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Electromagnetism to Yang-Mills theory.

Premotivation: Yang-Mills theory is about connecting (mediating) particles (photon, gluon, etc.). Fields are connections on principal G-bundles, where G is the gauge group of a particle under consideration. For electromagnetism we have $G = \mathrm{U}(1)$. Old F is the curvature of this connection.

Bohm-Aharanov effect: We have a long solenoid with magnetic field inside and no electric field outside. However we can see an interference pattern for electrons. This comes from a term $\exp(-i\hbar^{-1}q\int_\gamma A)$ in path integral, where A is the vector potential. Since the integral around solenoid is nonzero, we have interference.

So locally we get a function that assigns phases to paths in M. This is just like a connection on a trivialized bundle: $\gamma \mapsto \exp(i\int_\gamma A)$. Problems: (a) $A + df$ gives different function; (b) We cannot expect a canonical trivialization.

We have gauge symmetry for connections. We can construct a new connection as follows: $D'(\gamma) = g(\gamma(t))D(\gamma)g^{-1}(\gamma(0))$. Here $\gamma\colon [0,t] \to M$ is a path and $g\colon M \to U(1)$ is a gauge transformation. We have $D'(\gamma) = \exp(i\int_\gamma(A + df))$, where $g = \exp(if)$.

Notice that holonomy is gauge invariant (a) because $\int_\gamma df = 0$ by Stokes; (b) if $\gamma(t) = \gamma(0)$, then $D'(\gamma) = D(\gamma)$ because $\mathrm{U}(1)$ is abelian.

In fact, holonomy uniquely characterizes connections up to gauge symmetry.

F is a unique 2-form such that for a disc $B \to M$ we have $D(\partial B) = \exp(i\int_B F)$.

We have exact sequences $0 \to H^1(\mathrm{U}(1)) \to R \to \Omega^2_{\mathbf{Z}} \to 0$. $0 \to \Omega^1_{\mathbf{Z}} \to \Omega^1 \to R \to H^2(\mathbf{Z}) \to 0$. We have maps $H^1(\mathrm{U}(1)) \to H^2(\mathbf{Z})$ and $d\colon \Omega^1 \to \Omega^2_{\mathbf{Z}}$.

References: Baez-Munian and Freed's paper on differential cohomology and Maxwell's equations.