Konrad Waldorf.

Classical Gauge Theory and Matter Fields.

References:

G. Naber: Topology, Geometry, and Gauge Fields.

D. Bleecker: Gauge Theory and Variational Principles.

Notation: M is a spacetime (oriented Lorentz manifold).

Definition. A Yang-Mills theory over M consists of a Lie group G (gauge group), Ad-invariant bilinear form $\kappa\colon g\otimes g \to \mathbf{R}$ on the Lie algebra g of G, principal G-bundle P over M, fields are connections $\omega \in \Omega^1(P, g)$, action functional is $S_{\mathrm{YM}}(\omega) = 2^{-1}\int_M \|F_\omega\|^2_\kappa$.

Yang-Mills equation: $D^\omega * F_\omega = 0$ and $D^\omega F_\omega = 0$.

Definition: A gauge transformation is a bundle automorphism $g\colon P \to P$. Remark: We can identify gauge transformations with G-equivariant map $\tilde{g}\colon P \to G$. We have $r_{\tilde{g}} = g$. Equivariance means $\tilde{g}(p\gamma) = \gamma^{-1}\tilde{g}(p)\gamma$. Remark: If P is a principal G-bundle, $\rho\colon G \to \mathrm{GL}(V)$, then we have a bijection between $\Omega^k_\rho(P, V)$ and $\Omega^k(M, P \times_\rho V)$. We have $\psi \in \Omega^k_\rho(P, V)$ if $r^*_\gamma\psi = \rho(\gamma^{-1})(\psi)$.

Theorem: S_{YM} is gauge-invariant: $S_{\mathrm{YM}}(\omega) = S_{\mathrm{YM}}(g^*\omega)$.

Definition: A classical electromagnetic field theory is a Yang-Mills theory with $G = U(1)$. If we remove a line from 3-dimensional space, then we observe Aharonov-Bohm effect. If we remove a point, then we observe magnetic monopoles.

Definition: If G is a gauge group, then a matter type for G is a quadruple $T = (V, h, \rho, f)$ such that V is a finite-dimensional real vector space (internal state space), $h\colon V \otimes V \to \mathbf{R}$ is a real bilinear form on V, $\rho\colon G \to \mathrm{GL}(V)$ is an isometric representation of G: $h(\rho(g)v, \rho(g)w) = h(v, w)$ (transformation behavior), $f\colon V \to \mathbf{R}$ is a smooth ρ-invariant function: $f(\rho(g)v) = f(v)$ (self-interaction potential).

Definition: Let (P, G) be a Yang-Mills theory over M, T a matter type for G. A field for P of type T is a section $\phi\colon M \to P \times_\rho V$. The action functional is $S_T(\omega, \phi) = 2^{-1}\int_M \|D^\omega\phi\|^2_h + *(f(\phi))$.

Remark: We have two options: (a) Keep ω fixed, consider $S_\omega(\phi) = S_T(\omega, \phi)$. (b) Consider $S(\omega, \phi) = S_{\mathrm{YM}}(\omega) + S_T(\omega, \phi)$. Notice that (ω, ϕ) extremizes S iff (1) ϕ extremizes S_ω (Euler-Lagrange); (2) $\delta^\omega F_\omega = J^\omega(\phi)$ (inhomegeneous field equation). Here $\delta^\omega = *D^\omega *$.

Example 1: $G = \{e\}$, $P = M$. A scalar field is a field of matter type $(\mathbf{R}, h, \mathrm{id}, f)$ with $f(x) = -m^2x^2$. $S(\phi) = \int_M \|d\phi\| + m^2\phi^2$. Euler-Lagrange equation in this case is Klein-Gordon equation $(\Delta^2 + m^2)\phi = 0$.

Example 2: π^--meson: charge $n \in \mathbf{Z}$, no spin. Yang-Mills theory $(P, \mathrm{U}(1))$. Matter type $(\mathbf{C}, h, \rho_n, f)$. Here ρ_n is the nth power represetation of $\mathrm{U}(1)$ on $\mathbf{C}$, $f(z) = -m^2|z|^2$ and $h(z_1, z_2)$ is left as an exercise. Euler-Lagrange equation is $(\Delta^\omega + m^2)\phi = 0$. Here $\Delta^\omega = \delta^\omega D^\omega$.