Konrad Waldorf.

Classical Gauge Theory and Matter Fields II.

No elementary particle can be described in the framework of the previous lecture. Some non-elementary particles, like $\pi^-$-meson, can be described in these terms.

Dirac's starting point was that Schrödinger's equation is first order, whereas Klein-Gordon's equation $(\Delta^\omega + m^2)\phi = 0$ is second order.

We consider Clifford algebra $\mathrm{Cl}(p,q)$ of $\mathbf{R}^{p,q}$. This algebra has two involutive anti-automorphisms: $\alpha$ is induced by $v \mapsto -v$ on $\mathbf{R}^{p,q}$ and $\cdot^t\colon v_1 \otimes \cdots \otimes v_r \to v_r \otimes \cdots \otimes v_1$. The Clifford algebra has a real valued metric $H(v,w) = (v^t w)_0$.

Spin. We have $\mathrm{SO}(p,q)$ and $\mathrm{Spin}(p,q) = \{v_1 \cdots v_{2r} \mid v_i \in \mathbf{R}^{p,q} \wedge \|v_i\| = 1\}$. $\Lambda\colon \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$ is given by $\Lambda(\phi)(v) = \alpha(\phi)v\phi^{-1}$. We have an exact sequence $1 \to \mathbf{Z}/(2) \to \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q) \to 1$. $\mathrm{Cl}(p,q) \otimes_{\mathbf{R}} \mathbf{C}$ decomposes into $k$ copies of a subrepresentation $\Sigma$. If $p+q$ is odd, then $\Sigma$ is irreducible, $k = 2^{(p+q-1)/2}$, if $p+q$ is even, then $k = 2^{(p+q)/2}$ and $\Sigma$ decomposes into sum of two irreducible representations $\Sigma^+ \oplus \Sigma^-$. For example, if $p+q = 4$, then $\dim \Sigma^\pm = 2$.

Spin structure. Suppose $M$ is a spacetime with signature $(p,q)$. $FM$ (frame bundle of $M$) is an $\mathrm{SO}(p,q)$-bundle. A spin structure on $M$ is a $\mathrm{Spin}(p,q)$-bundle $SM$ with $\lambda\colon SM \to FM$ such that $\lambda(X\phi) = \lambda(X)\Lambda(\phi)$. If $\theta \in \Omega^1(FM, \mathrm{so}(p,q))$ is the Levi-Civita connection on $FM$, then $\Theta = d\Lambda^{-1}(\lambda^*\theta) \in \Omega^1(SM, \mathrm{spin}(p,q))$.

Dirac operator. Let $\rho\colon \mathrm{Spin}(p,q) \to \mathrm{GL}(V)$ be a representation of Spin-group on a subrepresentation $V \subset \mathbf{C}l(p,q)$ of $\mathbf{C}l(p,q)$. The spinor bundle $\Sigma M$ is defined as $SM \times_\rho V$. We have Clifford multiplication $TM \otimes \Sigma M \to \Sigma M$. Now we can define the Dirac operator $D\colon \Gamma(M, \Sigma M) \to \Gamma(M, \Sigma M)$: $\psi \mapsto \sum_i e_i \cdot D^\Theta \psi(e_i)$.

Spinors. A spinor is a field for $SM$ of matter type $(V, h, \rho, f)$ where $V$ is the same as above, $\rho$ is the restruction of Clifford multiplication to $\mathrm{Spin}(p,q)$, $h(v,w) = (H(v,w) + H(w,v))/2$. Fields are sections of $\Sigma M$ as usual, but we have a different action functional $S(\psi) = \int_M D\psi \wedge_h *\psi + *(f\psi)$.

Example. (Weyl spinor.) $f = 0$. $V = \Sigma^\pm$. $\dim_{\mathbf{C}} V = 2$. In the standard model we have "neutrinos".

Example. (Dirac spinor.) $V = \Sigma$, $f(v) = mh(v,v)/2$. $\dim_{\mathbf{C}} V = 4$. Standard model: "electrons". Euler-Lagrange equations: $D\psi + im\psi = 0$. (Dirac equation.) Particles that satisfy Dirac equation also satisfy Klein-Gordon equation.