We also discuss briefly Ocneanu's Fourier transform and give a simple application to illustrate the usefulness of this map.

In Section 3 we show that the higher relative commutants associated to a subfactor can be viewed as spaces of N-N resp. N-M resp. M-N resp. M-M bimodule intertwiners. After recalling the definition of the principal graphs of a subfactor in some detail, we prove that each even (resp. odd) vertex of the principal graphs represents in a unique way a reduced N-N (M-M) (resp. N-M (M-N)) bimodule, i.e., a bimodule of the form $pL^2(M_n)$, where p is a projection in $N' \cap M_{2n+1}$ (resp. $M' \cap M_{2n}$, $N' \cap M_{2n}$, $M' \cap M_{2n+1}$). This is done by writing down an explicit isomorphism between these bimodules. We identify the contragredient (or conjugate) of a reduced bimodule as another reduced bimodule by calculating the projection in the higher relative commutants to which this conjugate reduced bimodule is associated (Proposition 3.11). We give then the definition of the (full) fusion algebra associated to a subfactor, including all possible bimodule products, i.e. the products of N-N (resp. M-M) bimodules with themselves (called the *even part* of the *full fusion algebra*, or simply the *fusion algebra*) and the products of N-M (resp. M-N) with M-N (resp. N-M) bimodules, called the *odd part* (where we form the relative bimodule tensor product over N resp. M of course). We list briefly the properties of the structure constants (which are dimensions of spaces of bimodule intertwiners) appearing in this definition.

We prove in Section 4 that the edges (including multiplicities) of the principal graphs can be recovered by computing dimensions of certain intertwiner spaces of (reduced) bimodules, thus, together with the results of Section 3 (the identification of the vertices of the principal graphs as reduced bimodules), establishing the bimodule picture of the prinicipal graphs as "principal" fusion rule matrices, due to Ocneanu. This uses some of the results of Section 2, in particular various formulas involving the representations of the tower of II_1 factors associated to a subfactor coming from different k-step basic constructions. We have to work spatially all the time, since reduced bimodules are obtained by an action of a projection in a certain higher relative commutant on a Hilbert space $L^2(M_k)$ (Propositions 4.1 and 4.3). In Theorem 4.6 we prove that the bimodule tensor product of two reduced bimodules (over N or M), associated to projections p and q say, can be calculated as a product of projections in the higher relative commutants, involving the modular conjugations and the shift from $N' \cap M_{2n+1}$ to $M'_{2n+1} \cap M_{4n+3}$ for example. Roughly, we obtain the tensor product of these two reduced bimodules by fixing p and shifting q far enough in a higher relative commutant so that the shifted q commutes with p and consider then the reduced bimodule associated to this new projection. This reduced bimodule turns out to be the bimodule tensor product of the two reduced bimodules associated to p and q. We do this for all cases in Theorem 4.6. Combining this theorem with the explicit formulas that we proved in Section 2 for the shift and the $J \cdot J$-map (in terms of e_i's and orthonormal bases), we obtain an explicit procedure, that allows us to calculate the fusion algebra of a subfactor whenever the higher relative commutants are well understood. Applications of this to the subfactors in (Bisch and Jones [1995]) for instance will be presented elsewhere.

Section 5 contains a discussion of the basic construction of reduced subfactors and their relation to reduced bimodules. We give a simple method to compute the (full) fusion algebra associated to a subfactor by solving matrix equations and then calculating products of the resulting matrices. When the principal graphs of the subfactor are not too complicated (for instance if they contain at most triple points), the fusion algebra can be computed completely in this way. For instance, all the calculations in (Bisch [1994(b)]) were performed using the method presented in this section. We show how the principal graphs of the reduced subfactors associated to an inclusion of II_1 factors can be determined from the fusion algebra and remark that their fusion algebra can be read off the fusion algebra of the original inclusion. Let us point out that this is rather obvious, when we use the endomorphism picture for bimodules (see for instance Longo [1989] and [1990], Izumi [1991]) (properly infinite case). However, to keep the paper self-contained, we stay in the II_1 setting. To illustrate the method presented in this section, we discuss in detail the example of a subfactor with principal graph E_6, calculate the full fusion algebra associated to such a subfactor, and give a full discussion of the associated reduced subfactors.

1 Preliminaries on bimodules

Let A and B be II_1 factors. We denote by B^{op} the *opposite algebra* of B, i.e. $B^{\mathrm{op}} = B$ as Banach spaces and the multiplication is defined by $b_1 \cdot b_2 = b_2 b_1$, $b_1, b_2 \in B$. B^{op} is of course a II_1 factor. Recall that an A-B bimodule H is by definition a pair of commuting normal (unital) representations of A and B^{op} on the Hilbert space H. We usually denote the left action of A and the right action of B

(which is by definition the left action of $B^{\rm op}$) by $a\cdot\xi\cdot b$, where $a \in A$, $b\in B$, $\xi \in H$. The notion of (unitary) equivalence of bimodules is recalled in the next definition.

Definition 1.1 *Let A and B be II_1 factors and let H and K be two A-B bimodules. We say that H and K are (unitarily) equivalent if there is a unitary $u : H \to K$ such that $u(a\cdot\xi\cdot b) = a\cdot u(\xi)\cdot b$, for all $a \in A$, $b \in B$, $\xi \in H$. We write ${}_AH_B \cong {}_AK_B$. Furthermore we denote by*

$$Hom_{A\text{-}B}(H,K) = \{T \in B(H,K) | T(a\cdot\xi\cdot b) = a\cdot T(\xi)\cdot b, \text{ for all } a \in A,\ b\in B,\ \xi\in H\}$$

the space of A-B intertwiners from H to K. If $H = K$, we write $Hom_{A\text{-}B}(H)$ for $Hom_{A\text{-}B}(H,H)$.

Observe that $\mathrm{Hom}_{A\text{-}B}(H) = A' \cap (B^{\rm op})' \cap B(H)$ is a von Neumann algebra. Recall that an A-B bimodule H is called *irreducible*, if $A' \cap (B^{\rm op})' \cap B(H) = \mathbb{C}$. Suppose that $(B^{\rm op})' \cap B(H)$ is again a II_1 factor (i.e. the coupling constant of $B^{\rm op}$ on H is finite), then one defines the *index* of the A-B bimodule H to be the Jones index of the subfactor $A \subset (B^{\rm op})' \cap B(H)$. If $[(B^{\rm op})' : A] < \infty$, then $\mathrm{Hom}_{A\text{-}B}(H)$ is a finite dimensional C*-algebra (Jones [1983]). We will sometimes say that the two A-B bimodules H and K are *isomorphic* (as A-B bimodules), which means that there is a bijective A-B intertwiner T from H to K. The unitary in the polar decomposition of T implements a unitary equivalence as defined above. We will mostly be concerned with equivalence classes of A-B bimodules.

Let $A \subset B$ be an inclusion of II_1 factors with finite Jones index and denote by tr the trace on B. As usual we let $L^2(B)$ be the completion of B in the norm $\|\cdot\|_2$ induced by the trace, in other words, $L^2(B)$ is the GNS Hilbert space with respect to tr. Then $\hat{1} = 1_B \in L^2(B)$ is the cyclic and separating vector, and we write $\hat{b}$ for $b(\hat{1})$, i.e. $b \in B$ viewed as a vector in $L^2(B)$ is denoted by $\hat{b}$. As usual we let $J : L^2(B) \to L^2(B)$ be the conjugate linear isometry obtained by extending the map $\hat{b} \to \widehat{b^*}$ to all of $L^2(B)$ by continuity. $L^2(B)$ is a left B-module (hence a left A-module), where B acts by left multiplication. $L^2(B)$ is also a right B-module (and hence a right A-module) with the action $\xi\cdot b = Jb^*J(\xi)$, $b \in B$, $\xi \in L^2(B)$. $L^2(B)$ becomes in this way a B-B (resp. A-B, B-A, A-A) bimodule.

Lemma 1.2 *Let A, B be II_1 factors and let $\pi_i : A \to B(H_i)$, $\psi_i : B^{\rm op} \to B(H_i)$, $i = 1, 2$, be (nonzero) normal representations of A and $B^{\rm op}$ such that $\pi_i(A) \subset \psi_i(B^{\rm op})'$, $i = 1, 2$. We define two A-B bimodules ${}_AH_{1B}$ and ${}_AH_{2B}$ by $a\cdot\xi\cdot b = \pi_i(a)\psi_i(b)\xi$, for all $\xi \in H_i$, $a \in A$, $b \in B^{\rm op}$, $i = 1, 2$.*
i) If ${}_AH_{1B} \cong_A H_{2B}$ as A-B bimodules, then there is a surjective $*$-isomorphism $\theta : \psi_1(B^{\rm op})' \to \psi_2(B^{\rm op})'$ such that $\theta(\pi_1(A)) = \pi_2(A)$. Furthermore, the Murray-von Neumann coupling constants of $\psi_1(B^{\rm op})$ on H_1 and of $\psi_2(B^{\rm op})$ on H_2 coincide.
ii) Assume that the Murray-von Neumann coupling constants satisfy $\dim_{\psi_1(B^{\rm op})} H_1 = \dim_{\psi_2(B^{\rm op})} H_2 < \infty$. Suppose that there is a surjective $*$-isomorphism $\theta : \psi_1(B^{\rm op})' \to \psi_2(B^{\rm op})'$ such that $\theta(\pi_1(A)) = \pi_2(A)$. Then there are automorphisms $\psi \in Aut\,A$, $\phi \in Aut\,B$ and a unitary $u : H_1 \to H_2$, such that $u(\pi_1(a)\psi_2(b)\xi) = \pi_2(\psi(a))\psi_2(\phi(b))(u(\xi))$, for all $\xi \in H_1$, $a \in A$ and $b \in B^{\rm op}$. Thus ${}_AH_{1B}$ is equivalent to ${}_AH_{2B}$, with A and B actions twisted by automorphisms of A resp. B.

Proof If ${}_AH_{1B} \cong {}_AH_{2B}$, then by definition of unitary equivalence of bimodules, there exists a unitary $u : H_1 \to H_2$ such that $u(\pi_1(a)\psi_1(b)\xi) = \pi_2(a)\psi_2(b)u(\xi)$, for all $a \in A$, $b \in B^{\rm op}$, $\xi \in H_1$. Thus $\theta = \mathrm{Ad}\,u$ does the job.

Suppose that ii) holds. The condition on the coupling constants implies that θ is spatial, i.e. $\theta = \operatorname{Ad} u$, $u : H_1 \to H_2$ a unitary. Hence $\operatorname{Ad} u$ is a surjective $*$-isomorphism from $\psi_1(B^{\text{op}})$ onto $\psi_2(B^{\text{op}})$ and therefore $u\psi_1(b)u^* = \psi_2(\phi(b))$, for all $b \in B^{\text{op}}$ and some automorphism $\phi \in \operatorname{Aut} B^{\text{op}} = \operatorname{Aut} B$. Similarly, there is a $\psi \in \operatorname{Aut} A$ such that $u\pi_1(a)u^* = \pi_2(\psi(a))$, $a \in A$. The rest is clear. □

Note that the lemma shows that the Murray-von Neumann coupling constants and the subfactor associated to a bimodule determine the bimodule only up to automorphisms. If we consider the B-B bimodule $L^2(B)$ with action $x \cdot \xi \cdot y = xJ\theta(y)^*J(\xi)$, $x, y \in B$, $\xi \in L^2(B)$, $\theta \in \operatorname{Aut} B$ an outer automorphism, we see that the automorphisms are indeed necessary.

We will define in Section 3 the fusion algebra associated to a subfactor and describe in Section 4 its multiplication law in terms of projections in the higher relative commutants. To understand this multiplication, let us start with recalling the definition of the relative tensor product (Connes [1994], see also Sauvageot [1983], Popa [1986], Ocneanu [1991(a)]).

Definition 1.3 *Let A be a II_1 factor with trace* tr *and let H be a left A-module. A vector $\xi \in H$ is called a (left A-) bounded vector in H if there is a constant $c = c(\xi)$ such that $\|a\xi\| \le c\|a\|_2$, for all $a \in A$, where $\|a\|_2 = \operatorname{tr}(a^*a)^{\frac{1}{2}}$. We denote by H^0 the set of left A-bounded vectors in H. Similarly for right A-modules (i.e. left A^{op}-modules) and right A-bounded vectors.*

Remark 1.4 *It is easy to see that $AH^0 \subset H^0$ and $A'H^0 \subset H^0$. From this one deduces immediately that H^0 is dense in H.*

Proposition 1.5 *Let $A \subset B$ be an inclusion of II_1 factors with $[B : A] < \infty$. Consider the B-B (resp. A-B, B-A, A-A) bimodule $L^2(B)$ as defined above. Then the left (right) A-bounded vectors and the left (right) B-bounded vectors coincide and are given by $\hat{B}$.*

Proof Let us prove that the left A-bounded vectors in $L^2(B)$ are given by $\hat{B}$. If $b \in B$, then $\|a\hat{b}\|^2_{L^2(B)} = \operatorname{tr}(abb^*a^*) \le \|b\|^2\|a\|_2^2$, for all $a \in A$. Thus $\hat{B} \subset L^2(B)^0$, the set of left A-bounded vectors. Conversely, let $\xi \in L^2(B)^0$. Then there is a constant $c(\xi)$ such that $\|a\xi\|_2 \le c(\xi)\|a\|_2$, for all $a \in A$. Thus $R_\xi : \hat{A} \to L^2(B)$, $R_\xi(\hat{a}) = a\xi$, extends by continuity to a bounded linear map $L^2(A) \to L^2(B)$. Let $A \subset B \subset^e B_1$ be the basic construction and consider the composition $R_\xi e : L^2(B) \to L^2(B)$. Note that $a_1R_\xi(\hat{a}) = a_1a\xi = R_\xi a_1(\hat{a})$, a, $a_1 \in A$, so that $R_\xi e \in A' \cap B(L^2(B))$. But $B_1 = JA'J$ and hence there is an $x \in B_1$ with $R_\xi e = Jx^*J \in B(L^2(B))$. By (Pimsner and Popa [1986]) there is a unique $y^* \in B$ with $x^*e = y^*e$. Thus $R_\xi e = Jx^*Je = Jx^*eJ = Jy^*Je$, which implies that $R_\xi e(\hat{1}) = R_\xi(\hat{1}) = \xi = Jy^*e(\hat{1}) = \hat{y}$, so that indeed $\xi \in \hat{B}$ as desired.– The same proof (with $e = 1$) shows that the set of left B-bounded vectors in $L^2(B)$ is also given by $\hat{B}$. From this it follows immediately that the set of right B-bounded (resp. right A-bounded) vectors in $L^2(B)$ is again $\hat{B}$. □

Remark 1.6 *The following slightly more general statement (which can be deduced from 1.5) holds: If H is an A-B bimodule with finite index, then the left A-bounded vectors in H and the right B-bounded vectors in H coincide (see for instance* Sunder [1992], *II. Proposition 4).*

Definition 1.7 *Let A be a II_1 factor and H a left A-module, let ξ, $\eta \in H^0$. We let $\langle \xi, \eta \rangle_A \in A$ be the operator in A defined by*

$$(x\xi, \eta)_H = \operatorname{tr}(x\langle \xi, \eta \rangle_A), \qquad \text{for all } x \in A.$$

Note that $\langle \xi, \eta \rangle_A$ exists and is unique by the Radon-Nikodym theorem (note that $|(x^*x\xi, \eta)| \leq c(\xi)c(\eta)\|x\|_2^2$). The uniqueness of the Radon-Nikodym derivative implies immediately the following lemma.

Lemma 1.8 *Let A be a II_1 factor and H a left A-module. Let ξ, $\eta \in H^0$, $a \in A$. Then we have*

i) *$\langle \cdot, \cdot \rangle_A$ is $\mathbb{C}$-linear in the first variable.*

ii) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$.

iii) $\langle a\xi, \eta \rangle_A = a\langle \xi, \eta \rangle_A$.

iv) $\langle \xi, \xi \rangle_A \geq 0$.

Observe that $\langle \xi, a\eta \rangle_A = \langle \xi, \eta \rangle_A a^*$, which is an easy consequence of ii) and iii). Furthermore, it is easy to show that if $\eta_1, \ldots, \eta_n \in H^0$, then $\big(\langle \eta_i, \eta_j \rangle_A\big)_{1 \leq i,j \leq n} \in A \otimes M_n(\mathbb{C})$ is a positive operator (consider $A \otimes M_n(\mathbb{C})$ on H^n and identify $\big(\langle \eta_i, \eta_j \rangle_A\big)_{i,j}$ as the Radon-Nikodym derivative $\langle (\eta_1, \ldots, \eta_n)^t, (\eta_1, \ldots, \eta_n)^t \rangle_{A \otimes M_n(\mathbb{C})}$, which is positive by 1.8 iv), applied to the left $A \otimes M_n(\mathbb{C})$-module H^n).

Similarly one can define a right Radon-Nikodym derivative, which is done in the next definition.

Definition 1.9 *Let A be a II_1 factor and H a right A-module (i.e., a left A^{op}-module), let ξ, $\eta \in H^0$. We let $\langle \xi, \eta \rangle_A^o \in A$ be the operator in A defined by*

$$(\eta x, \xi)_H = \operatorname{tr}(x\langle \xi, \eta \rangle_A^o), \qquad \text{for all } x \in A.$$

Let us collect the properties of the right Radon-Nikodym derivative as in Lemma 1.8 for the left Radon-Nikodym derivative.

Lemma 1.10 *Let A be a II_1 factor and H a right A-module. Let ξ, $\eta \in H^0$, $a \in A$. Then we have*

i) *$\langle \cdot, \cdot \rangle_A^o$ is $\mathbb{C}$-linear in the second variable.*

ii) $\langle \xi, \eta \rangle_A^o = (\langle \eta, \xi \rangle_A^o)^*$.

iii) $\langle \xi a, \eta \rangle_A^o = a^* \langle \xi, \eta \rangle_A^o$.

iv) $\langle \xi, \xi \rangle_A^o \geq 0$.

Observe that $\langle \xi, \eta a \rangle_A^o = \langle \xi, \eta \rangle_A^o a$, which follows immediately from ii) and iii). As above, we have that if $\eta_1, \ldots, \eta_n \in H^0$, then $\big(\langle \eta_i, \eta_j \rangle_A^o\big)_{1 \leq i,j \leq n} \in A \otimes M_n(\mathbb{C})$ is a positive operator (use again an amplification trick).

If $A \subset B$ is an inclusion of II_1 factors, then $L^2(B)$ is a natural A-B bimodule with the action $a \cdot \xi \cdot b = aJb^*J(\xi)$, $a \in A$, $b \in B$, $\xi \in L^2(B)$, as we have seen above. We have that $\langle b_1, b_2 \rangle_A = E_A(b_1 b_2^*)$, where $E_A : B \to A$ is the trace preserving conditional expectation. Similarly, $L^2(B)$ is a natural B-A bimodule and $\langle b_1, b_2 \rangle_A^o = E_A(b_1^* b_2)$, b_1, $b_2 \in B$.

We are now ready to recall the definition of the relative tensor product of two bimodules. Let A, B, C be II_1 factors, let H be an A-B bimodule and K a B-C

bimodule and denote by H^0 (resp. K^0) the right (resp. left) B-bounded vectors in H (resp. K). Consider the algebraic tensor product $H^0 \odot K^0$ and define for $\sum_i \xi_i \otimes \eta_i, \sum_j \xi'_j \otimes \eta'_j \in H^0 \odot K^0$

$$\langle \sum_i \xi_i \otimes \eta_i, \sum_j \xi'_j \otimes \eta'_j \rangle = \sum_{i,j} (\xi_i \langle \eta_i, \eta'_j \rangle_B, \xi'_j)_H, \tag{1.1}$$

where $\langle \eta_i, \eta'_j \rangle_B$ is the Radon-Nikodym derivative of $x \in B \to (x\eta_i, \eta'_j)$ with respect to tr_B as above (definition 1.7) and $(\cdot,\cdot)_H$ is the inner product on H. Similarly, let $\langle \xi_i, \xi'_j \rangle_A^o$ be the Radon-Nikodym derivative of $x \in B \to (\xi'_j x, \xi_i)$ with respect to tr_B. Then we have for $\xi_i, \xi'_j \in H^0$, $\eta_i, \eta'_j \in K^0$

$$\begin{aligned} (\xi_i \langle \eta_i, \eta'_j \rangle_B, \xi'_j)_H &= \mathrm{tr}_B(\langle \eta_i, \eta'_j \rangle_B \langle \xi'_j, \xi_i \rangle_B^o) \\ &= \mathrm{tr}_B(\langle \xi'_j, \xi_i \rangle_B^o \langle \eta_i, \eta'_j \rangle_B) \\ &= (\langle \xi'_j, \xi_i \rangle_B^o \eta_i, \eta'_j)_K \end{aligned}$$

and thus we have on $H^0 \odot K^0$

$$\langle \sum_i \xi_i \otimes \eta_i, \sum_j \xi'_j \otimes \eta'_j \rangle = \sum_{i,j} (\langle \xi'_j, \xi_i \rangle_B^o \eta_i, \eta'_j)_K. \tag{1.2}$$

We leave it to the reader to check that $\langle \cdot,\cdot \rangle$ as defined in (1.1) or (1.2) is a (possibly non-degenerate) inner product on $H^0 \odot K^0$. Note that (1.1) is actually defined on $H \odot K^0$ and (1.2) on $H^0 \odot K$.

We let $N_{\langle \cdot,\cdot \rangle}$ be the null space of this inner product and define $H \otimes_B K$ to be the completion of $H^0 \odot K^0 / N_{\langle \cdot,\cdot \rangle}$ in the norm induced by $\langle \cdot,\cdot \rangle$ on $H^0 \odot K^0 / N_{\langle \cdot,\cdot \rangle}$ (Connes [1994], see also Sauvageot [1983]).

Definition 1.11 *Let A, B and C be II_1 factors, then $H \otimes_B K$ as defined above is called the bimodule tensor product (or the relative tensor product over B) of the A-B bimodule ${}_AH_B$ and the B-C bimodule ${}_BK_C$. We denote the equivalence class of the the vector $\sum_i \xi_i \otimes \eta_i$ in $H^0 \odot K^0 / N_{\langle \cdot,\cdot \rangle}$ by $\sum_i \xi_i \otimes_B \eta_i$ or $[\sum_i \xi_i \otimes \eta_i]$.*

It is easy to see that $H \otimes_B K$ is an A-C bimodule: If we let $\sum_i \xi_i \otimes \eta_i \in H^0 \odot K^0$, then we have an A-C action via $a \cdot (\sum_i \xi_i \otimes \eta_i) \cdot c = \sum_i (a\xi_i) \otimes (\eta_i c)$. To show that this defines a left A-action and a right C-action on the relative tensor product, one proves that $\|a \cdot \sum_i \xi_i \otimes \eta_i\| \leq \|a\| \|\sum_i \xi_i \otimes \eta_i\|$ and similarly for the right C-action. This inequality is shown by using an amplification trick and the remark about positivity of $\left(\langle \eta_i, \eta_j \rangle_A\right)_{i,j}$ after Lemma 1.8 (resp. Lemma 1.10). We leave the simple details to the reader. Since this induces in a natural way an A-C bimodule structure on the relative tensor product $H \otimes_B K$, we denote this A-C bimodule by ${}_AH \otimes_B K_C$.

Consider the quotient map $\phi : H^0 \odot K^0 \to {}_AH \otimes_B K_C$, defined by $\phi(\sum_i \xi_i \otimes \eta_i) = \sum_i \xi_i \otimes_B \eta_i$. Then

$$\begin{aligned}
\|\sum_i \xi_i \otimes_B \eta_i\|^2_{H\otimes_B K} &\leq \langle \sum_i \xi_i \otimes \eta_i, \sum_i \xi_i \otimes \eta_i \rangle \\
&= \sum_{i,j} \big(\xi_i \langle \eta_i, \eta_j \rangle_B, \xi_j\big)_H \\
&\leq \sum_{i,j} \|\xi_i \langle \eta_i, \eta_j \rangle_B\|_H \|\xi_j\|_H \\
&\leq (\max_{i,j} \|\langle \eta_i, \eta_j \rangle_B\|) \sum_{i,j} \|\xi_i\|_H \|\xi_j\|_H .
\end{aligned}$$

Thus ϕ extends by continuity to a map, still denoted by $\phi : H \odot K^0 \to {}_AH \otimes_B K_C$. Similarly, using (1.2) instead of (1.1) in the above estimates, we see that ϕ extends by continuity to a map $\psi : H^0 \odot K \to {}_AH \otimes_B K_C$. In particular, we get that $\phi(H \odot K^0)$ and $\psi(H^0 \odot K)$ are dense in ${}_AH \otimes_B K_C$ and we could have defined ${}_AH \otimes_B K_C$ equally well by using (1.1), defined on $H \odot K^0$ and taking the separated completion as above, or by using (1.2) on $H^0 \odot K$ and taking the separated completion. As we have just shown, all three ways of defining ${}_AH \otimes_B K_C$ coincide (see also Popa [1986]). Associativity of the bimodule tensor product is now immediate and we include a proof for the convenience of the reader.

Proposition 1.12 *Let A, B, C and D be II_1 factors and let ${}_AH_B$, ${}_BK_C$ and ${}_CL_D$ be bimodules. Then*

$$({}_AH \otimes_B K_C) \otimes_C L_D \cong {}_AH \otimes_B (K_C \otimes_C L_D)$$

as A-D bimodules.

Proof It follows from the paragraph preceding the proposition that the composition of the quotient maps $H^0 \odot (K^0 \odot L^0) \to H^0 \odot (K \otimes_C L) \to H \otimes_B (K \otimes_C L)$ is continuous with dense image. Similarly, the composition of the quotient maps $(H^0 \odot K^0) \odot L^0 \to (H \otimes_B K) \odot L^0 \to (H \otimes_B K) \otimes_C L$ is continuous with dense image. The result follows now from $(H^0 \odot K^0) \odot L^0 \cong H^0 \odot (K^0 \odot L^0)$. □

Lemma 1.13 *Let A, B, C be II_1 factors and let ${}_AH_B$ and ${}_BK_C$ be A-B resp. B-C bimodules. Then $\xi b \otimes_B \eta = \xi \otimes_B b\eta$ for all $\xi \in H^0$, $\eta \in K^0$ (or $\xi \in H$, $\eta \in K^0$, or $\xi \in H^0$, $\eta \in K$), where H^0 (resp. K^0) denotes the right (resp. left) B-bounded vectors in H (resp. K).*

Proof By definition of $\xi \otimes_B \eta$ (Definition 1.11 and the remarks afterwards) we need to show that $\xi b \otimes \eta - \xi \otimes b\eta \in N_{\langle \cdot,\cdot \rangle}$, which follows immediately from Lemma 1.8 (resp. 1.10). □

Lemma 1.14 *Let A and B be II_1 factors and let ${}_AH_B$ be an A-B bimodule. Consider the A-A bimodule $L^2(A)$ and the B-B bimodule $L^2(B)$. Then ${}_AL^2(A) \otimes_A H_B \cong {}_AH_B$ and ${}_AH \otimes_B L^2(B)_B \cong {}_AH_B$ (all equivalences as A-B bimodules). Furthermore, if $A \subset B$ is an inclusion of II_1 factors and we regard $L^2(B)$ as a B-A bimodule, then ${}_AH_B \otimes_B L^2(B)_A \cong {}_AH_A$ (as A-A bimodules).*

Proof We know that $L^2(A)^0 = \hat{A}$. Define a linear map $T : \hat{A} \odot H \to H$ by $T(\hat{a} \otimes \xi) = a\xi$, $\hat{a} \in A$, $\xi \in H$. Observe that $\langle \hat{a}_j, \hat{a}_i \rangle^o_A = a_j^* a_i$, for $\hat{a}_i$, $\hat{a}_j \in \hat{A}$ (we regard $L^2(A)$ as a right A-module here). Thus

$$\begin{aligned}\|T(\sum_i \hat{a}_i \otimes \xi_i)\|_H^2 &= \sum_{i,j}(a_i\xi_i, a_j\xi_j)_H = \sum_{i,j}(\langle \hat{a}_j, \hat{a}_i\rangle_A^o \xi_i, \xi_j)_H \\ &= \langle \sum_i \hat{a}_i \otimes \xi_i, \sum_i \hat{a}_i \otimes \xi_i\rangle,\end{aligned}$$

which shows that T is well-defined, factors through $N_{\langle\cdot,\cdot\rangle}$ and hence induces a surjective (since A is unital) isometry, still denoted by T, from ${}_AL^2(A) \otimes_A H_B \to {}_AH_B$, which is clearly an A-B intertwiner. This establishes the first equivalence and the remaining ones are shown in the same way, using the left Radon-Nikodym derivative. □

Next we discuss briefly morphisms (intertwiners) between bimodules.

Proposition 1.15 *Let A, B, C be II_1 factors and let ${}_AH_B$, ${}_AK_B$, ${}_AH_{iB}$, ${}_BK_{iC}$, $i = 1, 2$, be bimodules. Let $R : {}_AH_B \to {}_AK_B$ and $S : {}_AH_{1B} \to {}_AH_{2B}$ be A-B intertwiners and let $T : {}_BK_{1C} \to {}_BK_{2C}$ be a B-C intertwiner. Then*

i) *R maps left A-bounded (resp. right B-bounded) vectors of H to left A-bounded (resp. right B-bounded) vectors of K.*

ii) *If η, η' are left B-bounded vectors in K_1, then the Radon-Nikodym derivatives satisfy $\langle T(\eta), T(\eta')\rangle_B = \langle T^*T(\eta), \eta'\rangle_B = \langle \eta, T^*T(\eta')\rangle_B$.*

iii) *There is a unique A-C intertwiner $S \otimes_B T : {}_AH_1 \otimes_B K_{1C} \to {}_AH_2 \otimes_B K_{2C}$ such that $S \otimes_B T(\xi \otimes_B \eta) = [S \otimes T(\xi \otimes \eta)]$, for all $\xi \in H^0$ (right B-bounded vectors) and $\eta \in K^0$ (left B-bounded vectors), where $[\ ]$ is as in Definition 1.11. Furthermore, if S and T are bimodules isomorphisms, then so is $S \otimes_B T$.*

Proof The proof is straightforward. If $\xi \in H$ is a left A-bounded vector, then $\|aR(\xi)\|_K \le \|R\|\|a\xi\|_H \le c(\xi)\|a\|_2$, for all $a \in A$ (we used the notation of 1.3). Thus $R(\xi)$ is a left A-bounded vector in K. The same argument applies for right B-bounded vectors and hence i) is shown.

Let us show ii). We have seen in i) that $T(\eta)$, $T(\eta')$ are left B-bounded vectors in K_2. By definition of the Radon-Nikodym derivative we have $\big(bT(\eta), T(\eta')\big)_{K_2} = \mathrm{tr}_B\big(b\langle T(\eta), T(\eta')\rangle_B\big)$, for all $b \in B$. But $\big(bT(\eta), T(\eta')\big)_{K_2} = \big(T(\eta), T(b^*\eta')\big)_{K_2} = \big(T^*T(\eta), b^*\eta'\big)_{K_1} = \mathrm{tr}_B\big(b\langle T^*T(\eta), \eta'\rangle_B\big)$, $b \in B$, which implies the first equality by uniqueness of the Radon-Nikodym derivative. The second equality is shown in the same way.

We proceed with the proof of iii). Let us denote by $\langle\cdot,\cdot\rangle_i$ the inner product on ${H_i}^0 \odot {K_i}^0$, $i = 1, 2$ (see Definition 1.11). Let $\sum_i \xi_i \otimes \eta_i \in {H_1}^0 \odot {K_1}^0$, and let $S \otimes T$ be the usual tensor product of S and T. Then $S \otimes T\big(\sum_i \xi_i \otimes \eta_i\big) \in {H_2}^0 \odot {K_2}^0$ by i) and using ii) we obtain,

$$\langle S\otimes T(\sum_i \xi_i\otimes\eta_i), S\otimes T(\sum_i \xi_i\otimes\eta_i)\rangle_2$$

$$= \sum_{i,j}\big(S(\xi_i)\langle T(\eta_i),T(\eta_j)\rangle_B, S(\xi_j)\big)_{H_2} = \sum_{i,j}\big(S^*S(\xi_i\langle T^*T(\eta_i),\eta_j\rangle_B),\xi_j\big)_{H_1}$$

$$= \langle\sum_i S^*S(\xi_i)\otimes T^*T(\eta_i), \sum_i \xi_i\otimes\eta_i\rangle_1 = \langle S^*S\otimes T^*T(\sum_i\xi_i\otimes\eta_i),\sum_i\xi_i\otimes\eta_i\rangle_1$$

$$\leq \|S^*S\otimes T^*T\|\,\|\sum_i \xi_i\otimes\eta_i\|^2_{H_1{}^0\odot K_1{}^0},$$

so that $S\otimes T\big(N_{\langle\cdot,\cdot\rangle_1}\big)\subset N_{\langle\cdot,\cdot\rangle_2}$. Thus we get an induced map, still denoted by $S\otimes T$, from $H_1{}^0\odot K_1{}^0/N_{\langle\cdot,\cdot\rangle_1}\to H_2{}^0\odot K_2{}^0/N_{\langle\cdot,\cdot\rangle_2}$, which is continuous by the above computation and therefore extends to a map from ${}_AH_1\otimes_B K_{1C}\to {}_AH_2\otimes_B K_{2C}$, denoted by $S\otimes_B T$. $S\otimes_B T$ is clearly an A-C bimodule morphism and satisfies by definition $S\otimes_B T\big(\sum_i\xi_i\otimes\eta_i\big) = [S\otimes T\big(\sum_i\xi_i\otimes\eta_i\big)]$. The uniqueness is clear by a density argument. Finally, if S and T are bimodule isomorphisms, $S\otimes_B T$ is also, which can be easily deduced from the above calculation. □

Definition 1.16 *Let A, B be II_1 factors and let H be an A-B bimodule with actions denoted by $a\xi b$, $a\in A$, $b\in B$, $\xi\in H$. We define a B-A bimodule ${}_B\overline{H}_A$ (also denoted by $\overline{{}_AH_B}$) as $\overline{H}$, the conjugate Hilbert space, with the B-A action defined by $b\cdot\bar{\xi}\cdot a = \overline{a^*\xi b^*}$, where $\bar{\xi}$ denotes the vector ξ, considered as an element in $\overline{H}$. The B-A bimodule ${}_B\overline{H}_A$ is called the conjugate (or contragredient or adjoint) of the A-B bimodule ${}_AH_B$. An A-A bimodule H is called selfcontragredient if $\overline{{}_AH_A}\cong {}_AH_A$ as A-A bimodules.*

Remark 1.17 *i) Let H be a right A-module with right A action denoted by ξa, $\xi\in H$, $a\in A$. Then $\overline{H_A}$ is a left A-module with left A action $a\cdot\bar{\xi}=\overline{\xi a^*}$ (Definition 1.16). Clearly, the right A-bounded vectors H^0 and the left A-bounded vectors $\overline{H}^0$ coincide. Thus $\mathrm{tr}_A(a\langle\bar{\xi},\bar{\eta}\rangle_A) = \big(a\cdot\bar{\xi},\bar{\eta}\big)_{\overline{H}} = \big(\eta,\xi a^*\big)_H = \big(\eta a,\xi\big)_H = \mathrm{tr}_A(a\langle\xi,\eta\rangle_A^o)$, for all ξ, $\eta\in H^0$, $a\in A$. Hence $\langle\bar{\xi},\bar{\eta}\rangle_A = \langle\xi,\eta\rangle^o_A$, for all ξ, $\eta\in H^0$.*
ii) Let A, B, C be II_1 factors, and let ${}_AH_B$, ${}_BK_C$ be bimodules. Then $\overline{{}_AH\otimes_B K_C}\cong {}_C\overline{K}\otimes_B\overline{H}_A$ as C-A bimodules. Let us briefly sketch a proof of this statement. Define a linear map $T:\overline{K}^0\odot\overline{H}^0\to {}_A\overline{H\otimes_B K_C}$, by $T(\bar{\eta}\otimes\bar{\xi}) = \overline{\xi\otimes_B\eta}$, $\bar{\xi}\in\overline{H}^0$, $\bar{\eta}\in\overline{K}^0$. Then we get for $\xi_i\in H^0$, $\eta_i\in K^0$, $i=1,2$,

$$\big(\overline{\xi_1\otimes_B\eta_1},\overline{\xi_2\otimes_B\eta_2}\big)_{\overline{{}_AH\otimes_B K_C}}$$

$$= \big(\xi_2\otimes_B\eta_2,\xi_1\otimes_B\eta_1\big)_{{}_AH\otimes_B K_C} = \langle\xi_2\otimes\eta_2,\xi_1\otimes\eta_1\rangle = \big(\langle\xi_1,\xi_2\rangle_B^o\eta_2,\eta_1\big)_K$$

$$= \big(\langle\bar{\xi}_1,\bar{\xi}_2\rangle_B\eta_2,\eta_1\big)_K = \big(\eta_2,\big(\langle\bar{\xi}_1,\bar{\xi}_2\rangle_B\big)^*\eta_1\big)_K = \big(\overline{\big(\langle\bar{\xi}_1,\bar{\xi}_2\rangle_B\big)^*\eta_1},\bar{\eta}_2\big)_{\overline{K}}$$

$$= \big(\bar{\eta}_1\cdot\langle\bar{\xi}_1,\bar{\xi}_2\rangle_B,\bar{\eta}_2\big)_{\overline{K}} = \langle\bar{\eta}_1\otimes\bar{\xi}_1,\bar{\eta}_2\otimes\bar{\xi}_2\rangle_{\overline{K}^0\odot\overline{H}^0}.$$

Thus T is well-defined and induces an isometry $\overline{K}^0\odot\overline{H}^0/N_{\langle\cdot,\cdot\rangle_{\overline{K}^0\odot\overline{H}^0}}\to\overline{{}_AH\otimes_B K_C}$, which is clearly a C-A bimodule morphism (note that $c\cdot\big(\bar{\eta}\otimes\bar{\xi}\big)\cdot a = \overline{\eta c^*}\otimes\overline{a^*\xi}$,

which maps under T to $\overline{a^*\xi \otimes_B \eta c^*} = c \cdot \overline{\xi \otimes_B \eta} \cdot a$, $c \in C$, $a \in A$) and extends by continuity to an injective C-A bimodule morphism ${}_C\overline{K} \otimes_B \overline{H}_A \to \overline{{}_A H \otimes_B K_C}$. It is easy to see that this map is also surjective.
iii) It is obvious that $\overline{\overline{{}_A H_B}} \cong {}_A H_B$ as A-B bimodules.

Recall that if $A \subset B$ is an inclusion of II_1 factors, then $\overline{{}_A L^2(B)_B} \overset{B-A}{\cong} {}_B L^2(B)_A$, where the bimodule equivalence is realized by the modular conjugation $J : L^2(B) \to L^2(B)$, since J, viewed as an operator $\overline{L^2(B)} \to L^2(B)$, is a linear surjective isometry, satisfying $J(b \cdot \bar{\hat{x}} \cdot a) = J(\widehat{a^*xb^*}) = \widehat{bx^*a} = bJ(\hat{x})a = b \cdot J(\bar{\hat{x}}) \cdot a$, for all $a \in A$, $b \in B$ and $\hat{x} \in \hat{B} \subset L^2(B)$ (note that the inner product on $\overline{L^2(B)}$ is defined as $(\bar{\xi}, \bar{\eta}) = (\eta, \xi)_{L^2(B)}$, ξ, $\eta \in L^2(B)$ as usual). Finally, let us state the *Frobenius reciprocity* theorem.

Theorem 1.18 *Let A, B and C be II_1 factors.*

i) *Let ${}_A H_B$, ${}_A K_B$ be A-B bimodules. Then the following vector spaces are naturally isomorphic: $Hom_{A\text{-}B}({}_A H_B, {}_A K_B) \cong Hom_{B\text{-}A}({}_B\overline{K}_A, {}_B\overline{H}_A)$.*

Let ${}_A H_B$, ${}_B K_C$ and ${}_A L_C$ be bimodules with finite index. Then the following vector spaces are naturally isomorphic:

ii) *$Hom_{A\text{-}C}({}_A H \otimes_B K_C, {}_A L_C) \cong Hom_{A\text{-}B}({}_A H_B, {}_A L \otimes_C \overline{K}_B)$.*

iii) *$Hom_{A\text{-}C}({}_A H \otimes_B K_C, {}_A L_C) \cong Hom_{B\text{-}C}({}_B K_C, {}_B\overline{H} \otimes_A L_C)$.*

Proof We will only give a hint of the proof. Let $T \in \mathrm{Hom}_{A\text{-}B}({}_A H_B, {}_A K_B)$ and consider $T^* : K \to H$ defined by $(\xi, T^*(\eta))_H = (T(\xi), \eta)_K$, $\xi \in H$, $\eta \in K$. Then it is immediate that $T^*(a \cdot \eta \cdot b) = a \cdot T^*(\eta) \cdot b$, for all $a \in A$, $b \in B$ and $\eta \in K$. Thus $T^* \in \mathrm{Hom}_{A\text{-}B}({}_A K_B, {}_A H_B)$. Now we consider T^* as an operator $\bar{K} \to \bar{H}$, defined by $T^*(\bar{\eta}) = \overline{T^*(\eta)}$. T^* is clearly linear and satisfies $T^*(b \cdot \bar{\eta} \cdot a) = T^*(\overline{a^*\eta b^*}) = \overline{T^*(a^*\eta b^*)} = \overline{a^*T^*(\eta)b^*} = b \cdot \overline{T^*(\eta)} \cdot a = b \cdot T^*(\bar{\eta}) \cdot a$, for all $a \in A$, $b \in B$, $\bar{\eta} \in \bar{K}$. Thus $T^* \in \mathrm{Hom}_{B\text{-}A}(\overline{{}_A K_B}, \overline{{}_A H_B})$ and the map $T \to T^*$ is the desired natural isomorphism in i), which is an algebra isomorphism if $H = K$. This proves i).

Using the properties of of the contragredient bimodule and i), it is immediate that ii) and iii) are equivalent. Using again i) and Lemma 1.14, we see that it is enough to show that $\mathrm{Hom}_{A\text{-}B}({}_A H_B, {}_A K_B) \cong \mathrm{Hom}_{A\text{-}A}({}_A L^2(A)_A, {}_A K \otimes_B {}_B\overline{H}_A)$ which is left as an exercise (see for instance Ocneanu [1991(a)], Sunder [1992] for details). □

Remark 1.19 *Let ${}_A H_B$, ${}_A K_B$ be A-B bimodules with finite index. Then $Hom_{A\text{-}B}({}_A H_B, {}_A K_B)$ and $Hom_{A\text{-}B}({}_A K_B, {}_A H_B)$ are naturally anti-isomorphic and hence isomorphic. From the first part of the proof of Theorem 1.18, we see that the map $T \in Hom_{A\text{-}B}({}_A H_B, {}_A K_B) \to T^* \in Hom_{A\text{-}B}({}_A K_B, {}_A H_B)$ is a conjugate linear, surjective isomorphism, which implies the claim.*

Before we end this section, let us introduce some notations, which will be used later on.

Definition 1.20 *Let A, B, C be II_1 factors and let $\alpha = {}_A H_B$, $\beta = {}_A K_B$, $\gamma = {}_B L_C$ be A-B resp. B-C bimodules. The A-C bimodule ${}_A H_B \otimes_B {}_B L_C$ will be denoted by $\alpha\gamma$. Furthermore, we let $\langle \alpha, \beta \rangle = \dim_{\mathbb{C}} Hom_{A\text{-}B}({}_A H_B, {}_A K_B)$.*

Observe that if α and β were endomorphisms, one would write $\gamma\alpha = \gamma \circ \alpha$, rather than $\alpha\gamma$. Frobenius reciprocity can now be rewritten in the following way.

Corollary 1.21 *Let A, B, C be II_1 factors.*

i) *Let $\alpha = {}_AH_B$ and $\beta = {}_AK_B$ be A-B bimodules with finite index. Then $\langle \alpha, \beta \rangle = \langle \bar{\beta}, \bar{\alpha} \rangle$, $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$.*

Consider the bimodules $\alpha = {}_AH_B$, $\beta = {}_BK_C$ and $\gamma = {}_AL_C$ as in Theorem 1.18. Then

ii) $\langle \alpha\beta, \gamma \rangle = \langle \alpha, \gamma\bar{\beta} \rangle$.

iii) $\langle \alpha\beta, \gamma \rangle = \langle \beta, \bar{\alpha}\gamma \rangle$.

From these identities it follows that,

iv) $\langle \alpha\beta, \gamma \rangle = \langle \gamma, \alpha\beta \rangle = \langle \bar{\gamma}, \overline{\alpha\beta} \rangle$.

v) $\langle \gamma, \alpha\beta \rangle = \langle \bar{\beta}, \bar{\gamma}\alpha \rangle = \langle \beta, \bar{\alpha}\gamma \rangle$.

vi) $\langle \gamma, \alpha\beta \rangle = \langle \bar{\alpha}, \beta\bar{\gamma} \rangle = \langle \alpha, \gamma\bar{\beta} \rangle$.

Proof i)-iii) is just a rewriting of Theorem 1.18 and Remark 1.19. iv)-vi) follows by applying the rules i)-iii). □

Finally, let us show that the relative tensor product is compatible with direct sums of bimodules.

Definition 1.22 *Let A and B be II_1 factors and let ${}_AH_B$, ${}_AK_B$ be A-B bimodule. The Hilbert space $H \oplus K$ becomes an A-B bimodule with the action $a \cdot (\xi \oplus \eta) \cdot b = (a \cdot \xi \cdot b) \oplus (a \cdot \eta \cdot b)$, $\xi \in H$, $\eta \in K$, $a \in A$, $b \in B$. It is called the direct sum of the bimodules ${}_AH_B$ and ${}_AK_B$ and denoted by ${}_AH_B \oplus_A K_B$.*

Recall that the inner product on $H \oplus K$ is given by $\big(\xi \oplus \eta, \xi' \oplus \eta'\big) = (\xi, \xi')_H + (\eta, \eta')_K$, ξ, $\xi' \in H$, η, $\eta' \in K$.

Proposition 1.23 *Let A, B, C be II_1 factors and let ${}_AH_B$, ${}_AK_B$ and ${}_BL_C$ be bimodules. Then*

$$\big({}_AH_B \oplus {}_AK_B\big) \otimes_B L_C \cong \big({}_AH \otimes_B L_C\big) \oplus \big({}_AK \otimes_B L_C\big)$$

as A-C bimodules.

Proof First observe that $\big(H \oplus K\big)^0 = H^0 \oplus K^0$ (right B-bounded vectors). Then, let us define a linear map $T : \big(H \oplus K\big)^0 \odot L^0 \to \big(H^0 \odot L^0\big) \oplus \big(K^0 \odot L^0\big)$ by $T\big((\xi \oplus \eta) \otimes \zeta\big) = (\xi \otimes \zeta) \oplus (\eta \otimes \zeta)$, $\xi \in H^0$, $\eta \in K^0$, $\zeta \in L^0$. We have,

$$
\begin{aligned}
&\langle\, T\big(\sum_i(\xi_i\oplus\eta_i)\otimes\zeta_i\big), T\big(\sum_i(\xi_i\oplus\eta_i)\otimes\zeta_i\big)\rangle_{(H^0\odot L^0)\oplus(K^0\odot L^0)}\\
&\quad=\sum_{i,j}\langle(\xi_i\otimes\zeta_i)\oplus(\eta_i\otimes\zeta_i),(\xi_j\otimes\zeta_j)\oplus(\eta_j\otimes\zeta_j)\rangle\\
&\quad=\sum_{i,j}\big(\langle\xi_i\otimes\zeta_i,\xi_j\otimes\zeta_j\rangle_{H^0\oplus L^0}+\langle\eta_i\otimes\zeta_i,\eta_j\otimes\zeta_j\rangle_{K^0\oplus L^0}\big)\\
&\quad=\sum_{i,j}\big(\big(\xi_i\langle\zeta_i,\zeta_j\rangle_B,\xi_j\big)_H+\big(\eta_i\langle\zeta_i,\zeta_j\rangle_B,\eta_j\big)_K\big)\\
&\quad=\sum_{i,j}\big((\xi_i\oplus\eta_i)\cdot\langle\zeta_i,\zeta_j\rangle_B,\xi_j\oplus\eta_j\big)_{H\oplus K}\\
&\quad=\langle\sum_i(\xi_i\oplus\eta_i)\otimes\zeta_i,\sum_i(\xi_i\oplus\eta_i)\otimes\zeta_i\rangle_{(H\oplus K)^0\odot L^0}.
\end{aligned}
$$

Note that $N_{\langle\cdot,\cdot\rangle_{(H^0\odot L^0)\oplus(K^0\odot L^0)}} = N_{\langle\cdot,\cdot\rangle_{H^0\odot L^0}} \oplus N_{\langle\cdot,\cdot\rangle_{K^0\odot L^0}}$. Thus T is well-defined and induces a map between the quotient spaces, which is continuous and hence extends by continuity to a map $T:\big({}_AH_B\oplus{}_AK_B\big)\otimes_B L_C\to\big({}_AH\otimes_B L_C\big)\oplus\big({}_AK\otimes_B L_C\big)$. It is clearly injective by the above calculation and obviously an A-C bimodule morphism. Surjectivity of T can be shown in various ways. For instance, one can define a map $S:(H^0\odot L^0)\oplus(K^0\odot L^0)\to(H^0\oplus K^0)\otimes L^0$ by $S\big(\sum_i(\xi_i\otimes\eta_i)\oplus\big(\sum_j\nu_j\otimes\mu_j\big)\big)=\sum_i(\xi_i\oplus 0)\otimes\eta_i+\sum_j(0\oplus\nu_j)\otimes\mu_j$, $\xi_i\in H^0$, η_i, $\mu_j\in L^0$, $\nu_j\in K^0$, which is clearly the inverse of T. As above, one shows that S extends to an injective A-C bimodule morphism from $\big({}_AH\otimes_B L_C\big)\oplus\big({}_AK\otimes_B L_C\big)$ to $\big({}_AH_B\oplus{}_AK_B\big)\otimes_B L_C$, and hence the surjectivity of T follows. □

2 Shifts, Fourier transforms and the k-step basic construction

We discuss in this section the k-step basic construction, various natural representations of the higher relative commutants associated to it, and the natural shift on the higher relative commutants. Some of the material in this section is known to specialists and can be found partially in the literature (see, for instance (Ocneanu [1988], [1991(a)] and [1991(b)], Pimsner and Popa [1986] [1988], David, Choda and Hiai [1991]).

Let us fix an inclusion of II_1 factors $N\subset M$ with finite index and let

$$N\subset M\subset^{e_1}M_1\subset^{e_2}M_2\subset\dots$$

be the associated Jones tower with $e_k\in M_k$, obtained by iterating the basic construction $M_{k-1}\subset M_k\subset^{e_{k+1}}M_{k+1}=\{M_k,e_{k+1}\}''\subset B(L^2(M_k))$. The next result is quoted from (Pimsner and Popa [1988]) and follows from the abstract characterization of the basic construction given there.

Proposition 2.1 *Let $N\subset M$ be an inclusion of II_1 factors with finite index. Let*

$$
\begin{aligned}
f_k&=[M:N]^{k(k+1)/2}(e_{k+1}e_k\dots e_1)(e_{k+2}e_{k+1}\dots e_2)\cdots(e_{2k}\dots e_k)(e_{2k+1}\dots e_{k+1}),\\
g_k&=[M:N]^{k(k-1)/2}(e_{k+1}e_k\dots e_2)(e_{k+2}e_{k+1}\dots e_3)\cdots(e_{2k-1}\dots e_k)(e_{2k}\dots e_{k+1}).
\end{aligned}
$$

Then

i) *Let $N\subset M_k\subset^{\tilde f_k}\tilde M_{2k+1}$ be the basic construction, i.e., $\tilde f_k:L^2(M_k)\to L^2(N)$ is the orthogonal projection and $\tilde M_{2k+1}=\{M_k,\tilde f_k\}''\subset B(L^2(M_k))$. Then there is a surjective $*$-isomorphism $\phi:\tilde M_{2k+1}\to M_{2k+1}$ such that $\phi(\tilde f_k)=f_k$*

and $\phi(x) = x$, for all $x \in M_k$. We will say briefly that $N \subset M_k \subset^{f_k} M_{2k+1}$ is the basic construction with projection f_k.

ii) *$M \subset M_k \subset^{g_k} M_{2k}$ is the basic construction with projection g_k.*

We will have to work with explicit representations of the higher relative commutants $N' \cap M_{2k+1}$ and $M' \cap M_{2k}$ on the Hilbert space $L^2(M_k)$. If $x \in M_k$, we denote by $\hat{x}$ the vector $x(\hat{1}_{M_k}) \in L^2(M_k)$, where $\hat{1}_{M_k}$ is the cyclic and separating vector in $L^2(M_k)$. Furthermore, let $J_k : L^2(M_k) \to L^2(M_k)$ be the modular conjugation, $J_k(\hat{x}) = \widehat{x^*}$, $x \in M_k$. J_k is then an antilinear isometry on $L^2(M_k)$ satisfying $J_k^2 = id$. Let us denote by π_k the (necessarily faithful) representation of M_k, M_{2k+1} obtained from the basic construction of $N \subset M_k \subset B(L^2(M_k))$. Let us be more precise (see also (David, 1.5.6), (Jones and Sunder [1996])).

Proposition 2.2 *The representation π_k of M_k, M_{2k+1} on $L^2(M_k)$ as defined above satisfies $\pi_k(x)(\hat{z}) = \widehat{xz}$, for all $\hat{z} \in \widehat{M_k} \subset L^2(M_k)$, $x \in M_k$ and $\pi_k(xf_ky)(\hat{z}) = xE_N^{M_k}(yz)^\wedge$, for all $xf_ky \in M_{2k+1}$, x, y, $z \in M_k$, f_k as in 2.1 and $E_N^{M_k} : M_k \to N$ the trace preserving conditional expectation. Thus $\tilde{M}_{2k+1} = \pi_k(M_{2k+1}) = J_k\pi_k(N)'J_k$. More generally, we have $\pi_k(x)(\hat{y}) = [M : N]^{k+1}E_{M_k}(xyf_k)^\wedge$, for all $x \in M_{2k+1}$, $\hat{y} \in \widehat{M_k}$. If $x \in M_{2k}$, $\hat{y} \in \widehat{M_k}$, then $\pi_k(x)(\hat{y}) = [M : N]^k E_{M_k}(xyg_k)^\wedge$ and hence $J_k\pi_k(M)'J_k = \pi_k(M_{2k})$.*

Proof The first part is precisely the definition of π_k, written in detail. Since $M_{2k+1} = \text{span}\{af_kb \,|\, a, b \in M_k\}$, it is enough to show the formula for $\pi_k(x)(\hat{y})$ for elements $x = af_kb$, a, $b \in M_k$ (we omit the hats to keep the notation simple). But $E_{M_k}(af_kbyf_k) = aE_N^{M_k}(by)E_{M_k}(f_k) = [M : N]^{-(k+1)}aE_N^{M_k}(by)$. Thus indeed $\pi_k(af_kb)(y) = aE_N^{M_k}(by) = [M : N]^{k+1}E_{M_k}(af_kbyf_k)$ as desired. Finally, since $M_{2k} = \text{span}\{ag_kb \,|\, a, b \in M_k\}$, it is enough to show the last formula for elements $x = ag_kb$, a, $b \in M_k$. Observe that $f_k = [M : N]^k g_k(e_1 \dots e_k)(e_{2k+1} \dots e_{k+1})$, so that

$$\begin{aligned}
\pi_k(ag_kb)(y) &= [M : N]^{2k+1}E_{M_k}\big(ag_kbyg_k(e_1 \dots e_k)(e_{2k+1} \dots e_{k+1})\big) \\
&= [M : N]^{2k+1}aE_M^{M_k}(by)E_{M_k}\big(g_k(e_1 \dots e_k)(e_{2k+1} \dots e_{k+1})\big) \\
&= [M : N]^{k+1}aE_M^{M_k}(by)E_{M_k}(f_k) = aE_M^{M_k}(by) \\
&= [M : N]^k E_{M_k}(ag_kbyg_k).
\end{aligned}$$

Thus $\pi_k|_{M_{2k}}$ is equal to the representation coming from the basic construction $M \subset M_k \subset \tilde{M}_{2k} \in B(L^2(M_k))$ and hence $J_k\pi_k(M)'J_k = \pi_k(M_{2k})$ as claimed. □

Observe that if $x \in M_{2k+1}$, $y \in M_k$, there is a unique element $w \in M_k$ such that $xyf_k = wf_k$, namely $w = [M : N]^{k+1}E_{M_k}^{M_{2k+1}}(xyf_k)$ (Pimsner and Popa [1986]). Thus $\pi_k(x)(\hat{y}) = \hat{w}$.

The fact shown above, that the representation coming from the basic construction of $M \subset M_k$ is equal to $\pi_k|_{M_{2k}}$ (note that π_k is a priori in the representation coming from the basic construction of $N \subset M_k$), will be used several times later on. Note also that by choosing a tunnel $M \supset N \supset N_1 \supset \dots$ and performing the basic construction for $N_i \subset M_k \subset B(L^2(M_k))$, we can actually represent each M_n on $L^2(M_k)$. The next lemma contains some useful identities, which will be used in

Proposition 2.5 and Lemma 3.6 to show how the above defined representations are related.

Lemma 2.3 *Let f_k be the projection for the k-step basic construction as in 2.1. Then*

i) $f_k e_{k+1} e_k \dots e_1 f_{k+1} = e_{k+1} e_k \dots e_1 f_{k+1}$, $k \geq 0$.

ii) $e_{k+2} e_{k+3} \dots e_{2k+1} f_k e_{2k+1} \dots e_{k+3} e_{k+2} = e_1 e_2 \dots e_k f_k e_k \dots e_2 e_1$, $k \geq 1$.

iii) $e_{k+2} e_{k+3} \dots e_{2k} f_k e_{2k} \dots e_{k+3} e_{k+2} = e_2 e_3 \dots e_k f_k e_k \dots e_3 e_2$, $k \geq 2$.

Proof i): We have $f_k = [M:N]^{k(k+1)/2}(e_{k+1} \dots e_{2k+1}) \cdots (e_1 \dots e_{k+1})$, since $f_k = f_k^*$. Before we start the proof, let us do a special case, which illustrates the mechanism of the proof:

$$\begin{aligned} f_2 e_3 e_2 e_1 f_3 &= [M:N]^3 [M:N]^6 (e_3 e_4 e_5)(e_2 e_3 e_4)(e_1 e_2 e_3)(e_3 e_2 e_1)(e_4 e_3 e_2 e_1) \\ &\quad (e_5 e_4 e_3 e_2)(e_6 e_5 e_4 e_3)(e_7 e_6 e_5 e_4) \\ &= [M:N]^6 [M:N](e_3 e_4 e_5) e_2 (e_3 e_4) e_1 (e_4 e_3 e_2 e_1)(e_5 e_4 e_3 e_2)(e_6 e_5 e_4 e_3) \\ &\quad (e_7 e_6 e_5 e_4) \\ &= [M:N]^6 e_3 (e_4 e_5) e_2 e_1 (e_3 e_2 e_1)(e_5 e_4 e_3 e_2)(e_6 e_5 e_4 e_3)(e_7 e_6 e_5 e_4) \\ &= [M:N]^6 e_3 e_2 e_1 (e_4 e_3 e_2 e_1)(e_5 e_4 e_3 e_2)(e_6 e_5 e_4 e_3)(e_7 e_6 e_5 e_4) \\ &= e_3 e_2 e_1 f_3. \end{aligned}$$

The proof in general works the same way:

$$\begin{aligned} & f_k e_{k+1} \; e_k \dots e_1 f_{k+1} \\ &= [M:N]^{k(k+1)/2} [M:N]^{(k+1)(k+2)/2} (e_{k+1} \dots e_{2k+1}) \cdots (e_2 \dots e_{k+2}) \\ &\quad (e_1 \dots e_{k+1})(e_{k+1} \dots e_1)(e_{k+2} \dots e_1) \cdots (e_{2k+3} \dots e_{k+2}) \\ &= [M:N]^{(k+1)(k+2)/2} [M:N]^{k(k-1)/2} (e_{k+1} \dots e_{2k+1}) \cdots (e_2 (e_3 \dots e_{k+2})) \\ &\quad e_1 (e_{k+2} \dots e_3 e_2 e_1) \cdots (e_{2k+3} \dots e_{k+2}) \\ &= [M:N]^{(k+1)(k+2)/2} [M:N]^{(k-1)(k-2)/2} (e_{k+1} \dots e_{2k+1}) \cdots \\ &\quad (e_3 (e_4 \dots e_{k+3})) e_2 e_1 (e_3 e_2 e_1)(e_{k+3} \dots e_2) \cdots (e_{2k+3} \dots e_{k+2}) \\ &= [M:N]^{(k+1)(k+2)/2} [M:N]^{(k-2)(k-3)/2} (e_{k+1} \dots e_{2k+1}) \cdots \\ &\quad (e_4 (e_5 \dots e_{k+4})) e_3 e_2 e_1 (e_4 e_3 e_2 e_1)(e_5 e_4 e_3 e_2)(e_{k+4} \dots e_3) \\ &\quad \cdots (e_{2k+3} \dots e_{k+2}) \\ &= [M:N]^{(k+1)(k+2)/2} [M:N]^{(k-3)(k-4)/2} (e_{k+1} \dots e_{2k+1}) \cdots \\ &\quad (e_5 \dots e_{k+5}) e_4 e_3 e_2 e_1 (e_5 e_4 \dots e_1)(e_6 e_5 \dots e_2)(e_7 \dots e_3) \\ &\quad (e_{k+5} \dots e_4) \cdots (e_{2k+3} \dots e_{k+2}) \\ &= [M:N]^{(k+1)(k+2)/2} [M:N]^{(k-j)(k-j-1)/2} (e_{k+1} \dots e_{2k+1}) \cdots \\ &\quad (e_{j+2} \dots e_{k+j+2}) e_{j+1} e_j \dots e_1 (e_{j+2} e_{j+1} \dots e_1)(e_{j+3} \dots e_2) \\ &\quad (e_{k+j+2} \dots e_{j+1}) \cdots (e_{2k+3} \dots e_{k+2}) \\ &= [M:N]^{(k+1)(k+2)/2} e_{k+1} \dots e_1 (e_{k+1} \dots e_1)(e_{k+2} \dots e_1) \cdots \\ &\quad (e_{2k+3} \dots e_{k+2}) \\ &= e_{k+1} \dots e_1 f_{k+1}. \end{aligned}$$

Thus the proof of i) is complete. ii) and iii) are proved in a similar way, using the commutation relations of the Jones projections e_i. Let us just indicate the proof of ii). Set $h_k = (e_{k+1} \dots e_3)(e_{k+2} \dots e_4) \cdots (e_{2k-2} \dots e_k)(e_{2k-1} \dots e_{k+1})$, i.e. h_k is $[M:N]^{-\frac{(k-2)(k-1)}{2}} \times$ the Jones projection for $M_1 \subset M_k \subset M_{2k-1}$, then $e_{k+2} \dots e_{2k+1} f_k e_{2k+1} \dots e_{k+2} = e_1 \dots e_k f_k e_k \dots e_1 = [M:N]^{-1} h_{k+1} e_1$. □

Next we review the shift on the higher relative commutants. Let us change notation for a while and consider an inclusion of II_1 factors $A \subset B$ with finite index. Let $A \subset B \subset^{e_1} B_1$ be the basic construction and denote by $(\pi_0, L^2(B))$ the standard representation of $A \subset B \subset B_1$ on $L^2(B)$. Let $J_0 : L^2(B) \to L^2(B)$ be the modular conjugation and define $\gamma_0(x) = \pi_0^{-1} J_0 \pi_0(x)^* J_0$, $x \in A' \cap B_1$, i.e., $\pi_0(\gamma_0(x)) = J_0 \pi_0(x)^* J_0$. The properties of the basic construction imply that $J_0 B J_0 = B'$ and $J_0 A' J_0 = B_1$ (Jones [1983]), so that $J_0 \big(A' \cap B_1\big) J_0 = A' \cap B_1$ (note that all algebras are represented on $L^2(B)$, so that we omit π_0. If we treat $A' \cap B_1$ as an abstract (finite dimensional) algebra and we want to emphasize that it is represented on $L^2(B)$, we write $\pi_0(A' \cap B_1)$). Thus γ_0 defines a map $\gamma_0 : A' \cap B_1 \to A' \cap B_1$ (viewed abstractly as finite dimensional algebras). It is easy to see that γ_0 is a surjective, unital, linear $*$-anti-isomorphism from $A' \cap B_1 \to A' \cap B_1$ such that $\gamma_0^2(x) = x$, for all $x \in A' \cap B_1$. Note that one often writes briefly $\gamma_0(x) = J_0 x^* J_0$, $x \in A' \cap B_1$, when the representation is understood.

The next lemma contains a number of identities which will be useful in the next section.

Lemma 2.4 *Let $A \subset B$ be an inclusion of II_1 factors with $[B : A] < \infty$. Let $A \subset B \subset^{e_1} B_1$ be the basic construction and denote by $(\pi_0, L^2(B))$ the standard representation of $A \subset B \subset B_1$ on $L^2(B)$. Let $\gamma_0 : A' \cap B_1 \to A' \cap B_1$ be defined by $\gamma_0(x) = \pi_0^{-1} J_0 \pi_0(x)^* J_0$, $x \in A' \cap B_1$ as above. Then,*

i) *$\pi_0(x)(a_1 \cdot \xi \cdot a_2) = a_1 \cdot \pi_0(x)(\xi) \cdot a_2$, for all a_1, $a_2 \in A$, $\xi \in L^2(B)$, $x \in A' \cap B_1$, where $a_1 \cdot \xi \cdot a_2 = \pi_0(a_1) J_0 \pi_0(a_2)^* J_0(\xi)$. Thus $\pi_0(x)$ is an A-A intertwiner of the A-A bimodule $L^2(B)$.*

ii) *Let $x \in B_1$, $y \in B$, then $\pi_0(x)(\hat{y}) \in \hat{B}$. If we view the element $\pi_0(x)(\hat{y})$ as an element in B, denoted by $\pi_0(x)(y)$, then $\pi_0(x)(y) e_1 = x y e_1$ (in B_1).*

iii) *We have $\gamma_0(x) y e_1 = y x e_1$ for all $x \in A' \cap B$, $y \in B$.*

iv) *We have $\pi_0(\gamma_0(p))(y^*) = \big(\pi_0(p)(y)\big)^*$, for all projections $p \in A' \cap B_1$ and $y \in B$.*

Proof We set $\pi = \pi_0$, $e = e_1$, $\gamma = \gamma_0$ and $J = J_0$.

i): Let $x \in A' \cap B_1$, a_1, $a_2 \in A$ and $b \in B$. Then $\pi(x)(a_1 \cdot \hat{b} \cdot a_2) = \pi(x)\pi(a_1) J \pi(a_2)^* J(\hat{b}) = \pi(a_1) J \pi(a_2)^* J \pi(x)(\hat{b})$, since $x \in A' \cap B_1$.

ii): By Proposition 2.2 we have that $\pi(x)(\hat{y}) = [B : A] E_B(x y e_1)^\wedge$, for all $y \in B$, $x \in B_1$. Thus $\pi(x)(y) e_1 = [B : A] E_B(x y e_1) e_1 = x y e_1$ by the comment after Proposition 2.2.

iii): Let $x \in A' \cap B$ and $y \in B$. We verify the equality in the representation π. Hence $\pi(\gamma(x))\pi(y)\pi(e)(\hat{b}) = J\pi(x)^* J(\widehat{y E_A(b)}) = J(\widehat{x^* E_A(b)^* y^*}) = \widehat{y x E_A(b)}$, $b \in B$, where we used that $\pi|_B$ is left multiplication. On the other hand $\pi(yxe)(\hat{b}) = \widehat{y x E_A(b)}$, $b \in B$, so that indeed $\gamma(x) y e = y x e$ as claimed.

iv): Let $p \in A' \cap B_1$ be a projection and let $y \in B$. Then $\big(\pi(p)(\hat{y})\big)^* = J\big(\pi(p)(\hat{y})\big) = J\pi(p) J(\hat{y}^*) = \pi(\gamma(p))(\hat{y}^*)$.

□

The following proposition shows how the representations π_k, π_{k+1} (see Proposition 2.2) are related, when restricted to the higher relative commutants associated to $N \subset M$.

Proposition 2.5 *Let π_k (resp. π_{k+1}) be the representations of M_k, M_{2k+1} (resp. M_{k+1}, M_{2k+3}) on $L^2(M_k)$ (resp. $L^2(M_{k+1})$) as in 2.2. Let $x \in N' \cap M_{2k+1}$, $y \in M_k$ and $z \in M$. Then $\pi_k(x)(y)e_{k+1}e_k \dots e_1 z = \pi_{k+1}(x)(ye_{k+1}e_k \dots e_1 z)$ as vectors in $L^2(M_{k+1})$.*

Proof First observe that $\pi_k(x)(y) \in M_k$ and $\pi_{k+1}(x)(ye_{k+1} \dots e_1 z) \in M_{k+1}$ (proposition 2.2), so that the above equality indeed makes sense in $\hat{M}_{k+1}$ (to keep the notation simple we omit "hats" as usual). Furthermore, as in 2.4 i), we see that $\pi_{k+1}(x)(wz) = \pi_{k+1}(x)J_{k+1}\pi_{k+1}(z)^* J_{k+1}(w) = J_{k+1}\pi_{k+1}(z)^* J_{k+1}\pi_{k+1}(x)(w) = \pi_{k+1}(x)(w)z$, for all $x \in N' \cap M_{2k+1}$, $w \in M_{k+1}$, $z \in M$, since $\pi_{k+1}|_M$ is left multiplication and $J_{k+1}\pi_{k+1}(M)J_{k+1} = \pi_{k+1}(M_{2k+2})'$ (by the last part of Proposition 2.2). Thus we have $\pi_{k+1}(x)(ye_{k+1}e_k \dots e_1 z) = \pi_{k+1}(x)(ye_{k+1}e_k \dots e_1)z$, and it is therefore enough to show the identity in the proposition for $z = 1$. Next, recall that $M_{2k+1} = \text{span}\{af_k b \,|\, a, b \in M_k\}$, so that it will be sufficient to show the identity for $x = af_k b$, a, $b \in M_k$. By Proposition 2.2, we have,

$$
\begin{aligned}
\pi_k(af_k b)(y)e_{k+1}e_k \dots e_1 &= [M:N]^{k+1} E_{M_k}^{M_{2k+1}}(af_k by f_k)e_{k+1}e_k \dots e_1 \\
&= [M:N]^{k+1} a E_N^{M_k}(by) E_{M_k}^{M_{2k+1}}(f_k)e_{k+1}e_k \dots e_1 \\
&= a E_N^{M_k}(by)e_{k+1}e_k \dots e_1.
\end{aligned}
$$

The right-hand side is calculated using Lemma 2.3 as follows,

$$
\begin{aligned}
\pi_{k+1}(af_k b)(ye_{k+1}e_k \dots e_1) &= [M:N]^{k+2} E_{M_{k+1}}^{M_{2k+3}}(af_k bye_{k+1}e_k \dots e_1 f_{k+1}) \\
&= [M:N]^{k+2} E_{M_{k+1}}^{M_{2k+3}}(af_k byf_k e_{k+1}e_k \dots e_1 f_{k+1}) \\
&= [M:N]^{k+2} a E_N^{M_k}(by) E_{M_{k+1}}^{M_{2k+3}}(e_{k+1}e_k \dots e_1 f_{k+1}) \\
&= a E_N^{M_k}(by)e_{k+1}e_k \dots e_1,
\end{aligned}
$$

which completes the proof. □

Next we give a short proof of a well-known formula for γ_0 (see for instance (Ocneanu [1991(a)], David)).

Theorem 2.6 *Let $A \subset B$ be an inclusion of II_1 factors with finite index, let $A \subset B \subset^{e_1} B_1$ be the basic construction and let $\{m_i\}_{i \in I} \subset B$ be a finite basis of B over A, such that $b = \sum_i m_i E_A(m_i^* b)$, for all $b \in B$. Let $x \in A' \cap B_1$, then*

$$
\gamma_0(x) = [B:A] \sum_{i \in I} E_B(e_1 m_i x) e_1 m_i^*. \tag{2.1}
$$

Proof As above, let $\pi = \pi_0$ denote the representation of B_1 on $L^2(B)$ coming from the basic construction, let $e = e_1$ and $J = J_0$. Then we have for all $b \in B$,

$x \in A' \cap B_1$,

$$[B:A]\sum_i \pi(E_B(em_ix))\pi(e)\pi(m_i^*)(\hat{b}) = [B:A]\sum_i \pi(E_B(em_ix))(\widehat{E_A(m_i^*b)})$$
$$= [B:A]\sum_i (E_B(em_ixE_A(m_i^*b)))^\wedge = [B:A](E_B(e\sum_i m_iE_A(m_i^*b)x))^\wedge$$
$$= [B:A]\widehat{E_B(ebx)} = [B:A](E_B(e(\pi(x^*)(b^*))^*))^\wedge = (\pi(\widehat{x^*)(b^*}))^*$$
$$= J\pi(x)^*J(\hat{b}) = \pi(\gamma(x))(\hat{b}),$$

where we used that $(ebx)^* = x^*b^*e = \pi(x^*)(b^*)e$ by Lemma 2.4 ii). □

Recall that $\{m_i\}_{i=1,\dots,n+1} \subset B$ $(n \leq [B:A] < n+1)$ is called an *orthonormal basis* of B over A if $E_A(m_i^*m_j) = 0$, $i \neq j$, $E_A(m_i^*m_i) = 1$, $1 \leq i \leq n$, and $E_A(m_{n+1}^*m_{n+1})$ is a projection in A (Pimsner and Popa [1986]). Any such basis satisfies $\sum_i m_ie_1m_i^* = 1$, $b = \sum_i m_iE_A(m_i^*b)$, for all $b \in B$. Note that in the above theorem we do not require that $\{m_i\}_{i\in I}$ is an *orthonormal* basis of B over A. Any finite basis will do. Furthermore, observe that the formula for $\gamma_0(x)$ does not depend on the choice of the basis, since the left-hand side in (2.1) is independent of such a choice.

Proposition 2.7 *Let $A \subset B$ be an inclusion of II_1 factors with finite index and let $\{m_i\}_{i\in I}$ be a finite orthonormal basis of B over A. Let $\mathrm{tr}_{A'}$ be the normalized trace on $A' = A' \cap B(L^2(B))$ and consider the map*

$$\phi(x) = [B:A]^{-1}\sum_i m_ixm_i^*,$$

$x \in A'$. Then ϕ is the unique $\mathrm{tr}_{A'}$-preserving conditional expectation from $A' \to B'$ $(\subset B(L^2(B))$.

Proof We first show that $\phi(A') \subset B'$. Let $A \subset B \subset^e B_1$ be the basic construction, $J: L^2(B) \to L^2(B)$ the modular conjugation, and recall that $[J,e] = 0$, $B_1 = JA'J$, $B_1 = \mathrm{span}\{aeb \,|\, a, b \in B\}$. Let $aeb \in B_1$, a, $b \in B$, then $\phi(JaebJ) = [B:A]^{-1}\sum_i m_iJaJeJbJm_i^* = [B:A]^{-1}\sum_i JaJm_iem_i^*JbJ = [B:A]^{-1}JabJ \in B'$. Thus indeed $\phi(A') \subset B'$. We clearly have $\phi(b') = b'$, for all $b' \in B'$ (since $\sum_i m_im_i^* = [B:A]$), $\phi(x^*) = \phi(x)^*$, $\phi((A')_+) \subset (B')_+$ and $\phi(bac) = b\phi(a)c$, b, $c \in B'$, $a \in A'$. Furthermore, if $x = JaebJ \in A'$, a, $b \in B$, then $\mathrm{tr}_{A'}(\phi(x)) = [B:A]^{-1}\mathrm{tr}_{A'}(JabJ) = [B:A]^{-1}\mathrm{tr}_{B_1}(J(JabJ)^*J) = [B:A]^{-1}\mathrm{tr}_{B_1}(b^*a^*) = \mathrm{tr}_{B_1}(b^*ea^*) = \mathrm{tr}_{A'}(JaebJ) = \mathrm{tr}_{A'}(x)$. □

Since the $\mathrm{tr}_{A'}$-preserving conditional expectation is unique, the above formula does not depend on the choice of the orthonormal basis.

Corollary 2.8 *Let $A \subset B$ be II_1 factors with finite index and let $\gamma_0 : A'\cap B_1 \to A' \cap B_1$ be defined as above. Then $\mathrm{tr}_{B_1}(\gamma_0(x)) = \mathrm{tr}_{A'}(x)$, for all $x \in A' \cap B_1$.*

Proof By Theorem 2.6 we have,

$$\begin{aligned}
\mathrm{tr}_{B_1}(\gamma_0(x)) &= [B:A]\sum_i \mathrm{tr}_{B_1}(E_B(e_1 m_i x)e_1 m_i^*) = \sum_i \mathrm{tr}_{B_1}(e_1 m_i x m_i^*) \\
&= [B:A]\mathrm{tr}_{B_1}(e_1\phi(x)) = [B:A]\mathrm{tr}_{A'}\big(J\phi(x)^* e_1 J\big) \\
&= [B:A]\mathrm{tr}_{A'}\big(J\phi(x)^* J e_1\big) = [B:A]\mathrm{tr}_{A'}\big(E_A^B(J\phi(x)^* J)e_1\big) \\
&= [B:A]\mathrm{tr}_B(J\phi(x)^* J)\mathrm{tr}_{A'}(e_1) = \mathrm{tr}_B(J\phi(x)^* J),
\end{aligned}$$

since $J\phi(x)^*J \in A' \cap B$ and hence $E_A^B(J\phi(x)^*J) = \mathrm{tr}_B(J\phi(x)^*J)$. Thus $\mathrm{tr}_{B_1}(\gamma_0(x)) = \mathrm{tr}_{B_1}(J\phi(x)^*J)) = \mathrm{tr}_{A'}(\phi(x)) = \mathrm{tr}_{A'}(x)$ by the previous proposition. □

Next we will discuss the shift from $A' \cap B_1 \to B_1' \cap B_3$. Let $J_1 : L^2(B_1) \to L^2(B_1)$ be the modular conjugation and consider the basic construction $A \subset B_1 \subset \pi_1(B_3) = J_1 A' J_1$, where we denote, as before, by π_1 the representation of B_1, B_3 on $L^2(B_1)$. Then, as above, we get a linear $*$-antiisomorphism $\gamma_1 : A' \cap B_3 \to A' \cap B_3$ such that $\pi_1(\gamma_1(x)) = J_1\pi_1(x)^* J_1$, $x \in A' \cap B_3$. Since $J_1\pi_1(A' \cap B_1)J_1 = \pi_1(B_1' \cap B_3)$, we have that $\gamma_1(A' \cap B_1) = B_1' \cap B_3$. Hence the composition $\gamma_1\gamma_0 : A' \cap B_1 \to B_1' \cap B_3$ is a surjective $*$-isomorphism, which is called the *shift* from $A' \cap B_1$ to $B_1' \cap B_3$ (here it is just the 2-shift, but later we will take for instance $A = N$, $B = M_k$ and we will obtain a shift from $N' \cap M_{2k+1}$ to $M_{2k+1}' \cap M_{4k+3}$) (see Ocneanu [1988], Ocneanu [1991(a)]). Note that one sometimes writes $J_1 J_0 x J_0 J_1$ instead of $\gamma_1\gamma_0(x)$, when all the representations are understood. We will deduce an explicit formula for the shift in terms of a basis and the Jones projections e_1, e_2 and e_3.

Lemma 2.9 *Let $A \subset B$ be an inclusion of II_1 factors with finite index and let $A \subset B \subset^{e_1} B_1$ be the basic construction. Then*

i) $E_A^{B_1}(xe_1yb_1) = E_A(xE_B(e_1yb_1))$, *for all* $x, y \in B$, $b_1 \in B_1$ ($E_A = E_A^B$, $E_B = E_B^{B_1}$).

ii) $E_A(xE_B(e_1yb_1)) = E_A(E_B(b_1xe_1)y)$, *for all* $x, y \in B$, $b_1 \in A' \cap B_1$.

Proof The proof of i) is trivial and ii) follows from the uniqueness of the trace preserving conditional expectation $E_A^{B_1} : B_1 \to A$. □

Next we prove some identities which will be needed later on.

Proposition 2.10 *Let $A \subset B$ be an inclusion of II_1 factors with finite index and let $A \subset B \subset^{e_1} B_1$ be the basic construction. Let $\{m_i\}_{i\in I}$ be a finite orthonormal basis of B over A. Then $\{[B:A]^{\frac{1}{2}} m_i e_1 m_j\}_{i,j\in I}$ is a (not necessarily orthonormal) basis of B_1 over A satisfying*

$$\begin{aligned}
b_1 &= [B:A]\sum_{i,j} m_i e_1 m_j E_A(m_j^* E_B(e_1 m_i^* b_1)) \\
&= [B:A]\sum_{i,j} m_i e_1 m_j E_A^{B_1}((m_i e_1 m_j)^* b_1) \\
&= [B:A]\sum_{i,j} E_A(E_B(b_1 m_i e_1)m_j) m_j^* e_1 m_i^*, \qquad \textit{for all } b_1 \in B_1.
\end{aligned}$$

Futhermore, if $x \in A' \cap B_1$, then $x = [B:A]\sum_{i,j} E_A(m_i E_B(e_1 m_j x)) m_j^* e_1 m_i^*$.

Proof It is easy to see that $\{[B:A]^{\frac{1}{2}} m_i e_1\}_{i\in I}$ is an orthonormal basis of B_1 over B. Thus, if $b_1 \in B_1$, we get $b_1 = [B:A]\sum_i m_i e_1 E_B(e_1 m_i^* b_1)$. But $E_B(e_1 m_i^* b_1) = \sum_j m_j E_A(m_j^* E_B(e_1 m_i^* b_1))$, so that $b_1 = [B:A]\sum_{i,j} m_i e_1 m_j E_A(m_j^* E_B(e_1 m_i^* b_1))$, which shows the first equality. The second one follows from Lemma 2.9 i). Since $b_1 = [B:A]\sum_i E_B(b_1 m_i e_1)e_1 m_i^*$ and $E_B(b_1 m_i e_1) = \sum_j E_A(E_B(b_1 m_i e_1)m_j)m_j^*$, the third formula follows. The expression for $x \in A' \cap B_1$ follows from this and Lemma 2.9 ii). □

The next theorem gives an expression of the "spatially defined" shift (using J's) (Ocneanu [1988] and [1991(a)]) in terms of an orthonormal basis and the Jones projections e_i (Pimsner and Popa [1986]). The second formula below is well-known (Ocneanu [1991(a)], Pimsner and Popa [1986], see also David).

Theorem 2.11 *Let $A \subset B$ be an inclusion of II_1 factors with finite index and let $\gamma_1\gamma_0$ be the shift from $A'\cap B_1$ to $B_1'\cap B_3$ as defined above. Then*

$$\gamma_1\gamma_0(x)e_1 = [B:A]^2 e_1 e_2 x e_3 e_2 e_1,$$

for all $x \in A'\cap B_1$. Thus, if $\{m_i\}_{i\in I}$ is a finite orthonormal basis of B over A we have

$$\gamma_1\gamma_0(x) = [B:A]^2 \sum_{i\in I} m_i e_1 e_2 x e_3 e_2 e_1 m_i^*,$$

for all $x \in A'\cap B_1$.

Proof Let $\{m_i\}_{i\in I}$ be an orthonormal basis of B over A with $m_1 = 1$, so that $E_A(m_i^*) = 0$, if $i \neq 1$, and $E_A(m_1^*) = 1$ (this can be assumed without loss of generality since A is a factor). Then $\{[B:A]^{\frac{1}{2}} m_i e_1 m_j\}_{i,j\in I}$ is a basis of B_1 over A satisfying the hypothesis of Theorem 2.6 (with B_1 in place of B) by Proposition 2.10. Set $r_{ij} = [B:A]^{\frac{1}{2}} m_i e_1 m_j$, and let $A \subset B_1 \subset^f B_3$ be the basic construction with projection $f = [B:A]e_2e_1e_3e_2$ (Proposition 2.1). If $x \in A'\cap B_1$, we have by Theorem 2.6 that,

$$\begin{aligned}
\gamma_1\gamma_0(x)e_1 &= [B_1:A]\Big(\sum_{i,j} E_{B_1}(f r_{ij}\gamma_0(x)) f r_{ij}^*\Big)e_1 \\
&= [B:A]^2\Big(\sum_{i,j} E_{B_1}(f) r_{ij}\gamma_0(x) f r_{ij}^*\Big)e_1 \\
&= [B:A]\Big(\sum_{i,j} m_i e_1 m_j \gamma_0(x) f m_j^* e_1 m_i\Big)e_1 \\
&= [B:A]^3 \sum_{i,j,k} m_i e_1 m_j E_B(e_1 m_k x) e_1 m_k^* e_2 e_1 e_3 e_2 m_j^* e_1 m_i^* e_1 \\
&= [B:A]^3 \sum_{j,k} e_1 m_j E_B(e_1 m_k x) e_1 m_k^* e_2 e_1 m_j^* e_3 e_2 e_1 \\
&= [B:A]^2 e_1 e_2 \Big([B:A]\sum_{j,k} E_A(m_j E_B(e_1 m_k x)) m_k^* e_1 m_j^*\Big) e_3 e_2 e_1 \\
&= [B:A]^2 e_1 e_2 x e_3 e_2 e_1,
\end{aligned}$$

where the last equality follows from Proposition 2.10. The formula for $\gamma_1\gamma_0$ is now immediate since $\sum_i m_i e_1 m_i^* = 1$. □

Corollary 2.12 *The shift $\gamma_1\gamma_0 : A'\cap B_1 \to B_1'\cap B_3$ is trace preserving, i.e., $\mathrm{tr}_{B_3}(\gamma_1\gamma_0(x)) = \mathrm{tr}_{B_1}(x)$, for all $x \in A'\cap B_1$.*

Proof We calculate $\mathrm{tr}_{B_3}(\gamma_1\gamma_0(x)) = [B : A]^2 \sum_i \mathrm{tr}_{B_3}(m_i e_1 e_2 x e_3 e_2 e_1 m_i^*) = [B : A] \sum_i \mathrm{tr}_{B_3}(m_i e_1 E_B(x) e_2 e_1 m_i^*) = \sum_i \mathrm{tr}_{B_3}(m_i E_A^{B_1}(x) e_1 m_i^*) = \mathrm{tr}_{B_1}(\sum_i m_i e_1 m_i^*)\mathrm{tr}_{B_1}(x) = \mathrm{tr}_{B_1}(x)$, for all $x \in A' \cap B_1$. □

Although not needed in this paper, we include for the sake of completeness a few words about the 2-shift and the Fourier transforms, both of which are particular operators in the affine Hecke algebra associated to a subfactor (Jones [1994], Jones [1996]). For the next theorem, see (Pimsner and Popa [1988]).

Theorem 2.13 *Let $A \subset B$ be an inclusion of II_1 factors with $[B : A] < \infty$. Let $A \subset B \subset^{e_1} B_1 \subset^{e_2} B_2 \subset \ldots$ be the basic construction and define maps $T_n : A' \cap B_{2n+1} \to A' \cap B_{2n+3}$ by $T_n(x) = [B : A]^{2n+2} e_1 e_2 \ldots e_{2n+2} x e_{2n+3} e_{2n+2} \ldots e_2 e_1$, for all $x \in A' \cap B_{2n+1}$. Let $\{m_i\}_{i \in I}$ be a finite orthonormal basis of B over A and set $S_n(x) = \sum_i m_i T_n(x) m_i^*$, $x \in A' \cap B_{2n+1}$. Then we have,*

i) *$T_n(x^*) = (T_n(x))^*$, $T_n(x^*x) \geq 0$, $T_n(xy) = T_n(x)T_n(y)$, for all $x \in A' \cap B_{2n+1}$, $T_n(1) = e_1$. Furthermore $\mathrm{tr}_{B_{2n+1}}(T_n(x)) = [B : A]^{-1}\mathrm{tr}_{B_{2n+1}}(x)$, for all $x \in A' \cap B_{2n+1}$.*

ii) *$S_n(x) \in B_1' \cap B_{2n+3}$ for all $x \in A' \cap B_{2n+1}$. Thus $S_n : A' \cap B_{2n+1} \to B_1' \cap B_{2n+3}$ is a unital $*$-isomorphism, which is onto and trace preserving, $\mathrm{tr}_{B_{2n+3}}(S_n(x)) = \mathrm{tr}_{B_{2n+1}}(x)$, $x \in A' \cap B_{2n+1}$.*

iii) *$S_n(x) \in B_1' \cap B_{2n+2}$ for all $x \in A' \cap B_{2n}$. Thus $S_n|_{A' \cap B_{2n}} : A' \cap B_{2n} \to B_1' \cap B_{2n+2}$ is a unital $*$-isomorphism, which is onto and trace preserving.*

iv) *The definition of S_n does not depend on the choice of the orthonormal basis.*

v) *$S_{n+1}|_{A' \cap B_{2n+1}} = S_n$, $S_{n+1}|_{A' \cap B_{2n}} = S_n|_{A' \cap B_{2n}}$. Thus $(S_n)_{n \in \mathbb{N}}$ defines a trace-preserving surjective $*$-isomorphism $S : \overline{\cup_n A' \cap B_n}^w \to \overline{\cup_n B_1' \cap B_n}^w$ such that $S(A' \cap B_n) = B_1' \cap B_{n+2}$.*

vi) *$S_0 : A' \cap B_1 \to B_1' \cap B_3$ satisfies $S_0 = \gamma_1\gamma_0$.*

Proof The easy verification of i) is left to the reader. To show ii), let $x \in A' \cap B_{2n+1}$, $b_1, b_2 \in B$. Then,

$$\begin{aligned} b_1 e_1 b_2 S_n(x) &= [B : A]^{2n+2} \sum_i b_1 E_A(b_2 m_i) e_1 e_2 \ldots e_{2n+2} x e_{2n+3} \ldots e_2 e_1 m_i^* \\ &= [B : A]^{2n+2} \sum_{i,j} m_j E_A(m_j^* b_1 E_A(b_2 m_i)) e_1 e_2 \ldots x e_{2n+3} \ldots e_2 e_1 m_i^* \\ &= [B : A]^{2n+2} \sum_{i,j} m_j e_1 e_2 \ldots e_{2n+2} x e_{2n+3} \ldots e_2 e_1 E_A(m_j^* b_1 E_A(b_2 m_i)) m_i^* \\ &= [B : A]^{2n+2} \sum_j m_j e_1 e_2 \ldots e_{2n+2} x e_{2n+3} \ldots e_2 e_1 m_j^* b_1 e_1 b_2 \\ &= S_n(x) b_1 e_1 b_2, \end{aligned}$$

since $\sum_i E_A(b_2 m_i) m_i^* = b_2$. Hence $S_n(x) \in B_1' \cap B_{2n+3}$ as claimed. The fact that S_n is a unital $*$-homomorphism, which preserves the trace, follows immediately from i). Thus S_n is injective. Since $\dim A' \cap B_{2n+1} = \dim B_1' \cap B_{2n+3}$, S_n is also surjective and the proof of ii) is complete. If $x \in A' \cap B_{2n}$, then $S_n(x) \in B_1' \cap B_{2n+3}$ by ii). But $e_{2n+2} x e_{2n+3} e_{2n+2} = [B : A]^{-1} x e_{2n+2}$, so that indeed $S_n(x) \in B_1' \cap B_{2n+2}$. The remaining statement in iii) follows as in ii). The fact that S_n does not depend on the choice of the orthonormal basis can be deduced easily from the fact that two orthonormal bases of B over A differ by a unitary in an amplification of A (Pimsner and Popa [1986]). The first two equaltities in v) are straightforward and the rest of v) is a consequence of ii) and iii). Finally, vi) is obvious by Theorem 2.11. □

Definition 2.14 *The map* $S_n : A' \cap B_{2n+1} \to B_1' \cap B_{2n+3}$ *(resp.* $S_n|_{A' \cap B_{2n}} : A' \cap B_{2n} \to B_1' \cap B_{2n+2}$*) is called the 2-shift on* $A' \cap B_{2n+1}$ *(resp. on* $A' \cap B_{2n}$*).*

Remark 2.15 *One can of course define in a similar way a 2-shift from* $B' \cap B_n$ *to* $B_2' \cap B_{n+2}$*. The "orthonormal basis approach" to the 2-shift can be found in (*Pimsner and Popa [1986]*) and the "spatial approach" using* J*'s in (*Ocneanu [1988], Ocneanu [1991(a)], *see also* Popa [1990], Choda and Hiai [1991]*). It is shown in (*David*) that both points of view coincide. Note that the 2-shift actually defines a trace preserving* $*$*-isomorphism between the standard invariants of* $A \subset B$ *and* $B_1 \subset B_2$ *(resp.* $B \subset B_1$ *and* $B_2 \subset B_3$*).*

Let us discuss briefly another set of special operators contained in the affine Hecke algebra associated to a subfactor, the so-called Fourier transforms (Ocneanu [1988] and [1991(b)], see also Bisch [1994(a)]).

Definition 2.16 *Let* $A \subset B$ *be an inclusion of* II_1 *factors with finite index and let* $A \subset B \subset^{e_1} B_1 \subset^{e_2} B_2 \subset \ldots$ *be the basic construction. We define linear maps* ϕ, ϕ_l *and* $\phi_r : B' \cap B_{k+1} \to A' \cap B_k$ *by* $\phi_r(x) = [B : A]^{\frac{k+2}{2}} E_{B_k}(xe_1e_2 \ldots e_ke_{k+1})$, $\phi_l(x) = [B : A]^{\frac{k+2}{2}} E_{B_k}(e_{k+1}e_k \ldots e_2e_1x)$ *and* $\phi_{lr}(x) = [B : A]^{\frac{k+2}{2}} E_{B_k}(e_{k+1}e_k \ldots e_2e_1xe_1e_2 \ldots e_ke_{k+1})$*. We call* ϕ_r, ϕ_l *and* ϕ_{lr} *the left, resp. right, resp. 2-sided Fourier transform on* $B' \cap B_{k+1}$*.*

Let us remark that other normalizations of the above defined maps can also be found in the literature and that these maps are the classical Fourier transforms for finite groups in the case where the subfactor is obtained as a crossed product by a finite group (see Ocneanu [1991(b)]). Various combinations of shifts and Fourier transforms will give interesting linear maps on the higher relative commutants. For instance, the maps γ_k can be written as such a composition. To see the usefulness of the maps defined in 2.16, let us prove the following (well-known) proposition.

Proposition 2.17 *Let* $A \subset B \subset B_1 \subset B_2 \subset \ldots$ *be as above. Then* $\dim A' \cap B_k = \dim B' \cap B_{k+1}$*, for all* k*.*

Proof Let $\phi_r : B' \cap B_{k+1} \to A' \cap B_k$ be the Fourier transform as defined in 2.16. If $\phi_r(x) = 0$, then $e_{k+2}xe_1 \ldots e_{k+1}e_{k+2} = 0$ and hence $e_{k+2}xe_1 = 0$. Using an orthonormal basis of B over A, we get $e_{k+2}x = 0$ and hence $x = 0$. Thus $\dim A' \cap B_k \geq \dim B' \cap B_{k+1}$. Using the Fourier transform on $A' \cap B_k$, we see that $\dim A_1' \cap B_{k-1} \geq \dim A' \cap B_k \geq \dim B' \cap B_{k+1}$, where $B \supset A \supset A_1$ is the downward basic construction. But the 2-shift is an isomorphism between $A_1' \cap B_{k-1}$ and $B' \cap B_{k+1}$ (Theorem 2.13), so that all dimensions coincide. □

3 Principal graphs, reduced bimodules and the fusion algebra associated to a subfactor

Let $N \subset M$ be an inclusion of II_1 factors with finite index and let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ be the associated Jones tower of II_1 factors. We have seen above that the Hilbert spaces $L^2(M_n)$, $n \geq 0$ ($M_0 = M$), are natural N-N, M-M, M-N resp. N-M bimodules via the action $a \cdot \hat{x} \cdot b = \widehat{axb}$, a, b in N or M, $\hat{x} \in \hat{M}_n \subset L^2(M_n)$ (extended to all of $L^2(M_n)$ by continuity). Observe that if $J_n : L^2(M_n) \to L^2(M_n)$ denotes the modular conjugation, we could have written this action as $a \cdot \hat{x} \cdot b = \pi_n(a) J_n \pi_n(b)^* J_n(\hat{x})$, where π_n denotes the representations introduced in Proposition 2.2, since $\pi_n|_M$ is left multiplication. In this section, ρ will denote the N-M bimodule ${}_N L^2(M)_M$ with the above action $a \cdot \xi \cdot b = a J_0 b^* J_0(\xi)$, $a \in N$, $b \in M$ and $\xi \in L^2(M)$. As we have seen after Remark 1.17, the contragredient bimodule $\bar{\rho}$ is then just $L^2(M)$ as M-N Hilbert bimodule with the actions as above (exchanging N and M of course). Throughout this section we will work with N-N and N-M bimodules. It is a trivial exercise to rewrite all the statements for M-M and M-N bimodules. We will use the notation for bimodule multiplication introduced in 1.20.

The following two results are well-known. We include the proofs for the convenience of the reader.

Proposition 3.1 *Let $N \subset M$ be an inclusion of II_1 factors with finite index and associated N-N resp. N-M bimodules $L^2(M_n)$ as defined above. Then*

a) ${}_N L^2(M_n) \otimes_M {}_M L^2(M)_N \cong {}_N L^2(M_n)_N$ *as N-N bimodules.*

b) ${}_N L^2(M_n) \otimes_N {}_N L^2(M)_M \cong {}_N L^2(M_{n+1})_M$ *as N-M bimodules.*

Thus, if we let $\rho = {}_N L^2(M)_M$ be as above, we have

i) $(\rho\bar{\rho})^n \cong {}_N L^2(M_{n-1})_N$ *as N-N bimodules.*

ii) $(\bar{\rho}\rho)^n \cong {}_M L^2(M_n)_M$ *as M-M bimodules.*

iii) $(\rho\bar{\rho})^n \rho \cong {}_N L^2(M_n)_M$ *as N-M bimodules.*

iv) $(\bar{\rho}\rho)^n \bar{\rho} \cong {}_M L^2(M_n)_N$ *as M-N bimodules.*

Proof As in the proof of Lemma 1.14 we see that ${}_N L^2(M_n) \otimes_M L^2(M)_N \cong {}_N L^2(M_n)_N$ as N-N bimodules. Let us prove b). By Proposition 1.5 we know that the right N-bounded vectors in $L^2(M_n)$ (resp. left N-bounded vectors in $L^2(M)$) are given by $\hat{M}_n$ (resp. $\hat{M}$). As usual, we will abuse notation and write just x instead of $\hat{x}$ for the vector $\hat{x} = x(\hat{1}_{M_n}) \in \hat{M}_n \subset L^2(M_n)$.

Define a map $T : \hat{M}_n \odot \hat{M} \to L^2(M_{n+1})$ by

$$T\Big(\sum_i x_i \otimes y_i\Big) = [M:N]^{\frac{n+1}{2}} \sum_i x_i e_{n+1} e_n \dots e_2 e_1 y_i.$$

Note that $\langle a, b \rangle_N = E_N(ab^*)$, a, $b \in M$. We compute,

$$\begin{aligned} \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle &= \big(x_1 \langle y_1, y_2 \rangle_N, x_2\big)_{L^2(M_n)} \\ &= \big(x_1 E_N(y_1 y_2^*), x_2\big) = \mathrm{tr}_{M_n}\big(x_2^* x_1 E_N(y_1 y_2^*)\big), \end{aligned}$$

for all $x_i \in M_n$, $y_i \in M$, $i = 1, 2$. On the other hand we have,

$$\begin{aligned}[M:N]^{n+1} & \left(x_1 e_{n+1}\dots e_1 y_1, x_2 e_{n+1}\dots e_1 y_2\right)_{L^2(M_{n+1})} \\ &= [M:N]^{n+1}\mathrm{tr}\left(x_2^* x_1 e_{n+1}\dots e_1 y_1 y_2^* e_1 \dots e_{n+1}\right) \\ &= [M:N]^{n+1}\mathrm{tr}\left(x_2^* x_1 E_N(y_1 y_2^*) e_{n+1}\dots e_2 e_1 e_2 \dots e_{n+1}\right) \\ &= [M:N]\mathrm{tr}\left(x_2^* x_1 E_N(y_1 y_2^*) e_{n+1}\right) = \mathrm{tr}\left(x_2^* x_1 E_N(y_1 y_2^*)\right) \\ &= \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle .\end{aligned}$$

This implies that T is well-defined and induces an isometry, again denoted by T, from ${}_N L^2(M_n) \otimes_N {}_N L^2(M)_M$ to ${}_N L^2(M_{n+1})_M$ which is clearly a morphism of N-M (or even M-M) bimodules. Observe that $M_{n+1} = M_n e_{n+1} M_n$ (by which we mean the linear span of elements of the form $a e_{n+1} b$, a, $b \in M_n$), $M_n = M_{n-1} e_n M_{n-1}$, which implies that $M_{n+1} = M_n e_{n+1} e_n M_{n-1}$ and by induction we obtain $M_{n+1} = M_n e_{n+1} e_n \dots e_1 M$. From this we deduce that our map T is surjective and hence b) is shown.

The proof of i)-iv) is now immediate by induction using a), b) (and of course associativity of the bimodule tensor product, Proposition 1.12) □

Proposition 3.2 *The higher relative commutants associated to an inclusion of II_1 factors $N \subset M$ with finite index are spaces of bimodule intertwiners. More precisely, one has,*

$$\begin{aligned} N' \cap M_{2n+1} &\cong Hom_{N\text{-}N}({}_N L^2(M_n)_N), \\ N' \cap M_{2n} &\cong Hom_{N\text{-}M}({}_N L^2(M_n)_M), \\ M' \cap M_{2n} &\cong Hom_{M\text{-}M}({}_M L^2(M_n)_M), \\ M' \cap M_{2n+1} &\cong Hom_{M\text{-}N}({}_M L^2(M_n)_N). \end{aligned}$$

Proof Let us prove the first identity. Denote as in Section 2 by π_n the representation of $N \subset M_n$ on $L^2(M_n)$. Then by (Pimsner and Popa [1988], see Proposition 2.1), $\pi_n(N) \subset \pi_n(M_n) \subset \pi_n(M_{2n+1}) \subset B(L^2(M_n))$ is the basic construction. Thus,

$$\begin{aligned}\mathrm{Hom}_{N\text{-}N}({}_N L^2(M_n)_N) &= \pi_n(N)' \cap \pi_n(N^{\mathrm{op}})' \cap B(L^2(M_n)) \\ &= \pi_n(N)' \cap (J_n \pi_n(N) J_n)' \cap B(L^2(M_n)) \\ &= \pi_n(N' \cap M_{2n+1}) \cong N' \cap M_{2n+1},\end{aligned}$$

where J_n denotes as usual the modular conjugation on $L^2(M_n)$. For the second identity we use the last statement of Proposition 2.2, and then the proof is identical. The remaining two identities are shown in the same way. □

We will show next that the principal graphs of a subfactor can be viewed as "principal fusion rules" for certain bimodules associated to the vertices of these graphs (see also Ocneanu [1988], [1991(a)] and [1991(b)], Jones and Sunder [1996]). If we denote by (Γ, Γ') the principal graphs of $N \subset M$, then Γ will describe the principal part of the Bratteli diagram of $\mathbb{C} = N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset \dots$ and Γ' that of $\mathbb{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \dots$. Let us recall first the definition of the principal graphs (Goodman et al. [1989], see also (Popa [1990], 2.1, 2.2, 3.5), (Wenzl [1988], 1.1, 1.2)). We will describe the construction of Γ. Since $N' \cap M_{2n+1}$, $n \geq -1$, (resp. $N' \cap M_{2n}$, $n \geq 0$) are finite dimensional C*-algebras, they are multimatrix algebras and we suppose that their simple summands are indexed by a set K_n, $n \geq -1$ (resp. L_n, $n \geq 0$) (set $M_{-1} = N$, $M_0 = M$). Let z_{2n+3} be the central support of e_{2n+3} in $N' \cap M_{2n+3}$ (observe that $z_{2n+3} = \bigvee\{ue_{2n+3}u^* \mid u \in U(N' \cap M_{2n+2})\}$ (Popa [1990], 2.1)), then $(N' \cap M_{2n+3})z_{2n+3} = \text{span}(N' \cap M_{2n+2})e_{2n+3}(N' \cap M_{2n+2}) \overset{def}{\to}= Y_{2n+3}$ and $N' \cap M_{2n+2} \cong (N' \cap M_{2n+2})z_{2n+3} \subset (N' \cap M_{2n+3})z_{2n+3}$ is (isomorphic) to the (algebraic) basic construction of $N' \cap M_{2n+1} \subset N' \cap M_{2n+2}$. In other words, $N' \cap M_{2n+3}$ has a direct summand (which is the 2-sided ideal Y_{2n+3} generated by e_{2n+3} in $N' \cap M_{2n+3}$), that is isomorphic to the basic construction of the finite dimensional algebras $N' \cap M_{2n+1} \subset N' \cap M_{2n+2}$. Furthermore, note that $e_{2n+3}(N' \cap M_{2n+3})e_{2n+3} = (N' \cap M_{2n+1})e_{2n+3}$. Due to the properties of the basic construction we get a natural isomorphism $\phi : Z(N' \cap M_{2n+1}) \to Z(Y_{2n+3})$ (the centers), which sends a projection $q \in Z(N' \cap M_{2n+1})$ to the unique projection $\tilde{q} \in Z(Y_{2n+3})$ satisfying $qe_{2n+3} = \tilde{q}e_{2n+3}$ ($\tilde{q} = JqJ$, where J is the modular conjugation on $N' \cap M_{2n+2}$). Thus, if q_k is the identity of the k-th simple summand in $N' \cap M_{2n+1}$, $k \in K_n$, then $\tilde{q} = \phi(q_k)$ is the identity of a simple summand in Y_{2n+3} (and hence $N' \cap M_{2n+3}$), indexed by some $\tilde{k} \in K_{n+1}$. The map $k \in K_n \to \tilde{k} \in K_{n+1}$ identifies K_n with a subset of K_{n+1} and we will henceforth regard K_n as a subset of K_{n+1} using this identification. Recall that $K_{n+1} \backslash K_n$ is usually refered to as the "new stuff" (Goodman et al. [1989]). If $p_k \in N' \cap M_{2n+1}$ is a minimal projection $\leq q_k$, then $p_k e_{2n+3}(N' \cap M_{2n+3})p_k e_{2n+3} = p_k(N' \cap M_{2n+1})p_k e_{2n+3} = \mathbb{C}p_k e_{2n+3}$, i.e., $p_k e_{2n+3}$ is a minimal projection in $N' \cap M_{2n+3}$. Since $p_k e_{2n+3} = p_k q_k e_{2n+3} = p_k \tilde{q} e_{2n+3} = (p_k e_{2n+3})\tilde{q}$, where $\tilde{q} = \phi(q_k)$, we have that $p_k e_{2n+3} \leq \tilde{q}$, so that $p_k e_{2n+3}$ is contained in the simple summand of $N' \cap M_{2n+3}$ indexed by $\tilde{k}$. Thus, the identification $k \in K_n \to \tilde{k} \in K_{n+1}$ as described above, can be obtained in the following equivalent way: Let p_k be a minimal projection in the k-th simple summand of $N' \cap M_{2n+1}$, $k \in K_n$, let $\tilde{k} \in K_{n+1}$ be the index of the simple summand in $N' \cap M_{2n+3}$, which contains the minimal projection $p_k e_{2n+3}$, then the map $k \in K_n \to \tilde{k} \in K_{n+1}$ is precisely the above described identification.

The same analysis can be carried out for $N' \cap M_{2n} \subset N' \cap M_{2n+1}$, and we get an identification of L_n as a subset of L_{n+1}. Namely, if p_l is a minimal projection in the l-th simple summand of $N' \cap M_{2n}$, then $p_l e_{2n+2}$ is a minimal projection in a simple summand of $N' \cap M_{2n+2}$, indexed by some $\tilde{l} \in L_{n+1}$. The map $l \in L_n \to \tilde{l} \in L_{n+1}$ identifies as before L_n with a subset of L_{n+1}. Using these identifications, we set $K = \bigcup_{n \geq -1} K_n$ and $L = \bigcup_{n \geq 0} L_n$. Note that $K_{-1} = \{k_{-1}\}$ is a singleton, since $N' \cap N = \mathbb{C}$. Thus, if we denote by $G_n = (G_{kl})_{k \in K_n,\, l \in L_{n+1}}$ the inclusion matrix of the unital inclusion $N' \cap M_{2n+1} \subset N' \cap M_{2n+2}$, where G_{kl} denotes the multiplicity with which the k-th simple summand of $N' \cap M_{2n+1}$ sits in the l-th simple summand of $N' \cap M_{2n+2}$, then the transpose matrix G_n^t is the inclusion matrix for $N' \cap M_{2n+2} \hookrightarrow Y_{2n+3}$ (this follows again from the properties of the finite dimensional basic construction (Jones [1983])). In particular, G_n^t is a "submatrix" of the inclusion matrix $H_{n+1} = (H_{lk})_{l \in L_{n+1},\, k \in K_{n+1}}$, for $N' \cap M_{2n+2} \subset N' \cap M_{2n+3}$, where H_{lk} denotes as before the multiplicity with which the l-th simple summand of

$N' \cap M_{2n+2}$ sits in the k-th simple summand of $N' \cap M_{2n+3}$. More precisely, we have $H_{lk} = G_{kl}$, $l \in L_{n+1}$, $k \in K_n \subset K_{n+1}$. Thus, there is a unique "$K \times L$" matrix $G = (G_{kl})_{k \in K,\ l \in L}$, such that $(G_{kl})_{k \in K_n,\ l \in L_{n+1}}$ is the inclusion matrix for $N' \cap M_{2n+1} \subset N' \cap M_{2n+2}$ and $\left((G_{kl})_{k \in K_{n+1},\ l \in L_{n+1}}\right)^t$ is the one for $N' \cap M_{2n+2} \subset N' \cap M_{2n+3}$ (see Popa [1990]). We construct now a bipartite graph Γ in the following way. The vertices of Γ are defined to be the set $K \cup L$ and there are G_{kl}, $k \in K$, $l \in L$, edges between a vertex $k \in K$ and a vertex $l \in L$. Γ is then called the *principal graph* of $N \subset M$. Recall that Γ has a distinguished vertex $*$, which is the vertex denoted by k_{-1} above. Equivalently, we could have constructed Γ in the following way: The vertices of Γ are $\{k_{-1}\} \cup L_0 \cup K_0 \backslash K_{-1} \cup L_1 \backslash L_0 \cup K_1 \backslash K_0 \cdots = K \cup L$ as before. Let $*$ denote the vertex k_{-1} and connect $*$ to each $l \in L_0$ according to the multiplicity G_{*l}, $l \in L_0$. Then connect each $l \in L_0$ to the vertices $k \in K_0 \backslash K_{-1}$ according to their multiplicity G_{kl}, $k \in K_0 \backslash K_{-1}$, $l \in L_0$, then each $k \in K_0 \backslash K_{-1}$ to $l \in L_1 \backslash L_0$ etc. ("the new stuff is connected only to the old new stuff" (Goodman et al. [1989])). Observe that this procedure results in the same bipartite graph with distinguished vertex $*$ as the previous one, since for example each $k \in K_n$, viewed as an element of K_{n+1}, is connected to $l \in L_n$ with the same multiplicity G_{kl} as $l \in L_n$ is to that same k, viewed now as an element of K_n (this is just a rewording of what was described above as "G_n^t is a submatrix of H_{n+1}"). The details of all this can be found in (Goodman et al. [1989] or Popa [1990]).

The principal graph Γ' is constructed in the same way, using sets K'_n, $n \geq 0$, indexing the simple summands of $M' \cap M_{2n}$ and L'_n, $n \geq 0$, indexing the simple summands of $M' \cap M_{2n+1}$. We obtain a "$K' \times L'$" matrix $G' = (G'_{k'l'})_{k' \in K',\ l' \in L'}$ as above. Observe that Γ' has again a distinguished vertex $*' = k'_0$, if we denote $K'_0 = \{k'_0\}$ (i.e., k'_0 is the index of the simple summand $\mathbb{C}$ in $M' \cap M = \mathbb{C}$).

Suppose now that we are given the principal graphs (Γ, Γ') of $N \subset M$ with distinguished vertices $*$ resp. $*'$. Then we obtain the matrices G, G' via $\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$ x. resp. $\Delta_{\Gamma'} = \begin{pmatrix} 0 & G' \\ (G')^t & 0 \end{pmatrix}$, where Δ_Γ (resp. $\Delta_{\Gamma'}$) are the adjacency matrices of the graphs Γ resp. Γ'. If we denote by Γ_{even} (resp. Γ'_{even}) and Γ_{odd} (resp. Γ'_{odd}) the vertices with even (odd) distance from $*$ (resp. $*'$), then it is obvious that $K = \Gamma_{\text{even}}$ (resp. $K' = \Gamma'_{\text{even}}$) and $L = \Gamma_{\text{odd}}$ (resp. $L' = \Gamma'_{\text{odd}}$). In particular, $K_{-1} = \{*\}$ and K_n, $n \geq 0$, is the set of even vertices with distance $\leq 2(n+1)$ from $*$, whereas L_n is the set of odd vertices with distance $\leq 2n+1$ from $*$. Similarly for K'_n and L'_n (caution, the index of K'_n is shifted by one compared to K_n).

We proceed with showing that each index $k \in K$ (resp. $l \in L$) can be viewed as a uniquely determined irreducible N-N (resp. N-M) bimodule and we show in the next section that the numbers G_{kl} are dimensions of spaces of certain N-N resp. N-M bimodule intertwiners.

Let $p \in N' \cap M_{2n+1}$ (resp. $q \in N' \cap M_{2n}$) be a projection and let π_n be the representation of M_n, M_{2n+1} on $L^2(M_n)$ coming from the n-step basic construction for $N \subset M_n$ (Proposition 2.2). Recall that $\pi_n|_{M_{2n}}$ is equal to the representation of M_{2n} on $L^2(M_n)$ coming from the n-step basic construction for $M \subset M_n$ (Proposition 2.2, last statement). The Hilbert space $\pi_n(p)L^2(M_n)$ (resp. $\pi_n(q)L^2(M_n)$) is then an N-N (resp. N-M) bimodule in a natural way. Namely, since $\pi_n(p)$ (resp. $\pi_n(q)$) is an N-N (resp. N-M) intertwiner of the N-N (resp. N-M) bimodule $L^2(M_n)$, we have that $a \cdot \pi_n(p)(\hat{x}) \cdot b = \pi_n(p)(\widehat{axb}) = a\widehat{\pi_n(p)(x)}b$, for all

$a, b \in N$ and $x \in M_n$ (resp. $a \cdot \pi_n(q)(\hat{x}) \cdot b = \pi_n(q)(\widehat{axb}) = a\widehat{\pi_n(q)(x)}b$, for all $a \in N$, $b \in M$). Since $\pi_n|_M$ is left multiplication, this action can be rewritten as $a \cdot \pi_n(p)(\xi) \cdot b = \pi_n(a)J_n\pi_n(b)^*J_n\pi_n(p)(\xi)$, $a, b \in N$, $\xi \in L^2(M_n)$ and similarly for $\pi_n(q)$.

Definition 3.3 *Let $p \in N' \cap M_{2n+1}$, resp. $q \in N' \cap M_{2n}$, $n \geq 0$, be projections. We denote by $\pi_n(p)L^2(M_n)$, resp. $\pi_n(q)L^2(M_n)$ the N-N resp. N-M bimodules with above defined actions and we will call them the reduced bimodules associated to $N \subset M$. Similarly for $M' \cap M_{2n}$ and $M' \cap M_{2n+1}$.*

Note that one sometimes simply writes $pL^2(M_n)$ when the representation is understood. Also, observe that the space of left resp. right N (resp. M) bounded vectors of a reduced bimodule clearly contains $\pi_n(p)(\hat{M}_n)$ resp. $\pi_n(q)(\hat{M}_n)$ as a dense subspace, a remark that will be used later on, when we define bimodule morphisms on various tensor products of reduced bimodules.

Lemma 3.4 *Let $p \in N' \cap M_{2n+1}$ (resp. $p \in N' \cap M_{2n}$) be a projection. Then $\pi_n(p)L^2(M_n)$ is an irreducible N-N (resp. N-M) bimodule if and only if p is minimal in $N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$).*

Proof The space of N-N intertwiners of $\pi_n(p)L^2(M_n)$ is isomorphic to $(Np)' \cap pM_{2n+1}p$, which is equal to $\mathbb{C}p$, iff p is minimal. Similarly for the second statement. □

Lemma 3.5 *Let p_1 and p_2 be equivalent projections in $N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$). Then $\pi_n(p_1)L^2(M_n) \cong \pi_n(p_2)L^2(M_n)$ as N-N bimodules (resp. as N-M bimodules). Furthermore, if $p \in N' \cap M_{2n+1}$ (resp. $p \in N' \cap M_{2n}$) is a projection which is an orthogonal sum $p = \sum_{i=1}^k p_i$ of projections $p_i \in N' \cap M_{2n+1}$ (resp. $p_i \in N' \cap M_{2n}$), $1 \leq i \leq k$, then $\pi_n(p)L^2(M_n) \cong \bigoplus_{i=1}^k \pi_n(p_i)L^2(M_n)$ as N-N (resp. N-M) bimodules.*

Proof If $v \in N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$) is the partial isometry between p_1 and p_2, then $\pi_n(v)$ gives the desired bimodule equivalence. The rest is obvious. □

Next we will show that each $k \in K$ (resp. $l \in L$) labels precisely one irreducible N-N (resp. N-M) bimodule contained in $\oplus_n(\rho\bar{\rho})^n$ (resp. $\oplus_n(\rho\bar{\rho})^n\rho$). This will be accomplished by choosing a minimal projection in the k-th (resp. l-th) simple summand of a higher relative commutant and considering the associated reduced bimodule. The previous lemma shows that this reduced bimodule will not depend on the particular choice of the minimal projection as long as they are in the same simple summand. What we need to show however is that this reduced bimodule does not depend on the *level* of the higher relative commutant, i.e. minimal projections indexed by k (resp. l) in any higher relative commutant $N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$), $n \geq 0$, give equivalent reduced N-N (resp. N-M) bimodule. It is of course enough to establish this equivalence for reduced bimodules in two consecutive higher relative commutants $N' \cap M_{2n+1} \subset N' \cap M_{2n+3}$ (resp. $N' \cap M_{2n} \subset N' \cap M_{2n+2}$).

Lemma 3.6 *a) Let $p \in N' \cap M_{2n+1}$ be a projection, $n \geq 0$.*

i) *($n = 0$). $\pi_1(pe_3)(y) = pye_1$, for all $y \in M_1$.*

ii) *$\pi_{n+1}(pe_{2n+3})(y) = [M : N]^{2n} E^{M_{2n+1}}_{M_{n+1}}(pye_1 \ldots e_n f_n)e_n \ldots e_1$, for all $y \in M_{n+1}$, $n \geq 1$.*

b) Let $q \in N' \cap M_{2n}$, $n \geq 0$, be a projection ($M_0 = M$).

i) $(n=0)$. $\pi_1(qe_2)(y) = qE_M(y)$, *for all* $y \in M_1$.

ii) $(n=1)$. $\pi_2(qe_4)(y) = qye_2$, *for all* $y \in M_2$.

iii) $\pi_{n+1}(qe_{2n+2})(y) = [M:N]^{2n-1}E^{M_{2n+1}}_{M_{n+1}}(qye_2\dots e_nf_n)e_n\dots e_2$, *for all* $y \in M_{n+1}$, $n \geq 2$.

Proof The proof uses the explicit expression for the representations π_k given in Proposition 2.2. We prove first a). If $y \in M_1$, then $\pi_1(pe_3)(y) = [M:N]^2E_{M_1}(pe_3yf_1) = [M:N]^3pyE_{M_1}(e_3e_2e_1e_3e_2) = pye_1$ and a), i) is shown. Let $y \in M_{n+1}$, $n \geq 1$. Then $\pi_{n+1}(pe_{2n+3})(y) = [M:N]^{n+2}E^{M_{2n+3}}_{M_{n+1}}(pe_{2n+3}yf_{n+1})$ by 2.2. Observe that $f_{n+1} = [M:N]^{n+1}e_{n+2}\dots e_{2n+2}f_ne_{2n+3}\dots e_{n+2}$. We compute the conditional expectation $E^{M_{2n+3}}_{M_{n+1}}(pe_{2n+3}yf_{n+1})$. Let $x \in M_{n+1}$, then

$$\begin{aligned}\mathrm{tr}\big(xpye_{2n+3}f_{n+1}\big) &= [M:N]^{n+1}\mathrm{tr}\big(xpye_{2n+3}e_{n+2}\dots e_{2n+2}f_ne_{2n+3}\dots e_{n+2}\big)\\ &= [M:N]^{n-1}\mathrm{tr}\big(xpye_{n+2}\dots e_{2n+1}f_ne_{2n+1}\dots e_{n+2}e_{2n+3}\big)\\ &= [M:N]^{n-2}\mathrm{tr}\big(xpye_1\dots e_nf_ne_n\dots e_1\big)\\ &= [M:N]^{n-2}\mathrm{tr}\big(xE^{M_{2n+3}}_{M_{n+1}}(pye_1\dots e_nf_n)e_n\dots e_1\big),\end{aligned}$$

where we used Lemma 2.3 ii) in the third equality. Thus $\pi_{n+1}(pe_{2n+3})(y) = [M:N]^{2n}E^{M_{2n+1}}_{M_{n+1}}(pye_1\dots e_nf_n)e_n\dots e_1$ as claimed.

We prove b) next. If $y \in M_1$, then $\pi_1(qe_2)(y) = [M:N]^2E_{M_1}(qe_2yf_1) = [M:N]^3qE_M(y)E_{M_1}(e_2e_1e_3e_2) = qE_M(y)$ and b), i) is shown. If $y \in M_2$, then $\pi_2(qe_4)(y) = [M:N]^3E_{M_2}(qe_4yf_2) = [M:N]^3qyE_{M_2}(e_4f_2) = [M:N]^6qyE_{M_2}(e_4(e_3e_2e_1)(e_4e_3e_2)(e_5e_4e_3)) = qye_2$ and the proof of b), ii) is complete. Finally, let $n \geq 2$ and $y \in M_{n+1}$. Then $\pi_{n+1}(qe_{2n+2})(y) = [M:N]^{n+2}E^{M_{2n+3}}_{M_{2n+1}}(qe_{2n+2}yf_{n+1})$ (Proposition 2.2). We compute the conditional expectation as in a), ii). Let $x \in M_{n+1}$, then

$$\begin{aligned}\mathrm{tr}\big(xqe_{2n+2}yf_{n+1}\big) &= [M:N]^{n+1}\mathrm{tr}\big(xqye_{2n+2}e_{n+2}\dots e_{2n+2}f_ne_{2n+3}\dots e_{n+2}\big)\\ &= [M:N]^{n-2}\mathrm{tr}\big(xqye_{n+2}\dots e_{2n}f_ne_{2n}\dots e_{n+2}e_{2n+2}\big)\\ &= [M:N]^{n-3}\mathrm{tr}\big(xqye_2\dots e_nf_ne_n\dots e_2\big)\\ &= [M:N]^{n-3}\mathrm{tr}\big(xE^{M_{2n+3}}_{M_{n+1}}(qye_2\dots e_nf_n)e_n\dots e_2\big),\end{aligned}$$

where we used Lemma 2.3 iii). Thus $\pi_{n+1}(qe_{2n+2})(y) = [M:N]^{2n-1}E^{M_{2n+1}}_{M_{n+1}}(qye_2\dots e_nf_n)e_n\dots e_2$ as claimed. □

Theorem 3.7 *Let* $p \in N' \cap M_{2n+1}$ *(resp.* $q \in N' \cap M_{2n}$*) be a projection, let* π_n, π_{n+1} *be as in Proposition 2.2 and let* $M_{2n+1} \subset^{e_{2n+2}} M_{2n+2} \subset^{e_{2n+3}} M_{2n+3}$ *be a part of the tower associated to* $N \subset M$. *Then*

$$\begin{aligned}\pi_n(p)L^2(M_n) &\overset{N-N}{\cong} \pi_{n+1}(pe_{2n+3})L^2(M_{n+1})\\ \pi_n(q)L^2(M_n) &\overset{N-M}{\cong} \pi_{n+1}(qe_{2n+2})L^2(M_{n+1})\end{aligned}$$

as N-N resp. N-M bimodules, $n \geq 0$ $(M_0 = M)$.

Proof We will use Lemma 3.6 to define explicit bimodule isomorphisms. These maps will be related to the Fourier transforms defined in Definition 2.16. Let us begin with showing the equivalence of the N-N bimodules. Suppose first that $n \geq 1$. Define a linear map $T : \pi_{n+1}(pe_{2n+3})L^2(M_{n+1}) \to \pi_n(p)L^2(M_n)$ by $T(\pi_{n+1}(pe_{2n+3})(\hat{y})) = [M:N]^{\frac{3n+1}{2}} E_{M_n}^{M_{2n+1}}(pye_1 \dots e_n f_n)^\wedge$, $y \in M_{n+1}$. T will be later extended by continuity to all of $\pi_{n+1}(pe_{2n+3})L^2(M_{n+1})$. To simplify the notation, we will omit hats as usual. Recall that $M_{n+1} = M_n e_{n+1} e_n \dots e_1 M$. Thus, if $y = ae_{n+1} \dots e_1 b$, $a \in M_n$, $b \in M$, we get $E_{M_n}(pae_{n+1}e_n \dots e_1 b e_1 \dots e_n f_n) = E_{M_n}(paE_N(b)e_{n+1} \dots e_2 e_1 e_2 \dots e_{n+1} f_n) = [M:N]^{-n} E_{M_n}(paE_N(b)f_n) = [M:N]^{-(2n+1)}\pi_n(p)(aE_N(b))$. Hence $T(\pi_{n+1}(pe_{2n+3})(\hat{M}_{n+1})) \subset \pi_n(p)L^2(M_n)$ is a subspace, which is clearly dense. Note that $e_{n+1}f_n = f_n e_{n+1} = f_n$ (see Proposition 2.1). Recall that if $w \in M_{n+1}$, then there is a unique $w' \in M_n$ with $we_{n+1} = w'e_{n+1}$, namely $w' = [M:N]E_{M_n}(we_{n+1})$. Thus $we_{n+1} = [M:N]E_{M_n}(we_{n+1})e_{n+1}$. This will be used in the computations below. Let now x, $y \in M_{n+1}$. Then, using Lemma 3.6, we calculate the inner product,

$$\begin{aligned}
&\big(\pi_{n+1}(pe_{2n+3})(\hat{x}), \pi_{n+1}(pe_{2n+3})(\hat{y})\big)\\
&= [M:N]^{4n}\mathrm{tr}\big(e_1 \dots e_n \big(E_{M_{n+1}}(pye_1 \dots e_n f_n)\big)^* E_{M_{n+1}}(pxe_1 \dots e_n f_n)e_n \dots e_1\big)\\
&= [M:N]^{3n+1}\mathrm{tr}\big(e_n \big(E_{M_{n+1}}(pye_1 \dots e_n f_n)e_{n+1}\big)^* \big(E_{M_{n+1}}(pxe_1 \dots e_n f_n)e_{n+1}\big)\big)\\
&= [M:N]^{3n+3}\mathrm{tr}\big(e_n \big(E_{M_n}^{M_{n+1}}\big(E_{M_{n+1}}^{M_{2n+1}}(pye_1 \dots e_n f_n)e_{n+1}\big)e_{n+1}\big)^*\\
&\quad \big(E_{M_n}^{M_{n+1}}\big(E_{M_{n+1}}^{M_{2n+1}}(pxe_1 \dots e_n f_n)e_{n+1}\big)e_{n+1}\big)\big)\\
&= [M:N]^{3n+3}\mathrm{tr}\big(e_n e_{n+1} E_{M_n}^{M_{2n+1}}(pye_1 \dots e_n f_n)^* E_{M_n}^{M_{2n+1}}(pxe_1 \dots e_n f_n)e_{n+1}\big)\\
&= [M:N]^{3n+1}\mathrm{tr}\big(E_{M_n}^{M_{2n+1}}(pye_1 \dots e_n f_n)^* E_{M_n}^{M_{2n+1}}(pxe_1 \dots e_n f_n)\big).
\end{aligned}$$

Thus T extends to an isometry $\pi_{n+1}(pe_{2n+3})L^2(M_{n+1}) \to \pi_n(p)L^2(M_n)$, which is onto as seen above. T is an N-N bimodule morphism because,

$$\begin{aligned}
T\big(a \cdot \pi_{n+1}(pe_{2n+3})(\hat{y}) \cdot b\big) &= T\big(\pi_{n+1}(pe_{2n+3})(\widehat{ayb})\big)\\
&= [M:N]^{\frac{3n+1}{2}} E_{M_n}(paybe_1 \dots e_n f_n)^\wedge\\
&= [M:N]^{\frac{3n+1}{2}} (aE_{M_n}(pye_1 \dots e_n f_n)b)^\wedge\\
&= a \cdot T\big(\pi_{n+1}(pe_{2n+3})(\hat{y})\big) \cdot b,
\end{aligned}$$

for all a, $b \in N$. This shows the equivalence of the N-N bimodules in the theorem for $n \geq 1$. If $n = 0$, we define a linear map $T : \pi_1(pe_3)L^2(M_1) \to \pi_0(p)L^2(M)$ by $T(\pi_1(pe_3)(\hat{y})) = [M:N]^{\frac{1}{2}} E_M(pye_1)^\wedge$, $y \in M_1$. Since $M_1 = Me_1M$, we have for $y = ae_1 b$, a, $b \in M$, that $E_M(pae_1be_1) = E_M(paE_N(b)e_1) = [M:N]^{-1}\pi_0(p)(aE_N(b))$ $(f_0 = e_1)$. Hence $E_M(pye_1)^\wedge \in \pi_0(p)L^2(M)$, for all $y \in M_1$ and $T(\pi_1(pe_3)(\hat{M}_1))$ is clearly dense in $\pi_0(p)L^2(M)$. If x, $y \in M_1$, we compute (using Lemma 3.6),

$$\begin{aligned}\big(\pi_1(pe_3)(\hat{x}),\pi_1(pe_3)(\hat{y})\big) &= \mathrm{tr}\big(e_1y^*pxe_1\big) = \mathrm{tr}\big((pye_1)^*(pxe_1)\big)\\ &= [M:N]^2\mathrm{tr}\big(\big(E_M(pye_1)e_1\big)^*E_M(pxe_1)e_1\big)\\ &= [M:N]\mathrm{tr}\big(E_M(pye_1)^*E_M(pxe_1)\big).\end{aligned}$$

As before, T extends to a surjective isometry, which is an N-N bimodule morphism and establishes therefore the equivalence of the above N-N bimodules in the case $n=0$.

We proceed with the proof of the equivalence of the reduced N-M bimodules as in the statement of the theorem. According to Lemma 3.6, we have two special cases $n=0$, $n=1$ and the general case $n\geq 2$. Let us start with the general case. We define a linear map $T:\pi_{n+1}(qe_{2n+2})L^2(M_{n+1})\to\pi_n(q)L^2(M_n)$ by $T\big(\pi_{n+1}(qe_{2n+2})(\hat{y})\big)=[M:N]^{\frac{3n}{2}}E^{M_{2n+1}}_{M_n}(qye_2\dots e_nf_n)^\wedge$, $y\in M_{n+1}$. T will be later extended to all of $\pi_{n+1}(qe_{2n+2})L^2(M_{n+1})$ by continuity. Since $M_{n+1}=M_ne_{n+1}\dots e_2M_1$, we have for $y=ae_{n+1}\dots e_2b$, $a\in M_{n+1}$, $b\in M_1$, that $E^{M_{2n+1}}_{M_n}(qae_{n+1}\dots e_2be_2\dots e_nf_n) = E_{M_n}(qaE_M(b)e_{n+1}\dots e_3e_2e_3\dots e_{n+1}f_n) = [M:N]^{-n+1}E_{M_n}(qaE_M(b)f_n)=[M:N]^{-2n}\pi_n(q)(aE_M(b))$. Thus we get that $T\big(\pi_{n+1}(qe_{2n+2})(\hat{M}_{n+1})\big)\subset\pi_n(q)L^2(M_n)$ is a subspace, which is clearly dense.

Let x, $y\in M_{n+1}$ and compute, using Lemma 3.6, the inner product,

$$\begin{aligned}&\big(\pi_{n+1}(qe_{2n+2})(\hat{x}),\pi_{n+1}(qe_{2n+2})(\hat{y})\big)\\ &= [M:N]^{4n-2}\mathrm{tr}\big(e_2\dots e_n\big(E_{M_{n+1}}(qye_2\dots e_nf_n)\big)^*E_{M_{n+1}}(qxe_2\dots e_nf_n)e_n\dots e_2\big)\\ &= [M:N]^{3n}\mathrm{tr}\big(e_n\big(E_{M_{n+1}}(qye_2\dots e_nf_n)e_{n+1}\big)^*E_{M_{n+1}}(qxe_2\dots e_nf_n)e_{n+1}\big)\\ &= [M:N]^{3n+2}\mathrm{tr}\big(e_n\big(E^{M_{n+1}}_{M_n}\big(E^{M_{2n+1}}_{M_{n+1}}(qye_2\dots e_nf_n)e_{n+1}\big)e_{n+1}\big)^*\\ &\qquad E^{M_{n+1}}_{M_n}\big(E^{M_{2n+1}}_{M_{n+1}}(qxe_2\dots e_nf_n)e_{n+1}\big)e_{n+1}\big)\\ &= [M:N]^{3n+2}\mathrm{tr}\big(e_ne_{n+1}E^{M_{2n+1}}_{M_n}(qye_2\dots e_nf_n)^*E^{M_{2n+1}}_{M_n}(qxe_2\dots e_nf_n)e_{n+1}\big)\\ &= [M:N]^{3n}\mathrm{tr}\big(E^{M_{2n+1}}_{M_n}(qye_2\dots e_nf_n)^*E^{M_{2n+1}}_{M_n}(qxe_2\dots e_nf_n)\big).\end{aligned}$$

Thus T extends to an isometry $\pi_{n+1}(qe_{2n+2})L^2(M_{n+1})\to\pi_n(q)L^2(M_n)$, which is surjective as seen above. T is an N-M bimodule morphism, since

$$\begin{aligned}T\big(a\cdot\pi_{n+1}(qe_{2n+2})(\hat{y})\cdot b\big) &= T\big(\pi_{n+1}(qe_{2n+2})(\widehat{ayb})\big)\\ &= [M:N]^{\frac{3n}{2}}E_{M_n}(qe_{2n+2}aybe_2\dots e_nf_n)^\wedge\\ &= [M:N]^{\frac{3n}{2}}(aE_{M_n}(qe_{2n+2}ye_2\dots e_nf_n)b)^\wedge\\ &= a\cdot T\big(\pi_{n+1}(qe_{2n+2})(\hat{y})\big)\cdot b,\end{aligned}$$

for all $a \in N$ and $b \in M$. This establishes the desired N-M bimodule equivalences for $n \geq 2$. If $n = 0$, define a linear map $T : \pi_1(qe_2)L^2(M_1) \to \pi_0(q)L^2(M)$ by $T(\pi_1(qe_2)(\hat{y})) = q\widehat{E_M(y)} = \pi_0(q)(\widehat{E_M(y)})$, $y \in M_1$. By Lemma 3.6 b), i), T extends to a surjective isometry $\pi_1(qe_2)L^2(M_1) \to \pi_0(q)L^2(M)$, which is an N-M morphism by a similar computation as above. This completes the proof in the case $n = 0$. Finally, the case $n = 1$. Define a linear map $T : \pi_2(qe_4)L^2(M_2) \to \pi_1(q)L^2(M_1)$ by $T(\pi_2(qe_4)(\hat{y})) = [M : N]^{\frac{1}{2}} E_{M_1}^{M_2}(qye_2)^\wedge$, $y \in M_2$. Note that $E_{M_2}^{M_3}(qyf_1) = [M : N]^{-1} E_{M_1}^{M_2}(qye_2)$, for all $y \in M_2$. Hence $E_{M_1}^{M_2}(qye_2) \in \pi_1(q)L^2(M_1)$, for all $y \in M_2$, by a similar argument as above, using $M_2 = M_1 e_2 M_1$. Thus $T(\pi_2(qe_4)(\hat{M}_2)) \subset \pi_1(q)L^2(M_1)$ is a dense subspace. Let x, $y \in M_2$ and compute the inner product $(\pi_2(qe_4)(\hat{x}), \pi_2(qe_4)(\hat{y})) = \mathrm{tr}(e_2 y^* q x e_2) = [M : N]^2 \mathrm{tr}((E_{M_1}(qye_2)e_2)^* E_{M_1}(qxe_2)e_2) = [M : N]\mathrm{tr}(E_{M_1}(qye_2)^* E_{M_1}(qxe_2))$ (Lemma 3.6). Hence T extends to an isometry $\pi_1(qe_2)L^2(M_1) \to \pi_0(q)L^2(M)$, which clearly onto and an N-M bimodule morphism since $T(a \cdot \pi_2(qe_4) \cdot b) = T(\pi_2(qe_4)(\widehat{ayb})) = [M : N]^{\frac{1}{2}} E_{M_1}(qe_4 aybe_2) = [M : N]^{\frac{1}{2}} a E_{M_1}(qe_4 ye_2) b = a \cdot T(\pi_2(qe_4)(\hat{y})) \cdot b$, for all $a \in N$, $b \in M$. Thus the proof of the theorem is complete. □

Remark 3.8 *The above proof simplifies if $p = 1$ or $q = 1$. Let us consider $p = 1$. Since $E_{M_{n+1}}(f_n) = [M : N]^{-n} e_{n+1}$, we get for $y \in M_{n+1}$ that $\pi_{n+1}(e_{2n+3})(y) = [M : N]^{2n} y e_1 \dots e_n E_{M_{n+1}}(f_n) e_n \dots e_1 = ye_1$ (Lemma 3.6). It is then easy to see that the map $T : \pi_{n+1}(e_{2n+3})L^2(M_{n+1}) \to L^2(M_n)$, defined in the proof of the previous theorem, simplifies to $T(\pi_{n+1}(e_{2n+3})(\hat{y})) = [M : N]^{\frac{n+1}{2}} E_{M_n}(ye_1 \dots e_{n+1})^\wedge$, $y \in M_{n+1}$. Similarly in the case $q = 1$.*

Corollary 3.9 *Let $N \subset M$ be an inclusion of II_1 factors with finite index, and let Γ be the principal graph of $\mathbb{C} = N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset \dots$. Then there is a bijection between Γ_{even} (resp. Γ_{odd}) and the set of equivalence classes of irreducible N-N (resp. N-M) sub-bimodules of $\bigoplus_{n \geq -1} {}_N L^2(M_n)_N$ (resp. $\bigoplus_{n \geq 0} {}_N L^2(M_n)_M$), where $M_{-1} = N$, $M_0 = M$. More precisely, if $\alpha \in \Gamma_{\mathrm{even}}$ (resp. $\beta \in \Gamma_{\mathrm{odd}}$), choose a minimal projection p_α (resp. q_β) in the α-th (resp. β-th) simple summand of $N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$), where n is such that α (resp. β) occurs as the index of a simple summand of $N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$). Then the (equivalence class of the) irreducible N-N (resp. N-M) bimodule $\pi_n(p_\alpha)L^2(M_n)$ (resp. $\pi_n(q_\beta)L^2(M_n)$) does not depend on the choice of n and the choice of the minimal projection in the α-th (resp. β-th) simple summand of $N' \cap M_{2n+1}$ (resp. $N' \cap M_{2n}$) (where π_n is as in Proposition 2.2).*

Proof The fact that $\pi_n(p_\alpha)L^2(M_n)$ does not depend on which minimal projection one chooses in the α-th simple summand of $N' \cap M_{2n+1}$ was shown in Lemma 3.5. Furthermore, this bimodule is irreducible by Lemma 3.4. If we choose a minimal projection p_α in the α-th simple summand of $N' \cap M_{2n+1}$ and a minimal projection $\tilde{p}_\alpha$ in the α-th simple summand of $N' \cap M_{2m+1}$, some $n \leq m$, then $\tilde{p}_\alpha$ is equivalent to $p_\alpha e_{2n+3} e_{2n+5} \dots e_{2n+2k+1}$ (for an appropriate k), which is a minimal projection in the α-th simple summand of $N' \cap M_{2m+1}$. Applying Theorem 3.7 k times, we obtain $\pi_n(p_\alpha)L^2(M_n) \cong \pi_m(\tilde{p}_\alpha)L^2(M_m)$ as N-N bimodules. Note that by Propositions 3.1 and 3.2 we get all the irreducible N-N sub-bimodules of $\bigoplus_{n \geq -1} {}_N L^2(M_n)_N$ in this way. The statement about the N-M bimodules is shown in the same manner. □

All the statements in 3.4-3.8 and their proofs can of course be rewritten for the higher relative commutants $M' \cap M_{2n}$ and $M' \cap M_{2n+1}$, M-M and M-N bimodules. Note that we work then with the same representations π_n, since the representation of M_{2n} on $L^2(M_n)$ coming from the basic construction of $N \subset M_n$, restricted to M_{2n} is equal to the one coming from the basic construction of $M \subset M_n$ (Proposition 2.2). Let us state the result analogous to the previous corollary for the sake of completeness.

Corollary 3.10 *Let $N \subset M$ be an inclusion of II_1 factors with finite index, and let Γ' be the principal graph of $\mathbb{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \dots$. Then there is a bijection between Γ'_{even} (resp. Γ'_{odd}) and the set of equivalence classes of irreducible M-M (resp. M-N) sub-bimodules of $\bigoplus_{n\geq 0} {}_M L^2(M_n)_M$ (resp. $\bigoplus_{n\geq 0} {}_M L^2(M_n)_N$), where $M_0 = M$. More precisely, if $\alpha \in \Gamma'_{\text{even}}$ (resp. $\beta \in \Gamma'_{\text{odd}}$), choose a minimal projection p_α (resp. q_β) in the α-th (resp. β-th) simple summand of $M' \cap M_{2n}$ (resp. $M' \cap M_{2n+1}$), where n is such that α (resp. β) occurs as the index of a simple summand of $M' \cap M_{2n}$ (resp. $M' \cap M_{2n+1}$). Then the (equivalence class of the) irreducible M-M (resp. M-N) bimodule $\pi_n(p_\alpha)L^2(M_n)$ (resp. $\pi_n(q_\beta)L^2(M_n)$) does not depend on the choice of n and the choice of the minimal projection in the α-th (resp. β-th) simple summand of $M' \cap M_{2n}$ (resp. $M' \cap M_{2n+1}$).*

By Corollaries 3.9 and 3.10 it makes sense to talk about the N-N (resp. M-M, N-M, M-N) bimodule $\alpha \in \Gamma_{\text{even}}$ (resp. $\beta \in \Gamma'_{\text{even}}$, $\gamma \in \Gamma_{\text{odd}}$, $\delta \in \Gamma'_{\text{odd}}$), which is the terminology we will use below. We discuss next the conjugate of a reduced bimodule.

Proposition 3.11 *Let $p \in N' \cap M_{2n+1}$, $q \in N' \cap M_{2n}$ be projections and let $\gamma_n : N' \cap M_{2n+1} \to N' \cap M_{2n+1}$ be defined by $\pi_n(\gamma_n(x)) = J_n \pi_n(x)^* J_n$, $x \in N' \cap M_{2n+1}$ (see 2.4). Then,*

$$\overline{\pi_n(p)L^2(M_n)} \overset{N\text{-}N}{\cong} \pi_n(\gamma_n(p))L^2(M_n) \quad \text{as } N\text{-}N \text{ bimodules and,}$$
$$\overline{\pi_n(q)L^2(M_n)} \overset{M\text{-}N}{\cong} \pi_n(\gamma_n(q))L^2(M_n) \quad \text{as } M\text{-}N \text{ bimodules.}$$

Thus conjugation $^-$ defines an involution on Γ_{even} and a bijection between Γ_{odd} and Γ'_{odd}.

Proof We will show that the bimodule equivalences are implemented by the modular conjugation $J_n : L^2(M_n) \to L^2(M_n)$. Recall that $\pi_n(p)(\hat{x}) \in \hat{M}_n$ for all $x \in M_n$. Then $J_n\big(\pi_n(p)(\hat{x})\big) = J_n\pi_n(p)J_n(\hat{x}^*) = \pi_n(\gamma_n(p))(\hat{x}^*)$. Furthermore,

$$\begin{aligned}
&\Big(J_n\big(\pi_n(p)(\hat{x})\big), J_n\big(\pi_n(p)(\hat{y})\big)\Big)_{\pi_n(\gamma_n(p))L^2(M_n)} \\
&= \operatorname{tr}_{M_n}\big((\pi_n(\gamma_n(p))(y^*))^*\pi_n(\gamma_n(p))(x^*)\big) = \operatorname{tr}_{M_n}\big(\pi_n(p)(y)(\pi_n(p)(x))^*\big) \\
&= \big(\pi_n(p)(\hat{y}), \pi_n(p)(\hat{x})\big)_{\pi_n(p)L^2(M_n)} = \big(\overline{\pi_n(p)(\hat{x})}, \overline{\pi_n(p)(\hat{y})}\big)_{\overline{\pi_n(p)L^2(M_n)}},
\end{aligned}$$

for all $x, y \in M_n$, where the third equality follows from Lemma 2.4 iv). Thus J_n is a linear, surjective isometry from $\overline{\pi_n(p)L^2(M_n)} \to \pi_n(\gamma_n(p))L^2(M_n)$. Let $a, b \in N$, then $J_n\big(b \cdot \overline{\pi_n(p)(\hat{x})} \cdot a\big) = J_n\big(a^* \cdot \widehat{\pi_n(p)(x)} \cdot b^*\big) = J_n\big(\pi_n(p)(\widehat{a^*xb^*})\big) = \big((\pi_n(p)(a^*xb^*))^*\big)^\wedge = \big((a^*\pi_n(p)(x)b^*)^*\big)^\wedge = \big(b(\pi_n(p)(x))^*a\big)^\wedge = b \cdot J_n\big(\overline{\pi_n(p)(\hat{x})}\big) \cdot a$

for all $x \in M_n$. Thus J_n implements an N-N bimodule equivalence and the first identity is shown.

To prove the second one, observe that $\pi_n(\gamma_n(q)) = J_n\pi_n(q)J_n \in J_n\pi_n(N' \cap M_{2n})J_n = \pi_n(M' \cap M_{2n+1})$, since $J_n\pi_n(M)'J_n = \pi_n(M_{2n})$ by Proposition 2.2. Thus $\pi_n(\gamma_n(q))$ is an intertwiner of the M-N bimodule $L^2(M_n)$, so that the second equivalence indeed makes sense as an equivalence of M-N bimodules. As in the first part, one now shows that J_n implements this desired equivalence.

The first bimodule equivalence shows that, given the N-N bimodule $\alpha \in \Gamma_{\text{even}}$, the conjugate N-N bimodule is again a reduced bimodule and hence labelled by some $\bar{\alpha} \in \Gamma_{\text{even}}$. This map is well-defined by 3.9 and clearly an involution. Similarly, since $\gamma_n(q) \in M' \cap M_{2n+1}$, the conjugate of an N-M bimodule $\beta \in \Gamma_{\text{odd}}$ is an M-N bimodule indexed by a $\bar{\beta} \in \Gamma'_{\text{odd}}$. Again, this map is well-defined by 3.9 and 3.10 and clearly a bijection. □

It goes without saying that we also have an involution on Γ'_{even}. The *conjugation* (or *contragredient map*) $\gamma \in \Gamma_{\text{odd}} \to \bar{\gamma} \in \Gamma'_{\text{odd}}$ gives an identification of Γ_{odd} and Γ'_{odd} and induces a permutation on the even levels as we have shown above. Observe that the fact that the conjugate of a reduced N-N resp. N-M bimodule is indexed by the vertices of the principal graphs follows of course already from Proposition 3.1 and the Remark 1.17. Let us point out that Proposition 3.11 in conjunction with Theorem 2.6 (using $A = N$, $B = M_n$) gives an explicit formula for the conjugate bimodule by calculating a projection in a higher relative commutant.

Let us now define the (full) fusion algebra associated to a subfactor. Observe that this can be done without knowing what exactly the result of the relative tensor product of two reduced bimodules is. Namely, if α, $\beta \in \Gamma_{\text{even}}$ (resp. Γ'_{even}), then α and β is an N-N (resp. M-M) sub-bimodule of ${}_NL^2(N)_N$, $(\rho\bar{\rho})^k$ (resp. ${}_ML^2(M)_M$, $(\bar{\rho}\rho)^k$) for some $k \geq 1$. Hence $\alpha\beta = \alpha \otimes_N \beta$ (resp. $\alpha \otimes_M \beta$) is a sub-bimodule of $(\rho\bar{\rho})^{2k}$ (resp. $(\bar{\rho}\rho)^{2k}$) and can therefore be decomposed as (N-N case)

$$\alpha\beta = \sum_{\gamma \in \Gamma_{\text{even}}} N_{\alpha\beta}^{\gamma}\gamma,$$

where the integers $N_{\alpha\beta}^{\gamma}$ denote the multiplicity of the irreducible N-N bimodule γ in the N-N bimodule $\alpha\beta$, i.e., $N_{\alpha\beta}^{\gamma} = \dim \operatorname{Hom}_{N\text{-}N}(\gamma, \alpha\beta)$. Similarly for M-M bimodules with Γ_{even} replaced by Γ'_{even}.

Unfortunately, we cannot multiply two N-M bimodules α, $\beta \in \Gamma_{\text{odd}}$. However, the products $\bar{\alpha} \otimes_N \beta$ and $\alpha \otimes_M \bar{\beta}$ can be formed and will, with the same reasoning as above, be sub-bimodules of $(\bar{\rho}\rho)^k$ resp. $(\rho\bar{\rho})^k$, for some k. Hence they can be decomposed into an integer linear combination of the irreducible bimodules in Γ'_{even} resp. Γ_{even}.

Rather than giving a definition of an abstract fusion algebra and identifying the one coming from subfactors as such, let us stay with subfactors and define what we mean by the fusion algebra associated to a subfactor.

Definition 3.12 *Let $N \subset M$ be an inclusion of II_1 factors with finite index and denote by (Γ, Γ') the principal graphs as above.*

i) *(N-N part) Let* $\mathbb{Z}\Gamma_{\text{even}}$ *be the formal* $\mathbb{Z}$*-linear combinations of the set* $\{\alpha|\alpha \in \Gamma_{\text{even}}\}$. *We define a multiplication on* $\mathbb{Z}\Gamma_{\text{even}}$ *by*

$$\alpha\beta = \alpha \otimes_N \beta = \sum_{\gamma\in\Gamma_{\text{even}}} N_{\alpha\beta}^{\gamma}\gamma,$$

where $N_{\alpha\beta}^{\gamma} = \dim Hom_{N-N}(\gamma, \alpha\beta)$, $\alpha,\ \beta,\ \gamma \in \Gamma_{\text{even}}$, *and extend it to all of* $\mathbb{Z}\Gamma_{\text{even}}$ *by linearity, respecting the distributivity law.*

ii) *(M-M part) Let* $\mathbb{Z}\Gamma'_{\text{even}}$ *be the formal* $\mathbb{Z}$*-linear combinations of the set* $\{\alpha|\alpha \in \Gamma'_{\text{even}}\}$. *We define a multiplication on* $\mathbb{Z}\Gamma'_{\text{even}}$ *by*

$$\alpha\beta = \alpha \otimes_M \beta = \sum_{\gamma\in\Gamma'_{\text{even}}} M_{\alpha\beta}^{\gamma}\gamma,$$

where $M_{\alpha\beta}^{\gamma} = \dim Hom_{M-M}(\gamma, \alpha\beta)$, $\alpha,\ \beta,\ \gamma \in \Gamma'_{\text{even}}$, *and extend it to all of* $\mathbb{Z}\Gamma'_{\text{even}}$ *by linearity, respecting the distributivity law.*

Recall that the contragredient map induces an involution $^- : \mathbb{Z}\Gamma_{\text{even}} \to \mathbb{Z}\Gamma_{\text{even}}$ *and* $^- : \mathbb{Z}\Gamma'_{\text{even}} \to \mathbb{Z}\Gamma'_{\text{even}}$. *We denote by* $\mathfrak{F} = \mathfrak{F}_{N\subset M} = (\mathbb{Z}\Gamma_{\text{even}}, \mathbb{Z}\Gamma'_{\text{even}}, ^-)$ *the two* $\mathbb{Z}$*-algebras with involution defined as above and call* $\mathfrak{F}$ *the fusion algebra (or fusion ring) associated to* $N \subset M$.

Observe that we have two distinguished selfcontragredient bimodules $*$ $(={}_N\ L^2(N)_N)$ and $*'$ $(={}_M\ L^2(M)_M)$ in Γ_{even} resp. Γ'_{even}. Clearly, $\mathbb{Z}\Gamma_{\text{even}}$ and $\mathbb{Z}\Gamma'_{\text{even}}$ are unital (the units are $*$ resp. $*'$), associative algebras (since bimodule multiplication is associative !) over $\mathbb{Z}$ with a natural trace defined using the square root of the index of the bimodule (i.e., of the associated subfactor) in the usual way (see remark after Definition 1.1). We can replace $\mathbb{Z}$ by $\mathbb{C}$ to get complex algebras.

If we let L (resp. R) be the left (resp. right) regular representation of $\mathfrak{F}$, i.e., $L_\alpha(\beta) = \alpha\beta$ (resp. $R_\alpha(\beta) = \beta\bar{\alpha}$), $\alpha,\ \beta \in \mathbb{Z}\Gamma_{\text{even}}$ (resp. $\mathbb{Z}\Gamma'_{\text{even}}$), then the matrix representation of L_α (resp. R_α), $\alpha \in \Gamma_{\text{even}}$, in the basis $B = \{\gamma\,|\,\gamma \in \Gamma_{\text{even}}\}$ is given by $L_\alpha = (N_{\alpha\beta}^{\gamma})_{\gamma,\beta\in\Gamma_{\text{even}}}$ and $R_\alpha = (N_{\beta\bar{\alpha}}^{\gamma})_{\gamma,\beta\in\Gamma_{\text{even}}}$ and similarly for M-M bimodules and Γ'_{even} (γ is the row index and β the column index of the matrix). Observe that $N_{\beta\bar{\alpha}}^{\gamma} = N_{\gamma\alpha}^{\beta}$. Furthermore, since $N_{\bar{\alpha}\beta}^{\gamma} = N_{\alpha\gamma}^{\beta}$, we have $L_{\bar{\alpha}} = (L_\alpha)^t$ and similarly $R_{\bar{\alpha}} = (R_\alpha)^t$ (as matrices as above). In particular, the N-N bimodule $\alpha \in \Gamma_{\text{even}}$ is selfcontragredient iff the matrix L_α is symmetric. Similar statements hold for Γ'_{even} and M-M bimodules.

We let $\mathbb{Z}\Gamma_{\text{odd}}$ be the formal $\mathbb{Z}$-linear combinations of the set $\{\alpha|\alpha \in \Gamma_{\text{odd}}\}$ and similarly for $\mathbb{Z}\Gamma'_{\text{odd}}$. We define a multiplication $\mathbb{Z}\Gamma_{\text{odd}} \times \mathbb{Z}\Gamma'_{\text{odd}} \to \mathbb{Z}\Gamma_{\text{even}}$ by

$$\alpha\beta = \alpha \otimes_M \beta = \sum_{\gamma\in\Gamma_{\text{even}}} P_{\alpha\beta}^{\gamma}\gamma,$$

where $P_{\alpha\beta}^{\gamma} = \dim \mathrm{Hom}_{N-N}(\gamma, \alpha\beta)$, $\alpha \in \Gamma_{\text{odd}}$, $\beta \in \Gamma'_{\text{odd}}$, $\gamma \in \Gamma_{\text{even}}$, extended by linearity as before. Similarly we define a multiplication : $\mathbb{Z}\Gamma'_{\text{odd}} \times \mathbb{Z}\Gamma_{\text{odd}} \to \mathbb{Z}\Gamma'_{\text{even}}$ by

$$\alpha\beta = \alpha \otimes_N \beta = \sum_{\gamma\in\Gamma'_{\text{even}}} Q_{\alpha\beta}^{\gamma}\gamma,$$

where $Q_{\alpha\beta}^{\gamma} = \dim \mathrm{Hom}_{M-M}(\gamma, \alpha\beta)$, $\alpha \in \Gamma'_{\text{odd}}$, $\beta \in \Gamma_{\text{odd}}$, $\gamma \in \Gamma'_{\text{even}}$, extended by linearity. Furthermore we have products $\mathbb{Z}\Gamma_{\text{even}} \times \mathbb{Z}\Gamma_{\text{odd}} \to \mathbb{Z}\Gamma_{\text{odd}}$, $\mathbb{Z}\Gamma'_{\text{even}} \times \mathbb{Z}\Gamma'_{\text{odd}} \to \mathbb{Z}\Gamma'_{\text{odd}}$, $\mathbb{Z}\Gamma_{\text{odd}} \times \mathbb{Z}\Gamma'_{\text{even}} \to \mathbb{Z}\Gamma_{\text{odd}}$ and $\mathbb{Z}\Gamma'_{\text{odd}} \times \mathbb{Z}\Gamma_{\text{even}} \to \mathbb{Z}\Gamma'_{\text{odd}}$ defined in a similar way. Thus we can regard for instance $\alpha \in \mathbb{Z}\Gamma_{\text{odd}}$ as a matrix $L_\alpha : \mathbb{Z}\Gamma'_{\text{odd}} \to \mathbb{Z}\Gamma_{\text{even}}$ (the matrix representation is with respect to the obvious bases) or as a matrix $R_\alpha : \mathbb{Z}\Gamma'_{\text{odd}} \to \mathbb{Z}\Gamma'_{\text{even}}$ etc. This point of view allows us to perform fusion rule calculations by multiplying matrices and solving matrix equations (Section 5, see also Bisch and Haagerup [1996]).

Definition 3.13 *Let $N \subset M$, Γ, Γ' be as above. We call $\mathfrak{F}_{\text{full}} = \mathfrak{F}_{\text{full}}(N \subset M) = (\mathbb{Z}\Gamma_{\text{even}}, \mathbb{Z}\Gamma'_{\text{even}}, \mathbb{Z}\Gamma_{\text{odd}}, \mathbb{Z}\Gamma'_{\text{odd}}, ^{-})$ with involution and various multiplications ("even $\times$ even", "even $\times$ odd", "odd $\times$ even", "$\overline{\text{odd}}$ $\times$ odd" and "odd $\times$ $\overline{\text{odd}}$") defined as above the full fusion algebra associated to $N \subset M$. The N-N and M-M part of $\mathfrak{F}_{\text{full}}$ (what we called the fusion algebra associated to $N \subset M$ in Definition 3.12) is called the even part and the N-M (resp. M-N) parts are called the odd part of $\mathfrak{F}_{\text{full}}$.*

In the next lemma we collect some of the properties of the structure constants of the fusion algebra associated to a subfactor.

Proposition 3.14 *Let $N \subset M$ be an inclusion of II_1 factors with finite index. Let $(N_{\alpha\beta}^{\gamma})_{\alpha,\beta,\gamma\in\Gamma_{\text{even}}}$ be the structure constants of the N-N part of the fusion algebra $\mathfrak{F}_{N\subset M}$. Then*

i) *(unit)* $N_{\alpha *}^{\gamma} = N_{*\alpha}^{\gamma} = \delta_{\alpha\gamma}$, *for all* α, $\gamma \in \Gamma_{\text{even}}$.

ii) *(associativity)*

$$\sum_{\gamma,\epsilon\in\Gamma_{\text{even}}} N_{\alpha\beta}^{\gamma} N_{\gamma\delta}^{\epsilon} = \sum_{\gamma,\epsilon\in\Gamma_{\text{even}}} N_{\beta\delta}^{\gamma} N_{\alpha\gamma}^{\epsilon},$$

for all α, β, $\gamma \in \Gamma_{\text{even}}$.

iii) *(involution)* $N_{\alpha\beta}^{\bar{\gamma}} = N_{\bar{\beta}\bar{\alpha}}^{\gamma}$, *for all* α, β, $\gamma \in \Gamma_{\text{even}}$.

iv) *(Frobenius reciprocity)* $N_{\alpha,\beta}^{\gamma} = N_{\gamma\bar{\beta}}^{\alpha} = N_{\bar{\alpha}\gamma}^{\beta}$ *for all* α, β, $\gamma \in \Gamma_{\text{even}}$.

Proof The properties of the structure constants follow immediately from the indicated properties of the algebra. iv) is a reformulation of Corollary 1.21 iv)-vi). □

It is clear that similar statements hold for the M-M part of the fusion algebra and that various compatibility conditions between the N-N, M-M, M-N and N-M parts imply conditions on the other structure constants as well. We will leave it to the reader to list them. Note that we will usually identify Γ'_{odd} with Γ_{odd} using the contragredient map. We end this section with a definition.

Definition 3.15 *The matrices*

- $N_{\alpha}^{(l)} = (N_{\alpha\beta}^{\gamma})_{\gamma,\beta\in\Gamma_{\text{even}}}$ *(resp.* $N_{\bar{\alpha}}^{(r)} = (N_{\beta\bar{\alpha}}^{\gamma})_{\gamma,\beta\in\Gamma_{\text{even}}}$*),* $\alpha \in \Gamma_{\text{even}}$,
- $M_{\alpha}^{(l)} = (M_{\alpha\beta}^{\gamma})_{\gamma,\beta\in\Gamma'_{\text{even}}}$ *(resp.* $M_{\bar{\alpha}}^{(r)} = (M_{\beta\bar{\alpha}}^{\gamma})_{\gamma,\beta\in\Gamma'_{\text{even}}}$*),* $\alpha \in \Gamma'_{\text{even}}$,

are called the fusion matrices associated to $N \subset M$.

Note that the principal graphs Γ, Γ' of $N \subset M$ do not determine the fusion algebra completely in general (see, for instance (Bisch [1994(b)], $3+\sqrt{3}$ example)) and additional information is needed to calculate the fusion matrices (namely the "orthogonality information" contained in the *commuting squares* formed by the higher relative commutants). Let us remark that the fusion matrices can be interpreted as principal graphs of reduced subfactors - more on this in Section 5.

4 Bimodule tensor products and higher relative commutants

We show in this section how the bimodule tensor product of two reduced bimodules associated to a subfactor can be recovered as a product of projections in the higher relative commutants. Furthermore, we will identify the multiplicities G_{kl} occurring in the principal graphs as dimensions of spaces of bimodule intertwiners.

Recall that the simple summands of $N' \cap M_{2n+1}$ are indexed by K_n and those of $N' \cap M_{2n}$ by L_n. As before, we denote by π_n the representation of M_{2n}, M_{2n+1} on $L^2(M_n)$ coming from the basic construction of $N \subset M_n$ and G_{kl} are the entries of the matrix G associated to the principal graph Γ (see the discussion after Proposition 3.2).

Proposition 4.1 *Let $q \in N' \cap M_{2n} \subset N' \cap M_{2n+1}$ be a projection. Then,*

$$\begin{aligned} \pi_n(q)L^2(M_n) \otimes_M L^2(M)_N &\cong \left(\pi_n(q) \otimes_M id_{{}_M L^2(M)_N}\right)\left(L^2(M_n) \otimes_M L^2(M)\right) \\ &\cong \pi_n(q)L^2(M_n) \end{aligned}$$

as N-N bimodules. Thus, if $q = q_l$, $l \in L_n$, is a minimal projection in the l-th simple summand of $N' \cap M_{2n}$, then

$$\pi_n(q_l)L^2(M_n) \overset{N\text{-}N}{\cong} \bigoplus_{k \in K_n} G_{kl}\, \pi_n(p_k)L^2(M_n),$$

as N-N bimodules, and hence

$$\dim Hom_{N\text{-}N}\left(\pi_n(p_k)L^2(M_n), \pi_n(q_l)L^2(M_n) \otimes_M L^2(M)_N\right) = G_{kl},$$

where p_k is a minimal projection in the k-th simple summand of $N' \cap M_{2n+1}$, $k \in K_n$.

Proof Observe that since $q \in N' \cap M_{2n+1}$, $\pi_n(q)L^2(M_n)$ is indeed an N-N bimodule. The equivalence of the first two bimodules follows from Proposition 1.15 iii). The equivalence of the first and third bimodule is immediate by Lemma 1.14. Now let q_l be as in the statement of the proposition. Since the l-th simple summand of $N' \cap M_{2n}$ sits with multiplicity G_{kl} in the k-th simple summand of $N' \cap M_{2n+1}$ (by definition of the principal graph), the third equivalence is obvious by Lemma 3.5 (by $G_{kl}\, \pi_n(p_k)L^2(M_n)$ we mean G_{kl} copies of the bimodule $\pi_n(p_k)L^2(M_n)$ as usual). Since the reduced bimodules $\pi_n(p_k)L^2(M_n)$, $k \in K_n$, are irreducible N-N bimodules, the last statement is immediate from the third bimodule equivalence. □

Corollary 4.2 *The embedding from $Hom_{N\text{-}M}({}_N L^2(M_n)_M) \to Hom_{N\text{-}N}({}_N L^2(M_n)_N)$, given by $\pi_n(x) \to \pi_n(x) \otimes_M id_{{}_M L^2(M)_N}$, $x \in N' \cap M_{2n}$, coincides with the inclusion $N' \cap M_{2n} \subset N' \cap M_{2n+1}$, given by the principal graph Γ.*

Proposition 4.3 *Let $p \in N' \cap M_{2n+1}$ be a projection, then*

$$\begin{aligned} \pi_n(p)L^2(M_n) \otimes_N L^2(M)_M &\cong \pi_n(p) \otimes_N id_{{}_N L^2(M)_M}\left(L^2(M_n) \otimes_N L^2(M)\right) \\ &\cong \pi_{n+1}(p)L^2(M_{n+1}) \end{aligned}$$

as N-M bimodules. Thus, if $p = p_k$, $k \in K_n$, is a minimal projection in the k-th simple summand of $N' \cap M_{2n+1}$, then

$$\pi_{n+1}(p_k)L^2(M_{n+1}) \overset{N\text{-}M}{\cong} \bigoplus_{l \in L_{n+1}} G_{kl}\, \pi_{n+1}(q_l)L^2(M_{n+1})$$

as N-M bimodules, and hence

$$\dim Hom_{N\text{-}M}\big(\pi_{n+1}(q_l)L^2(M_{n+1}), \pi_n(p_k)L^2(M_n)\otimes_N L^2(M)_M\big) = G_{kl},$$

where q_l is a minimal projection in the l-th simple summand of $N'\cap M_{2n+2}$.

Proof The equivalence of the first and the second bimodule is obvious by Proposition 1.15 iii). We will show the equivalence between the first and the third bimodule. Recall that $\pi_{n+1}(p)(ye_{n+1}\dots e_1z) = \pi_n(p)(y)e_{n+1}\dots e_1z$, for all $y\in M_n$ and $z\in M$ by Proposition 2.5. Also, recall that $\pi_n(\hat{M}_n)$ is dense in the space of right N-bounded vectors of $\pi_n(p)L^2(M_n)$. Define a linear map $T:\pi_n(p)(\hat{M}_n)\odot\hat{M}\to\pi_{n+1}(p)L^2(M_{n+1})$ by $T\big(\pi_n(p)(\hat{y})\otimes\hat{z}\big) = [M:N]^{(n+1)/2}\pi_{n+1}(p)(ye_{n+1}e_n\dots e_1z)$, $y\in M_n$, $z\in M$ (as usual we will omit "hats" in the calculation below). Then we have for $y_i\in M_n$, $z_i\in M$, $i=1$, 2:

$$\begin{aligned}
[M:N]^{n+1}\ &\big(\pi_{n+1}(p)(y_1e_{n+1}\dots e_1z_1), \pi_{n+1}(p)(y_2e_{n+1}\dots e_1z_2)\big)_{L^2(M_{n+1})}\\
&= [M:N]^{n+1}\mathrm{tr}\big(z_2^*e_1\dots e_{n+1}(\pi_n(p)(y_2))^*\pi_n(p)(y_1)e_{n+1}\dots e_1z_1\big)\\
&= [M:N]^{n+1}\mathrm{tr}\big((\pi_n(p)(y_2))^*\pi_n(p)(y_1)e_{n+1}\dots e_1z_1z_2^*e_1\dots e_{n+1}\big)\\
&= [M:N]^{n+1}\mathrm{tr}\big((\pi_n(p)(y_2))^*\pi_n(p)(y_1)E_N(z_1z_2^*)\\
&\qquad e_{n+1}\dots e_2e_1e_2\dots e_{n+1}\big)\\
&= [M:N]\mathrm{tr}\big((\pi_n(p)(y_2))^*\pi_n(p)(y_1)E_N(z_1z_2^*)e_{n+1}\big)\\
&= \mathrm{tr}\big((\pi_n(p)(y_2))^*\pi_n(p)(y_1)E_N(z_2z_1^*)\big)\\
&= \big(\pi_n(p)(y_1)\langle z_1,z_2\rangle_N, \pi_n(p)(y_2)\big)_{L^2(M_n)}\\
&= \langle\pi_n(p)(y_1)\otimes z_1, \pi_n(p)(y_2)\otimes z_2\rangle,
\end{aligned}$$

which implies that T is well-defined, factors through $N_{\langle\cdot,\cdot\rangle}$ and induces an isometry from $\pi_n(p)L^2(M_n)\otimes_N L^2(M)_M$ to $\pi_{n+1}(p)L^2(M_{n+1})$, which is clearly onto since $M_{n+1} = \mathrm{span}\{ae_{n+1}e_n\dots e_1b\,|\,a\in M_n,\ b\in M\}$ (see proof of Proposition 3.1) is dense in $L^2(M_{n+1})$. It is easy to see that T is an N-M map, so that the equivalence of N-M bimodules between the first and the third bimodule is established.

Now let p_k be as in the statement of the proposition. Since the k-th simple summand of $N'\cap M_{2n+1}$ sits with multiplicity G_{kl} in the l-th simple summand of $N'\cap M_{2n+2}$, $l\in L_{n+1}$, the third equivalence is again obvious by Lemma 3.5. The last statement follows immediately from this equivalence and the fact that the reduced bimodules $\pi_{n+1}(q_l)L^2(M_{n+1})$ are irreducible N-M bimodules. □

Corollary 4.4 *The embedding from $Hom_{N\text{-}N}({}_NL^2(M_n)_N)\to Hom_{N\text{-}M}({}_NL^2(M_{n+1})_M)$, given by $\pi_n(x)\to\pi_n(x)\otimes_N id_{{}_NL^2(M)_M}$, $x\in N'\cap M_{2n+1}$, coincides with the inclusion $N'\cap M_{2n+1}\subset N'\cap M_{2n+2}$, given by the principal graph Γ.*

Remark 4.5 *i) Propositions 4.1 and 4.3 characterize the entries in the matrix $G=\big(G_{kl}\big)_{k\in K,l\in L}$ ($K=\Gamma_{\mathrm{even}}$, $L=\Gamma_{\mathrm{odd}}$) as dimensions of certain intertwiner spaces and we could of course define the principal graphs using these dimensions. Clearly, similar propositions hold for the principal graph Γ' and $M'\cap M_{2n}$, $M'\cap M_{2n+1}$, $M'\cap M_{2n+2}$ - the proofs are the same (use again Proposition 2.2,* *last statement). Namely, we have that the inclusion* $Hom_{M\text{-}M}({}_ML^2(M_n)_M) \to Hom_{M\text{-}N}({}_ML^2(M_n)_N)$, *given by* $\pi_n(x) \to \pi_n(x) \otimes_M id_{{}_ML^2(M)_N}$, *and the inclusion* $Hom_{M\text{-}N}({}_ML^2(M_n)_N) \to Hom_{M\text{-}M}({}_ML^2(M_{n+1})_M)$, *given by* $\pi_n(x) \to \pi_n(x) \otimes_N id_{{}_NL^2(M)_M}$, *coincides with the inclusions* $M' \cap M_{2n} \subset M' \cap M_{2n+1}$, *resp.* $M' \cap M_{2n+1} \subset M' \cap M_{2n+2}$. *In the notation introduced in Definition 1.20, the last statements of Propositions 4.1 and 4.3 (also formulated for* Γ'*) read*

$$\begin{array}{lll} G_{\gamma\delta} &= \langle\gamma, \delta\bar{\rho}\rangle = \langle\gamma\rho, \delta\rangle, & \gamma \in \Gamma_{\text{even}}, \quad \delta \in \Gamma_{\text{odd}}, \\ G'_{\gamma'\delta'} &= \langle\gamma', \delta'\rho\rangle = \langle\gamma'\bar{\rho}, \delta'\rangle, & \gamma' \in \Gamma'_{\text{even}}, \quad \delta' \in \Gamma'_{\text{odd}}. \end{array}$$

ii) The multiplicities of the embeddings $M' \cap M_{2n} \subset N' \cap M_{2n}$ *and* $M' \cap M_{2n+1} \subset N' \cap M_{2n+1}$ *can be described in a similar way. More precisely, these inclusions coincide with the embeddings* $Hom_{M\text{-}M}({}_ML^2(M_n)_M) \to Hom_{N\text{-}M}({}_NL^2(M_n)_M)$, *given by* $\pi_n(x) \to id_{{}_NL^2(M)_M} \otimes_M \pi_n(x)$, *and the one from* $Hom_{M\text{-}N}({}_ML^2(M_n)_N)$ *to* $Hom_{N\text{-}N}({}_NL^2(M_n)_N)$, *given by* $\pi_n(x) \to id_{{}_NL^2(M)_M} \otimes_M \pi_n(x)$ *(i.e., multiplication by* ${}_NL^2(M)_M$ *from the left). The multiplicities can then immediately be calculated using Frobenius reciprocity. We find that the multiplicity of the* α*-th simple summand of* $M' \cap M_{2n}$ *(resp.* $M' \cap M_{2n+1}$*) in the* β*-th simple summand of* $N' \cap M_{2n}$ *(resp.* $N' \cap M_{2n+1}$*),* $\alpha \in \Gamma'_{\text{even}}$ *(resp.* $\alpha \in \Gamma'_{\text{odd}}$*),* $\beta \in \Gamma_{\text{odd}}$ *(resp.* $\beta \in \Gamma_{\text{even}}$*) is given by* $\langle\rho\alpha, \beta\rangle = \langle\bar{\alpha}, \bar{\beta}\rho\rangle = G'_{\bar{\alpha},\bar{\beta}}$, *since* $\bar{\beta} \in \Gamma'_{\text{odd}}$ *(resp.* $\langle\rho\alpha, \beta\rangle = \langle\bar{\beta}, \bar{\alpha}\bar{\rho}\rangle = G_{\bar{\beta},\bar{\alpha}}$, *since* $\bar{\alpha} \in \Gamma_{\text{odd}}$*) (see i) and Corollary 1.21), where* $\rho = {}_NL^2(M)_M$ *as usual. Thus we have recovered all the multiplicities of the embeddings,*

$$\begin{array}{ccccc} N' \cap M_{2n} & \subset & N' \cap M_{2n+1} & \subset & N' \cap M_{2n+2} \\ \cup & & \cup & & \cup \\ M' \cap M_{2n} & \subset & M' \cap M_{2n+1} & \subset & M' \cap M_{2n+2} \end{array}$$

as dimensions of spaces of bimodule intertwiners. Observe that the role of the conjugate $^-$ *(or contragredient) map is displayed nicely in this bimodule picture of the above embeddings.*

We will now show how the bimodule tensor product of reduced bimodules can be determined by computing certain products of projections associated to these reduced bimodules. The idea of how to do this has been known to experts for some time (see also Goodman and Wenzl [1990]), we were however unable to find any references in the literature. We will obtain a procedure that allows us to calculate the fusion algebra associated to certain subfactors quite easily (as for instance the subfactors in (Bisch and Jones [1995])). Let $\gamma_k : N' \cap M_{2k+1} \to N' \cap M_{2k+1}$ be the surjective, linear $*$-antiisomorphism defined by $\pi_k(\gamma_k(x)) = J_k\pi_k(x)^*J_k$, $x \in N' \cap M_{2k+1}$ (apply Lemma 2.4 with $A = N$, $B = M_k$). Then $\text{sh}_{2k+1} = \gamma_{2k+1}\gamma_k : N' \cap M_{2k+1} \to M'_{2k+1} \cap M_{4k+3}$ is a trace preserving, surjective $*$-isomorphism (by the remark before Lemma 2.9 and Corollary 2.12 with $A = N$, $B = M_k$, $B_1 = M_{2k+1}$ and $B_3 = M_{4k+3}$), which we call the $((2k+2)$-$)$ shift on $N' \cap M_{2k+1}$. Note that sh_{2k+1} is a *shift* as in Definition 2.14: if we let $A = N$, $B = M_k$ in Theorem 2.13, then sh_{2k+1} is the 2-shift which we denoted there by S_0 (2.13 vi)). Similarly, we get a trace preserving, surjective $*$-isomorphism $\text{sh}_{2k} = \gamma_{2k}\gamma_k : M' \cap M_{2k} \to M'_{2k} \cap M_{4k}$ (by the remark before Lemma 2.9 and Corollary 2.12 with $A = M$, $B = M_k$), which is again a shift as in Definition 2.14. Note that $\gamma_{2k}(x) \in M'_{2k} \cap M_{4k}$ for all $x \in M' \cap M_{2k}$, since $\pi_{2k}(\gamma_{2k}(x)) \in J_{2k}\pi_{2k}(M' \cap M_{2k})J_{2k} = \pi_{2k}(M'_{2k} \cap M_{4k})$. $\gamma_{2k}|_{M' \cap M_{2k}}$ is therefore a linear, surjective $*$-antiisomorphism $M' \cap M_{2k} \to M'_{2k} \cap M_{4k}$.

Theorem 4.6 *a) Let* $p,\ q \in N' \cap M_{2n+1}$ *be projections and let* $\mathrm{sh}_{2n+1} : N' \cap M_{2n+1} \to M'_{2n+1} \cap M_{4n+3}$ *be the shift as defined above. Then,*

$$\pi_n(p)L^2(M_n) \otimes_N \pi_n(q)L^2(M_n) \overset{N\text{-}N}{\cong} \pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))L^2(M_{2n+1})$$

as N-N *bimodules. Furthermore* $p\ \mathrm{sh}_{2n+1}(q) \in N' \cap M_{4n+3}$ *is a projection with trace* $\mathrm{tr}_{M_{4n+3}}(p\ \mathrm{sh}_{2n+1}(q)) = \mathrm{tr}_{M_{2n+1}}(p)\mathrm{tr}_{M_{2n+1}}(q)$.
b) Let $p,\ q \in N' \cap M_{2n}$ *be projections and let* $\gamma_{2n} : N' \cap M_{2n} \to M'_{2n} \cap M_{4n+1}$ *be defined as above. Then,*

$$\begin{aligned} \pi_n(p)L^2(M_n) \otimes_M \overline{\pi_n(q)L^2(M_n)} &\overset{N\text{-}N}{\cong} \pi_n(p)L^2(M_n) \otimes_M \pi_n(\gamma_n(q))L^2(M_n) \\ &\overset{N\text{-}N}{\cong} \pi_{2n}(p\gamma_{2n}(q))L^2(M_{2n}) \end{aligned}$$

as N-N *bimodules. Furthermore,* $p\gamma_{2n}(q) \in N' \cap M_{4n+1}$ *is a projection with trace* $\mathrm{tr}_{M_{4n+1}}\big(p\gamma_{2n}(q)\big) = \mathrm{tr}_{M_{2n}}(p)\mathrm{tr}_{M_{4n+1}}(\gamma_{2n}(q))$.
c) Let $p,\ q \in M' \cap M_{2n}$ *be projections and let* $\mathrm{sh}_{2n} : M' \cap M_{2n} \to M'_{2n} \cap M_{4n}$ *be the shift as above. Then,*

$$\pi_n(p)L^2(M_n) \otimes_M \pi_n(q)L^2(M_n) \overset{M\text{-}M}{\cong} \pi_{2n}(p\ \mathrm{sh}_{2n}(q))L^2(M_{2n})$$

as M-M *bimodules. Furthermore,* $p\ \mathrm{sh}_{2n}(q) \in M' \cap M_{4n}$ *is a projection with trace* $\mathrm{tr}_{M_{4n}}(p\ \mathrm{sh}_{2n}(q)) = \mathrm{tr}_{M_{2n}}(p)\mathrm{tr}_{M_{2n}}(q)$.
d) Let $p,\ q \in N' \cap M_{2n}$ *be projections and* $\mathrm{sh}_{2n+1} : N' \cap M_{2n} \to M'_{2n} \cap M_{4n+2}$ *be the shift as above, restricted to* $N' \cap M_{2n}$*. Then,*

$$\begin{aligned} \overline{\pi_n(p)L^2(M_n)} \otimes_N \pi_n(q)L^2(M_n) &\overset{M\text{-}M}{\cong} \pi_n(\gamma_n(p))L^2(M_n) \otimes_N \pi_n(q)L^2(M_n) \\ &\overset{M\text{-}M}{\cong} \pi_{2n+1}(\gamma_n(p)\ \mathrm{sh}_{2n+1}(q))L^2(M_{2n+1}) \end{aligned}$$

as M-M *bimodules. Furthermore,* $\gamma_n(p)\ \mathrm{sh}_{2n+1}(q) \in N' \cap M_{4n+2}$ *is a projection with trace* $\mathrm{tr}_{M_{4n+2}}(\gamma_n(p)\ \mathrm{sh}_{2n+1}(q)) = \mathrm{tr}_{M_{2n+1}}(\gamma_n(p))\mathrm{tr}_{M_{2n}}(q)$.

Proof Let us start with the proof of a). Observe that p and $\mathrm{sh}_{2n+1}(q)$ are commuting projections, so that $p\ \mathrm{sh}_{2n+1}(q) \in N' \cap M_{4n+3}$ is again a projection with trace as stated in the theorem (by Corollary 2.12). We define a linear map $T : \pi_n(p)(\hat{M}_n) \odot \pi_n(q)(\hat{M}_n) \to \pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))L^2(M_{2n+1})$ by $T\big(\pi_n(p)(\hat{x}) \otimes \pi_n(q)(\hat{y})\big) = [M : N]^{(n+1)/2}\pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))(\widehat{xf_ny})$, $x,\ y \in M_n$, f_n the Jones projection for $N \subset M_n$ (Proposition 2.1 i)). The fact that T is well-defined will follow from a computation of the inner products below. Recall that $M_{2n+1} = M_nf_nM_n \overset{\mathrm{def}}{=} \mathrm{span}\{xf_ny \,|\, x,\ y \in M_n\}$ and that the Radon-Nikodym derivatives satisfy $\langle \pi_n(q)(\hat{y}_1), \pi_n(q)(\hat{y}_2)\rangle_N = \langle \hat{y}_1, \pi_n(q)(\hat{y}_2)\rangle_N$, $y_1,\ y_2 \in M_n$ (Proposition 1.15 ii)). Furthermore, we have

$$\begin{aligned} \pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))(\widehat{xf_ny}) &= \pi_{2n+1}(\gamma_{2n+1}\gamma_n(q))\pi_{2n+1}(p)(\widehat{xf_ny}) \\ &= J_{2n+1}\pi_{2n+1}(\gamma_n(q)^*)J_{2n+1}\big(\widehat{pxf_ny}\big) \\ &= J_{2n+1}\big((\gamma_n(q)^*y^*f_nx^*p)^\wedge\big) \\ &= (pxf_ny\gamma_n(q))^\wedge, \end{aligned}$$

since $\pi_{2n+1}|_{M_{2n+1}}$ is left multiplication. Let x_i, $y_i \in M_n$, $i = 1, 2$. We compute

$$\begin{aligned}
\langle\ &\pi_n(p)(\hat{x}_1) \otimes \pi_n(q)(\hat{y}_1), \pi_n(p)(\hat{x}_2) \otimes \pi_n(q)(\hat{y}_2)\rangle = \\
&= \big(\pi_n(p)(\hat{x}_1)\langle \pi_n(q)(\hat{y}_1), \pi_n(q)(\hat{y}_2)\rangle_N, \pi_n(p)(\hat{x}_2)\big)_{L^2(M_n)} \\
&= \big(\pi_n(p)\big((x_1 E_N^{M_n}(y_1(\pi_n(q)(y_2))^*))^\wedge\big), \pi_n(p)(\hat{x}_2)\big)_{L^2(M_n)} \\
&= \mathrm{tr}_{M_n}\big(x_2^*\pi_n(p)(x_1)E_N^{M_n}(y_1(\pi_n(q)(y_2))^*)\big),
\end{aligned}$$

where we used in the third and fourth equality that $\pi_n(p)$ is a right N-module intertwiner. Next we calculate,

$$\begin{aligned}
\big(\ &T\big(\pi_n(p)(\hat{x}_1) \otimes \pi_n(q)(\hat{y}_1)\big), T\big(\pi_n(p)(\hat{x}_2) \otimes \pi_n(q)(\hat{y}_2)\big)\big)_{L^2(M_{2n+1})} \\
&= [M:N]^{n+1}\big(\pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))(\widehat{x_1 f_n y_1}), \widehat{x_2 f_n y_2}\big)_{L^2(M_{2n+1})} \\
&= [M:N]^{n+1}\big((p x_1 f_n y_1 \gamma_n(q))^\wedge, (x_2 f_n y_2)^\wedge\big)_{L^2(M_{2n+1})} \\
&= [M:N]^{n+1}\mathrm{tr}_{M_{2n+1}}\big(y_2^* f_n x_2^* p x_1 f_n y_1 \gamma_n(q)\big) \\
&= [M:N]^{n+1}\mathrm{tr}_{M_{2n+1}}\big(x_2^* p x_1 f_n y_1 \gamma_n(q) y_2^* f_n\big) \\
&= [M:N]^{n+1}\mathrm{tr}_{M_{2n+1}}\big(x_2^*\pi_n(p)(x_1) f_n y_1(\pi_n(q)(y_2))^* f_n\big) \\
&= [M:N]^{n+1}\mathrm{tr}_{M_{2n+1}}\big(x_2^*\pi_n(p)(x_1)E_N^{M_n}(y_1(\pi_n(q)(y_2))^*) f_n\big) \\
&= \mathrm{tr}_{M_n}\big(x_2^*\pi_n(p)(x_1)E_N^{M_n}(y_1(\pi_n(q)(y_2))^*)\big) \\
&= \langle \pi_n(p)(\hat{x}_1) \otimes \pi_n(q)(\hat{y}_1), \pi_n(p)(\hat{x}_2) \otimes \pi_n(q)(\hat{y}_2)\rangle,
\end{aligned}$$

where we used in the fifth equality that $px_1 f_n = \pi_n(p)(x_1) f_n$ and $\gamma_n(q) y_2^* f_n = \pi_n(\gamma_n(q))(y_2^*) f_n = (\pi_n(q)(y_2))^* f_n$ by Lemma 2.4 ii) and iii) (applied to $A = N$, $B = M_n$, $B_1 = M_{2n+1}$, $e_1 = f_n$). Thus the above defined linear map T is well-defined and induces an isometry, still denoted by $T : \pi_n(p)L^2(M_n) \otimes_N \pi_n(q)L^2(M_n) \to \pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))L^2(M_{2n+1})$. Since $T\big(a \cdot (\pi_n(p)(\hat{x}) \otimes \pi_n(q)(\hat{y})) \cdot b\big) = T\big(\pi_n(p)(\widehat{ax}) \otimes \pi_n(q)(\widehat{yb})\big) = (pax f_n y b \gamma_n(q))^\wedge = (apx f_n y \gamma_n(q) b)^\wedge = a \cdot T\big(\pi_n(p)(\hat{x}) \otimes \pi_n(q)(\hat{y})\big) \cdot b$, for all a, $b \in N$, we have that the induced map T is an injective N-N bimodule morphism (by definition of the N-M bimodule structure on the relative tensor product, see the remark after Definition 1.11), which is onto, since $\pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))(\hat{M}_{2n+1})$ is dense in $\pi_{2n+1}(p\ \mathrm{sh}_{2n+1}(q))L^2(M_{2n+1})$. This completes the proof of a).

Since the arguments for b)-d) are similar, we will be brief. To keep the notation simple, we will omit all the "hats" (it should be clear from the proof of a) where they need to be used). Let us prove b). Recall that $\pi_n(q)L^2(M_n) \cong \pi_n(\gamma_n(q))L^2(M_n)$ as M-N bimodules (Proposition 3.11), which proves the first identity. To prove the second one, define a linear map $T : \pi_n(p)(\hat{M}_n) \odot \pi_n(\gamma_n(q))(\hat{M}_n) \to$

$\pi_{2n}(p\gamma_{2n}(q))L^2(M_n)$ by

$$T\big(\pi_n(p)(\hat{x}) \otimes \pi_n(\gamma_n(q))(\hat{y})\big) = [M:N]^{n/2}\pi_{2n}(p\gamma_{2n}(q))(\widehat{xg_ny}),$$

for all x, $y \in M_n$, g_n the Jones projection for $M \subset M_n$ (Proposition 2.1 ii)). Observe that $\pi_n(\gamma_n(q)) \in \pi_n(M' \cap M_{2n+1})$ and $\pi_{2n}(\gamma_{2n}(q)) \in J_{2n}\pi_{2n}(N' \cap M_{2n})J_{2n} = \pi_{2n}(M'_{2n} \cap M_{4n+1})$. In particular, $p\gamma_{2n}(q) \in N' \cap M_{4n+1}$ is a projection with above stated trace. Furthermore, recall that $M_{2n} = M_ng_nM_n$. We compute for x_i, $y_i \in M_n$, $i = 1$, 2 (omitting "hats")

$$\begin{aligned}&\langle\ \pi_n(p)(x_1) \otimes \pi_n(\gamma_n(q))(y_1), \pi_n(p)(x_2) \otimes \pi_n(\gamma_n(q))(y_2)\rangle\\ &= \big(\pi_n(p)\big(x_1E_M^{M_n}\big(y_1(\pi_n(\gamma_n(q))(y_2))^*\big)\big), x_2\big)\\ &= \mathrm{tr}_{M_n}\big(x_2^*\pi_n(p)(x_1)E_M^{M_n}\big(y_1\pi_n(q)(y_2^*)\big)\big)\end{aligned}$$

since $\pi_n(\gamma_n(q))(y_2)^* = \pi_n(q)(y_2^*)$ by Lemma 2.4 iv). We also used in the second equality that $\pi_n(p)$ is a right M-module map. Since $\pi_{2n}|_{M_{2n}}$ is left multiplication, we have $T\big(\pi_n(p)(\hat{x}) \otimes \pi_n(\gamma_n(q))(\hat{y})\big) = \widehat{pxg_nyq}$, x, $y \in M_n$. Thus

$$\begin{aligned}&\big(\ T\big(\pi_n(p)(x_1) \otimes \pi_n(\gamma_n(q))(y_1)\big), T\big(\pi_n(p)(x_2) \otimes \pi_n(\gamma_n(q))(y_2)\big)\big)_{L^2(M_{2n})}\\ &= [M:N]^n\mathrm{tr}_{M_{2n}}\big(x_2^*px_1g_ny_1qy_2^*g_n\big)\\ &= [M:N]^n\mathrm{tr}_{M_{2n}}\big(x_2^*\pi_n(p)(x_1)E_M^{M_n}\big(y_1\pi_n(q)(y_2^*)\big)g_n\big)\\ &= \mathrm{tr}_{M_n}\big(x_2^*\pi_n(p)(x_1)E_M^{M_n}\big(y_1\pi_n(q)(y_2^*)\big)\big)\\ &= \langle\pi_n(p)(x_1) \otimes \pi_n(\gamma_n(q))(y_1), \pi_n(p)(x_2) \otimes \pi_n(\gamma_n(q))(y_2)\rangle,\end{aligned}$$

where we used that $px_1g_n = \pi_n(p)(x_1)g_n$, $qy_2^*g_n = \pi_n(q)(y_2^*)g_n$, by Lemma 2.4 ii) (applied to $A = M$, $B = M_n$). Note that we also use that π_n is the representation coming from the basic construction of $M \subset M_n$ (Proposition 2.2). As before, T is well-defined and induces an isometry $\pi_n(p)L^2(M_n) \otimes_M \pi_n(\gamma_n(q))L^2(M_n) \to \pi_{2n}(p\gamma_{2n}(q))L^2(M_{2n})$, which is clearly an N-N bimodule morphism (note that $\gamma_n(q) \in M' \cap M_{2n+1}$) and surjective with the same argument as above.

Next we prove c). It is clear that $p\ \mathrm{sh}_{2n}(q)$ is a projection in $M' \cap M_{4n}$ with trace stated above (proof as in a)). The proof of c) proceeds now as the proof of a), namely one shows that the linear map $T : \pi_n(p)(\hat{M}_n) \odot \pi_n(q)(\hat{M}_n) \to \pi_{2n}(p\ \mathrm{sh}_{2n}(q))L^2(M_{2n})$ defined by $T\big(\pi_n(p)(\hat{x}) \otimes \pi_n(q)(\hat{y})\big) = \pi_{2n}(p\ \mathrm{sh}_{2n}(q))(\widehat{xg_ny}) = (pxg_ny\gamma_n(q))^\wedge$, x, $y \in M_n$, g_n as in Proposition 2.1 ii), is well-defined and induces the desired equivalence of M-M bimodules.

Finally, the proof of d). Since $\overline{\pi_n(p)L^2(M_n)} \cong \pi_n(\gamma_n(p))L^2(M_n)$ as M-N bimodules, by Proposition 3.11, the first identity follows. The second one is now shown precisely as in a) (with $\gamma_n(p)$ in place of p). Observe that the T defined there is actually an M-M bimodule morphism since $\gamma_n(p) \in M' \cap M_{2n+1}$ (thus $\pi_n(\gamma_n(p))$ is a left M-module map) and since $q \in N' \cap M_{2n}$ (thus $\pi_n(q)$ is a right M-module map). Since $\mathrm{sh}_{2n+1}(q) \in M'_{2n+1} \cap M_{4n+2}$, it is clear that $\gamma_n(p)\ \mathrm{sh}_{2n+1}(q) \in N' \cap M_{4n+2}$ is a projection with trace as stated above. This completes the proof of the theorem. □

Remark 4.7 *i) Observe that a) of the previous theorem gives a formula for the bimodule tensor product of two reduced N-N bimodules, b) for the one of a reduced N-M with a reduced M-N bimodule, c) for the one of two reduced M-M bimodules and finally d) for the tensor product of a reduced M-N with a reduced N-M bimodule. Since we have explicit formulas of the antiisomorphisms γ_k and the shifts sh_{2n+1} and sh_{2n} in terms of the e_i's and orthonormal bases (Theorem 2.6, Theorem 2.11, Theorem 2.13), we can calculate the (full) fusion algebra associated to a subfactor by computing products of projections in the higher relative commutants (as given by the theorem) and decomposing the resulting bimodules into irreducibles according to Lemma 3.5. This method proves to be very useful if the higher relative commutants of the subfactor are known in detail (for instance for the subfactors in (*Bisch and Jones [1995]*)). We will give some applications elsewhere.*
ii) If we use Theorem 3.7 in conjunction with Theorem 4.6, we have explicit formulas for the bimodule tensor product of any two reduced bimodules $\pi_n(p)L^2(M_n)$ and $\pi_m(q)L^2(M_m)$ (embed the projections in the same higher relative commutant using 3.7 and compute the bimodule tensor product according to 4.6) and we could abstractly define a product on the (equivalence classes of) projections in the higher relative commutants in this way.
iii) Associativity of the bimodule tensor product of reduced bimodules can be proved using 4.6 (see Remark ii)). It amounts to showing that certain projections, obtained by using the formulas in 4.6, are equivalent.

5 Fusion algebra calculations and reduced subfactors

5.1 Reduced subfactors Let $N \subset M$ be an inclusion of II_1 factors and let M_n be the II_1 factors in the associated Jones' tower. If $p \in N' \cap M_n$ (resp. $p \in M' \cap M_n$) is a projection, we call $Np \subset pM_np$ (resp. $Mp \subset pM_np$) a *reduced subfactor*. We will discuss in this section only reduced subfactors of the form $Np \subset pM_np$ and leave it to the reader to reformulate everything for those of the form $Mp \subset pM_np$. Note that $(Np)' \cap pM_np = p(N' \cap M_n)p$, so that the reduced subfactor is irreducible iff the projection p is minimal in $N' \cap M_n$. We will usually assume that p is minimal when we talk about reduced subfactors, although this is not necessary for the statements below. Furthermore, observe that $[pM_np : Np] = \mathrm{tr}_{M_n}(p)\mathrm{tr}_{N'}(p)[M_n : N]$ (which is called a *local index* of $N \subset M$). A good way to deal with fusion questions regarding reduced subfactors is to use the endomorphism picture (tensor $N \subset M$ with $B(H)$, H an infinite dimensional Hilbert space, consider the resulting algebras in their standard representation and define an endomorphism using a common cyclic and separating vector), which makes statements regarding the fusion algebra of reduced subfactors rather obvious. To keep this article self-contained, we will however stay in the II_1 setting.

Let us start with the basic construction for reduced subfactors. The following lemma is well-known (see, for instance Bisch [1994(a)] or Wenzl [1988]) - we include a proof here for the convenience of the reader.

Lemma 5.1 *Let $A \subset B$ be II_1 factors with finite index and let $A \subset B \subset B_1$ be the basic construction. Let $p \in A' \cap B$ be a projection, let $q = \gamma_0(p) = JpJ \in B' \cap B_1$, where $J : L^2(B) \to L^2(B)$ denotes the modular conjugation. Then,*

$$Apq \subset (pBp)q \subset pqB_1pq$$

is (isomorphic to) the basic construction for $(Ap \subset pBp) \cong (Apq \subset (pBp)q)$.

Proof Let us denote by $\mathrm{tr}_{A'}$, tr_B, tr_{B_1} the unique normalized traces on the II_1 factors $A' = A' \cap B(L^2(B))$, B, B_1. Recall that $\mathrm{tr}_{B_1}(Jx^*J) = \mathrm{tr}_{A'}(x)$, for all $x \in A'$, by uniqueness of the trace. Furthermore, we have that $[(pBp)q : Apq] = [pBp : Ap] = \mathrm{tr}_{A'}(p)tr_B(p)[B : A]$. Similarly, $[pqB_1pq : (pBp)q] = \mathrm{tr}_{B_1}(q)\mathrm{tr}_{B'}(q)[pB_1p : pBp] = \mathrm{tr}_{B_1}(q)\mathrm{tr}_{B'}(q)[B : A]$. But $\mathrm{tr}_{B_1}(q) = \mathrm{tr}_{A'}(p)$ and $\mathrm{tr}_{B'}(q) = \mathrm{tr}_{JB'J}(JqJ) = \mathrm{tr}_B(p)$, so that both indices coincide. Observe that $pq = qp$ and $pqe_A = pe_A$ (Lemma 2.4 iii)), where $e_A : L^2(B) \to L^2(A)$ is the orthogonal projection. Now define $e = \mathrm{tr}_{B_1}(p)^{-1}pqe_Apq$. We get that $\mathrm{tr}_{pqB_1pq}(e) = \mathrm{tr}_{B_1}(p)^{-1}\mathrm{tr}_{B_1}(pq)^{-1}\mathrm{tr}_{B_1}(e_Apq) = \mathrm{tr}_{B_1}(pq)^{-1}[B : A]^{-1} = \mathrm{tr}_B(p)^{-1}\mathrm{tr}_{A'}(p)^{-1}[B : A]^{-1}$. Furthermore, an easy calculation (using $pqe_A = pe_A$) shows that $e^2 = e \in pqB_1pq$. We will apply the abstract characterization of the basic construction (Pimsner and Popa [1988]) to show that pqB_1pq is the basic construction for $Apq \subset (pBp)q$. Since we clearly have that e commutes with Apq, we only need to check

$$E^{pqB_1pq}_{pqBpq}(e) = \mathrm{tr}_B(p)^{-1}\mathrm{tr}_{A'}(p)^{-1}[B : A]^{-1}pq.$$

Note that $E^{B_1}_B(pe_Ap) = [B : A]^{-1}p$ and $\mathrm{tr}_{pqB_1pq}(pqxpq) = \mathrm{tr}_{pB_1p}(pxp)$, for all $x \in B$. Thus, if $x \in B$, we have

$$\begin{aligned}\mathrm{tr}_{pqB_1pq}(pqxpqe_Apq) &= \mathrm{tr}_{B_1}(xpe_Ap)\mathrm{tr}^{-1}_{B_1}(q)\mathrm{tr}_{B_1}(p)^{-1}\\ &= [B : A]^{-1}\mathrm{tr}_{B_1}(q)^{-1}\mathrm{tr}_{B_1}(p)^{-1}\mathrm{tr}_{B_1}(xp)\\ &= [B : A]^{-1}\mathrm{tr}_{B_1}(q)^{-1}\mathrm{tr}_{pB_1p}(xp)\\ &= [B : A]^{-1}\mathrm{tr}_{B_1}(q)^{-1}\mathrm{tr}_{pqB_1pq}(pqxpq),\end{aligned}$$

so that $E^{pqB_1pq}_{pqBpq}(pqe_Apq) = [B : A]^{-1}\mathrm{tr}_{B_1}(q)^{-1}pq$ as desired. □

Recall that if $p \in A$ is a projection, the basic construction for $pAp \subset pBp$ is $pAp \subset pBp \subset pB_1p$ with Jones projection pe_Ap.

Next we identify the iterated basic construction for reduced subfactors. Let $N \subset M$ be an inclusion of II_1 factors with finite index. By (Pimsner and Popa [1988]) we have that $N \subset M_n \subset M_{2n+1} \subset M_{3n+2} \subset M_{4n+3} \subset \dots$ is the basic construction for $N \subset M_n$. Denote as in the discussion before Theorem 4.6 by $\gamma_{(r+1)n+r}$ the surjective, linear $*$-antiisomorphisms $N' \cap M_{2(r+1)n+2r+1} \to N' \cap M_{2(r+1)n+2r+1}$ (note that $N \subset M_{(r+1)n+r} \subset M_{2(r+1)n+2r+1}$ is the basic construction), restricted to $M'_{rn+(r-1)} \cap M_{(r+2)n+r+1}$, which yields a surjective $*$-antiisomorphism $M'_{rn+(r-1)} \cap M_{(r+2)n+r+1} \to M'_{rn+(r-1)} \cap M_{(r+2)n+r+1}$ ($M_{rn+r-1} \subset M_{(r+1)n+r} \subset M_{(r+2)n+r+1}$ is the basic construction).

Corollary 5.2 *Let $N \subset M$ be an inclusion of II_1 factor with finite index and let $p \in N' \cap M_n$ be a projection. Consider the reduced subfactor $Np \subset pM_np$, set $A = Np$ and $B = pM_np$ and let $A \subset B \subset B_1 \subset \dots \subset B_k$ be the iterated basic construction. Set $q_0 = p$, $q_1 = q_0\gamma_n(q_0), \dots, q_r = q_{r-1}\gamma_{rn+r-1}(q_{r-1})$. Then $q_r \in (q_{r-1}M_{rn+r-1}q_{r-1})' \cap q_{r-1}M_{(r+1)n+r}q_{r-1}$ and $A \subset B \subset B_1 \subset \dots \subset B_k$ is (isomorphic to) $Nq_k \subset q_kMq_k \subset \dots q_kM_{(k+1)n+k}q_k$.*

Proof The proof is a straightforward induction, using Lemma 5.1.1 and the comment after the lemma. □

Observe that it follows that reduced subfactors of reduced subfactors are reduced subfactors of the original inclusion. In particular, the vertices of the principal graphs of reduced subfactors can be identified with subsets of the vertices of the principal graphs of $N \subset M$, and the fusion algebra of a reduced subfactor can be read off the fusion algebra for $N \subset M$. Let $p \in N' \cap M_{2n+1}$ (resp. $q \in N' \cap M_{2n}$) be a projection and consider the reduced N-N (resp. N-M) bimodule $\alpha = \pi_n(p)L^2(M_n)$ (resp. $\beta = \pi_n(q)L^2(M_n)$) (see Proposition 2.2). Then $\mathrm{Hom}_{N\text{-}N}(\alpha) \cong (Np)' \cap pM_{2n+1}p$ and since $\alpha\bar{\alpha} = \pi_n(p)L^2(M_n) \otimes_N \pi_n(p)L^2(M_n) \cong \pi_{2n+1}(p\gamma_{2n+1}(p))L^2(M_{2n+1})$ by 3.11 and 4.6 a), we have $\mathrm{Hom}_{N\text{-}N}(\alpha\bar{\alpha}) \cong (Np\gamma_{2n+1}(p))' \cap p\gamma_{2n+1}(p)M_{4n+3}p\gamma_{2n+1}(p)$. Similarly, $\mathrm{Hom}_{N\text{-}M}(\beta) \cong (Nq)' \cap qM_{2n}q$ and since $\beta\bar{\beta} = \pi_n(q)L^2(M_n) \otimes_M \pi_n(q)L^2(M_n) \cong \pi_{2n}(q\gamma_{2n}(q))L^2(M_{2n})$ by 4.6 b), we have $\mathrm{Hom}_{N\text{-}N}(\beta\bar{\beta}) \cong (Nq\gamma_{2n}(q))' \cap q\gamma_{2n}(q)M_{4n+1}q\gamma_{2n}(q)$. Thus, tensoring repeatedly with α, $\bar{\alpha}$ (resp. β, $\bar{\beta}$) from the right we get ($M_{-1} = N$, $M_0 = M$).

Proposition 5.3 *Consider the reduced subfactors $Np \subset pM_{2n+1}p$, $p \in N' \cap M_{2n+1}$ a projection, and $Nq \subset qM_{2n}q$, $q \in N' \cap M_{2n}$ a projection and let the bimodules α and β be as above. Furthermore, define projections $p_0 = p$, $p_1 = p_0\gamma_{2n+1}(p_0), \ldots, p_r = p_{r-1}\gamma_{2rn+2r-1}(p_{r-1})$, $q_0 = q$, $q_1 = q_0\gamma_{2n}(q_0), \ldots, q_r = q_{r-1}\gamma_{2rn+r-1}(q_{r-1})$. Then,*

(i) $Hom_{N\text{-}N}\big((\alpha\bar{\alpha})^k\big) \cong \big(Np_{2k-1}\big)' \cap p_{2k-1}\big(M_{2k(2n+1)+2k-1}\big)p_{2k-1}$.

(ii) $Hom_{N\text{-}N}\big((\alpha\bar{\alpha})^k\alpha\big) \cong \big(Np_{2k}\big)' \cap p_{2k}\big(M_{(2k+1)(2n+1)+2k}\big)p_{2k}$.

(iii) $Hom_{N\text{-}N}\big((\beta\bar{\beta})^k\big) \cong \big(Nq_{2k-1}\big)' \cap q_{2k-1}\big(M_{4kn+2k-1}\big)q_{2k-1}$.

(iv) $Hom_{N\text{-}M}\big((\beta\bar{\beta})^k\beta\big) \cong \big(Nq_{2k}\big)' \cap q_{2k}\big(M_{2(2k+1)n+2k}\big)q_{2k}$.

As in Propositions 4.1 and 4.3 one can then determine the principal graphs for the reduced subfactors. Let us consider first a reduced subfactor of the form $Np \subset pM_{2n+1}p$, $p \in N' \cap M_{2n+1}$ a minimal projection. By Corollary 5.1.2 (or 5.1.3) we see that the even and odd vertices of the principal graphs are subsets of Γ_{even} (Γ, Γ' denote as before the principal graphs of $N \subset M$). The even (resp. odd) vertices are obtained by decomposing $(\alpha\bar{\alpha})^k$ (resp. $(\alpha\bar{\alpha})^k\alpha$), for all $k \in \mathbb{N}$, into irreducible N-N bimodules. Let γ be such an even vertex and δ an odd one. As in 4.1 and 4.3 one shows that the number of edges between these two vertices is obtained as $\langle\gamma\alpha, \delta\rangle$. Note that all computations here involve only N-N bimodules. Next, let us consider a reduced subfactor of the form $Nq \subset qM_{2n}q$, $q \in N' \cap M_{2n}$ a minimal projection. Again, by Corollary 5.1.2 (or 5.1.3) we see that the vertices of the principal graphs of this reduced subfactor are (identified with) subsets of Γ_{even} and Γ_{odd} (one graph) (resp. Γ'_{even} and Γ'_{odd} (the other graph)). The even (resp. odd) vertices are obtained by decomposing $(\beta\bar{\beta})^k$ (resp. $(\beta\bar{\beta})^k\beta$), $k \in \mathbb{N}$, into irreducible N-N (resp. N-M) bimodules. Let γ be such an even vertex and δ an odd one, then the number of edges is again given by $\langle\gamma\beta, \delta\rangle$ (N-M bimodules). We leave it to the reader to formulate the analogue of Proposition 5.1.3 and the above discussion for the other principal graph of a reduced subfactor (replacing α by $\bar{\alpha}$, β by $\bar{\beta}$). Furthermore, as mentioned above, observe that the fusion algebra can be read off the fusion algebra for $N \subset M$ for both types of reduced subfactors.

5.2 Computation of the fusion algebra associated to a subfactor

The calculation of the fusion algebra for an inclusion of II_1 factors $N \subset M$ can sometimes be carried out by purely (linear) algebraic methods (see, for instance

Bisch [1994(b)], Haagerup [1994], Izumi [1991], Ocneanu [1991(b)], Sunder [1992], Sunder and Vijayarajan [1993], for other methods see Bisch and Haagerup [1996], Goodman and Wenzl [1990], Kawahigashi [1996], Kosaki et al., Wassermann [1995], Remark 4.7), since algebraic/combinatorial properties of the fusion matrices (for instance the fact that they have only nonnegative integer entries) are sometimes enough to calculate the bimodule tensor products. However, as soon as the principal graphs of the subfactor have multiple edges or quadruple points, these algebraic manipulations may not be enough to determine the fusion algebra completely (see, for instance (Bisch [1994(b)], $3+\sqrt{3}$ example) and additional information must be used to determine the fusion algebra (see, for instance (Kawahigashi [1996]) for a solution for the $3+\sqrt{3}$ example). We will give a method that allows us to calculate in many cases the fusion algebra associated to a subfactor by solving matrix equations and we will illustrate this method with an example.

Let $N \subset M$ be an inclusion of II_1 factors with finite index and let $\rho = {}_N L^2(M)_M$ be as before. Let (Γ, Γ') be the principal graphs of $N \subset M$ and denote by G and G' the associated matrices (see the discussion after Proposition 3.2). As shown after Definition 3.12, we can regard R_ρ (resp. $R_{\bar\rho}$), right multiplication by $\bar\rho$ (resp. ρ), as a matrix $\mathbb{Z}\Gamma_{\text{odd}} \to \mathbb{Z}\Gamma_{\text{even}}$ (resp. $\mathbb{Z}\Gamma_{\text{even}} \to \mathbb{Z}\Gamma_{\text{odd}}$). If we let $B_1 = \{\alpha \,|\, \alpha \in \Gamma_{\text{even}}\}$ and $B_2 = \{\beta \,|\, \beta \in \Gamma_{\text{odd}}\}$ be the canonical bases of $\mathbb{Z}\Gamma_{\text{even}}$ resp. $\mathbb{Z}\Gamma_{\text{odd}}$, then the matrix representation of R_ρ (resp. $R_{\bar\rho}$) with respect to these bases is G (resp. G^t), i.e. $\left(R_\rho\right)_{\alpha\beta} = \langle \alpha, \beta\bar\rho\rangle = G_{\alpha\beta}$, and $\left(R_{\bar\rho}\right)_{\beta\alpha} = \langle \alpha\rho, \beta\rangle = G_{\alpha\beta}$, $\alpha \in \Gamma_{\text{even}}$, $\beta \in \Gamma_{\text{odd}}$ (Remark 4.5). Thus $R_{(\rho\bar\rho)^n} = \left(GG^t\right)^n$ ($R_{(\rho\bar\rho)^n} : \mathbb{Z}\Gamma_{\text{even}} \to \mathbb{Z}\Gamma_{\text{even}}$) and $R_{(\rho\bar\rho)^n\rho} = \left(GG^t\right)^n G$ ($R_{(\rho\bar\rho)^n\rho} : \mathbb{Z}\Gamma_{\text{odd}} \to \mathbb{Z}\Gamma_{\text{even}}$). Let $N' \cap M_{2n+1} = \bigoplus_{k\in K_n} A_k^{(2n+1)}$, $A_k^{(2n+1)} \cong M_{a_k^{(2n+1)}}(\mathbb{C})$, and $N' \cap M_{2n} \cong \bigoplus_{l\in L_n} A_l^{(2n)}$, $A_l^{(2n)} \cong M_{a_l^{(2n)}}(\mathbb{C})$. Denote the irreducible N-N (resp. N-M) bimodules indexed by $K_n \subset \Gamma_{\text{even}}$ (resp. $L_n \subset \Gamma_{\text{odd}}$) (see the discussion after Proposition 3.2 for the notation) by α_k, $k \in K_n$ (resp. β_l, $l \in L_n$). Then Proposition 3.1 and Lemma 3.5 imply

(a) $\left(\rho\bar\rho\right)^{n+1} \cong \bigoplus_{k\in K_n} a_k^{(2n+1)} \alpha_k$ as N-N bimodules, $n \geq 0$.

(b) $\left(\rho\bar\rho\right)^{n} \rho \cong \bigoplus_{l\in L_n} a_l^{(2n)} \beta_l$ as N-M bimodules, $n \geq 0$.

The right regular representation yields the following matrix equations:

(1) $\left(GG^t\right)^{n+1} = \sum_{k\in K_n} a_k^{(2n+1)} R_{\alpha_k}$, $n \geq 0$ (all matrices are "$\Gamma_{\text{even}} \times \Gamma_{\text{even}}$" matrices).

(2) $\left(GG^t\right)^{n} G = \sum_{l\in L_n} a_l^{(2n)} R_{\beta_l}$, $n \geq 0$ (all matrices are "$\Gamma_{\text{even}} \times \Gamma_{\text{odd}}$" matrices).

The matrices appearing in the equations (1.1) and (1.2) have nonnegative integer entries and the matrix R_{α_*} (corresponding to the distinguished vertex $*$) is the identity matrix. Note also, that if $N \subset M$ is irreducible, the irreducible N-M bimodule indexed by $l_0 \in L_0 = \{l_0\}$ is precisely $\rho = {}_N L^2(M)_M$, so that the correponding matrix $R_{\beta_{l_0}}$ is given by G. Furthermore, if the involution on $\mathbb{Z}\Gamma_{\text{even}}$ is trivial (this happens for instance if the indices of the N-N bimodules $\alpha \in \Gamma_{\text{even}}$, i.e., the Jones indices of the corresponding reduced subfactors, are distinct, since the indices of α and $\bar\alpha$ coincide), then all matrices R_{α_k} are symmetric. Furthermore, the dimension vector $\vec{a}^{(2n+1)}$ for the dimensions of the simple summand of

$N' \cap M_{2n+1}$ can be calculated as $\vec{a}^{(2n+1)} = \left(GG^t\right)^{n+1}(*)$, $n \geq 0$, and the one for $N' \cap M_{2n+2}$ is obtained as $\vec{a}^{(2n+2)} = G^t \vec{a}^{(2n+1)}$. In many situations, for instance when the principal graph has at most triple points, this information is enough to solve the matrix equations (1) and (2). In particular, a solution of (1) gives the right regular representation of the fusion algebra $\mathbb{Z}\Gamma_{\text{even}}$. Note that one obtains a similar set of matrix equations for the other principal graph Γ' with associated matrix G' (replacing $N' \cap M_k$ by $M' \cap M_k$ etc.). If we can solve both sets of matrix equations, the full fusion algebra (Definition 3.13) associated to $N \subset M$ can be determined simply by multiplying matrices (i.e., we compute the fusion algebra in its right regular representation). The fusion calculations in (Bisch [1994(b)]) were carried out using this procedure. Let us also point out that we can read off the principal graphs of the reduced subfactors from the matrices R_{α_k} resp. R_{β_l}. Namely, the matrices associated to these principal graphs are precisely the matrices R_{α_k} resp. R_{β_l} restricted to the vertices appearing in the decomposition of $\left(\alpha_k\bar{\alpha}_k\right)^n$, $\left(\alpha_k\bar{\alpha}_k\right)^n\alpha_k$, $n \in \mathbb{N}$, and similarly for β_l (by Proposition 5.1.3 and the discussion afterwards). Let us illustrate this algorithm with the example of a subfactor with principal graph E_6. (See Figure 1)

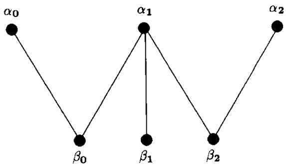

Figure 1 The principal graph E_6

We let $\Gamma_{\text{even}} = \{\alpha_0, \alpha_1, \alpha_2\}$, $\alpha_0 = *$, $\Gamma_{\text{odd}} = \{\beta_0, \beta_1, \beta_2\}$ and $G = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Since the principal graph is finite, we will have to solve only finitely many matrix equations. Note that a subfactor $N \subset M$ with principal graph E_6 has depth 4 (the depth is the longest distance of a vertex in the principal graph to the distinguished vertex $*$) and Jones index $[M : N] = 4\cos^2\frac{\pi}{12}$ (Goodman et al. [1989]). Recall that there are two non-isomorphic hyperfinite subfactors with this principal graph and that both principal graphs are given by E_6. The simple summands of $N' \cap M$ are indexed by β_0 or $L_0 = \{0\}$, those of $N' \cap M_2$ by β_0, β_1, β_2 or $L_1 = \{0, 1, 2\}$ and those of $N' \cap M_{2n}$ by β_0, β_1, β_2 or $L = L_n = L_1$, $n \geq 1$. Similarly, the simple summand of $N' \cap N = \mathbb{C}$ is indexed by α_0 or $K_{-1} = \{0\}$, the simple summands of $N' \cap M_1$ are indexed by α_0, α_1 or $K_0 = \{0, 1\}$ and those of $N' \cap M_{2n+1}$ are indexed by α_0, α_1, α_2 or $K = K_n = K_1 = \{0, 1, 2\}$, $n \geq 1$. The dimension vectors $\vec{a}^{(n)}$ for $N' \cap M_n$ are $\vec{a}^{(1)} = GG^t(*) = (1, 1, 0)$, $\vec{a}^{(2)} = G^t\vec{a}^{(1)} = (2, 1, 1)$, $\vec{a}^{(3)} = G\vec{a}^{(2)} = (2, 4, 1)$ and $\vec{a}^{(4)} = G^t\vec{a}^{(3)} = (6, 4, 5)$. Since $N \subset M$ has finite depth, the matrix equations (1) and (2) reduce to the following set of four equations:

(i) $GG^t = R_{\alpha_0} + R_{\alpha_1}$.

(ii) $\left(GG^t\right)^2 = 2R_{\alpha_0} + 4R_{\alpha_1} + R_{\alpha_2}$.

(iii) $(GG^t)G = 2R_{\beta_0} + R_{\beta_1} + R_{\beta_2}$.

(iv) $(GG^t)^2 G = 6R_{\beta_0} + 4R_{\beta_1} + 5R_{\beta_2}$.

Since $R_{\alpha_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we get $R_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and hence $R_{\alpha_2} = (GG^t)^2 - 2R_{\alpha_0} - 4R_{\alpha_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Since $N \subset M$ is irreducible we have $R_{\beta_0} = G$ as remarked above. It is then easy to solve the remaining equations and we obtain $R_{\beta_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $R_{\beta_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. This yields the following fusion rules (we write $\alpha_0 = 1$):

E_6			
	1	α_1	α_2
1	1	α_1	α_2
α_1	α_1	$1 + 2\alpha_1 + \alpha_2$	α_1
α_2	α_2	α_1	1

For instance we have $R^2_{\alpha_1} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 2 & 1 \end{pmatrix} = R_{\alpha_0} + 2R_{\alpha_1} + R_{\alpha_2} = R_{\alpha_0+2\alpha_1+\alpha_2}$.

The decomposition of $R^2_{\alpha_1}$ as a linear combination of R_{α_i}'s can be read off the first row of the matrix $R^2_{\alpha_1}$. In the same way we calculate the remaining fusion rules.

E_6			
	β_0	β_1	β_2
α_1	$\beta_0 + \beta_1 + \beta_2$	$\beta_0 + \beta_2$	$\beta_0 + \beta_1 + \beta_2$
α_2	β_2	β_1	β_0

E_6			
	β_0	β_1	β_2
β_0	$1 + \alpha_1$	α_1	$\alpha_1 + \alpha_2$
β_1	α_1	$1 + \alpha_2$	α_1
β_2	$\alpha_1 + \alpha_2$	α_1	$1 + \alpha_1$

Note that the involution on the N-N part (and M-M part) of the fusion algebra is trivial. Since $\Gamma = \Gamma' = E_6$, the above tables actually determine the full fusion algebra associated to $N \subset M$. Furthermore, as explained in Section 5.1, we can read off the principal graphs of the reduced subfactors easily. Before we do this, let us briefly work out the local indices. Recall that if $N \subset M$ is an extremal inclusion of II_1 factors with finite index, principal graph Γ and associated standard matrix G, and if we denote by $\vec{s} = (s_k)_{k\in K}$ and $\vec{t} = (t_l)_{l\in L}$ the normalized trace vectors on the higher relative commutants $N' \cap M_{2n+1}$ resp. $N' \cap M_{2n}$ (normalized such that $s_* = 1$), then $GG^t\vec{s} = [M : N]\vec{s}$, $G^t\vec{s} = [M : N]\vec{t}$, $G\vec{t} = \vec{s}$, $G^tG\vec{t} = [M : N]\vec{t}$.

If we let $\vec{\xi} = (\vec{s}, [M:N]^{\frac{1}{2}}\vec{t})$, then $\Delta_\Gamma \vec{\xi} = [M:N]^{\frac{1}{2}}\vec{\xi}$ and $\vec{\xi}$ is the normalized vector whose entries are the square roots of local indices, i.e., the indices of the reduced subfactors associated to minimal projections in the higher relative commutants (see for instance Popa [1994]). Let us now come back to the E_6 subfactor discussed above. If we let $t = [M:N] = \|G\|^2 = 4\cos^2\frac{\pi}{12}$, then $\vec{s} = (1, t-1, t^2-4t+2)$, $\vec{t} = (1, \frac{t-1}{t}, \frac{t^2-3t+1}{t})$ and $\vec{\xi} = (1, t-1, t^2-4t+2, \sqrt{t}, \frac{t-1}{\sqrt{t}}, \frac{t^2-3t+1}{\sqrt{t}})$, where the entries of $\vec{\xi}$ are in the order α_0, α_1, α_2, β_0, β_1, β_2. Note that $t^3-5t^2+5t-1=0$ (t is a zero of the characteristic polynomial of GG^t). Thus the local indices are the squares of the entries of $\vec{\xi}$, which are given by $\vec{ind} = \left(1, (t-1)^2 \approx 7.4641016\ldots, 1, 4\cos^2\frac{\pi}{12}, 2, 4\cos^2\frac{\pi}{12}\right)$. Let us now determine the principal graphs of the reduced subfactors with these indices. Choose a minimal projection p_i (resp. q_j) in the i-th (resp. j-th) simple summand of $N' \cap M_3$ (resp. $N' \cap M_2$) and consider the reduced subfactors $Np_i \subset p_i M_3 p_i$, $i \in K = \{0,1,2\}$ (resp. $Nq_j \subset q_j M_2 q_j$, $j \in L = \{0,1,2\}$). Note that all other reduced subfactors (associated to minimal projections) are isomorphic to one of these. Their indices are given by the vector $\vec{ind}$ above, where the first three entries are the indices of $Np_i \subset p_i M_3 p_i$, $i = 0, 1, 2$, and the last three those of $Nq_j \subset q_j M_2 q_j$, $j = 0, 1, 2$. The index of $Np_0 \subset p_0 M_3 p_0$ is one, so we are done (this also follows from the fact that R_{α_0} is the identity matrix). The even vertices of $Np_1 \subset p_1 M_3 p_1$ are computed by decomposing the tensor powers $\left(\alpha_1\bar{\alpha}_1\right)^k$, $k \in \mathbb{N}$, into irreducible N-N bimodules and the odd ones by doing the same to $\left(\alpha_1\bar{\alpha}_1\right)^k\alpha_1$, $k \in \mathbb{N}$ (note that $\bar{\alpha}_1 = \alpha_1$). We read from the first fusion table, that even and odd vertices of this reduced subfactor are therefore identified with α_0, α_1 and α_2. The number of edges between α_i and α_j are given by $\langle \alpha_i\alpha_1, \alpha_j \rangle$ (Section 5.1), so that the matrix associated to the principal graph is precisely R_{α_1} (both principal graphs actually coincide). Thus the principal graph is given by

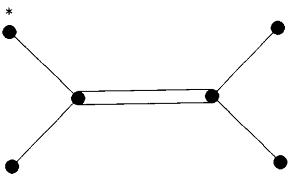

Figure 2

The same analysis for $Np_2 \subset p_2 M_3 p_2$ shows that the even vertices, which are calculated by decomposing $\left(\alpha_2\bar{\alpha}_2\right)^k = \alpha_2^{2k} = \alpha_0$ into irreducibles, are given by α_0, and similarly, we get that the odd ones are given by just α_2. The number of edges are $\langle \alpha_0\alpha_2, \alpha_2 \rangle = 1$, so that we have again an index 1 subfactor, confirming our index calculation above. The reduced subfactor $Nq_0 \subset q_0 M q_0$ is given by the bimodule β_0, which is equal to $\rho = {}_N L^2(M)_M$, as remarked above, so that we just get another copy of $N \subset M$. Let us consider the reduced subfactor $Nq_1 \subset q_1 M_2 q_1$. The even vertices are calculated by decomposing $\left(\beta_1\bar{\beta}_1\right)^k$ into irreducibles, which yields (using the second and third table) the labels α_0 and α_1. Similarly, decomposing $(\beta_1\bar{\beta}_1)^k\beta_1$ yields β_1 and the edges are calculated as $\langle\alpha_0\beta_1,\beta_1\rangle = 1$, $\langle\alpha_1\beta_1,\beta_1\rangle = 1$. Thus the principal graph is given by

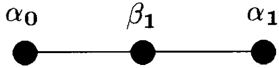

Figure 3

which confirms again our index calculation (we have an index 2 subfactor). Finally, we consider $Nq_2 \subset q_2Mq_2$. Decomposing tensor powers of β_2 and $\bar{\beta}_2$ into irreducibles results in the even vertices α_0, α_1 and α_2 and the odd vertices β_0, β_1 and β_2. Calculating the edges, we get $\langle\alpha_0\beta_2,\beta_0\rangle = 0$, $\langle\alpha_0\beta_2,\beta_1\rangle = 0$, $\langle\alpha_0\beta_2,\beta_2\rangle = 1$, $\langle\alpha_1\beta_2,\beta_2\rangle = 1$, $\langle\alpha_1\beta_2,\beta_1\rangle = 1$, $\langle\alpha_1\beta_2,\beta_0\rangle = 1$, $\langle\alpha_2\beta_2,\beta_0\rangle = 1$, $\langle\alpha_2\beta_2,\beta_1\rangle = 0$, $\langle\alpha_2\beta_2,\beta_2\rangle = 0$, from which it follows that the principal graph is given by Figure 1 (thus the other principal graph is given by the same figure). The fusion algebras of the reduced subfactors can be read off the above tables.

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