Peter Teichner.

Hamiltonian approach to classical field theory.

Data for a classical Lagrangian field theory: (1) Smooth (oriented) manifold $M$ (spacetime); (2) A smooth fiber bundle $E \to M$ (fields on $M$ are $\Gamma(M, E)$); (3) Lagrangian density $\lambda \in \Omega^{n,0}(JE)$.

Note: Get an action $S\colon \Phi \to \mathbf{R}$: $M \times \Phi \to JE$ is the jet map. We have $\Omega^{n,0}(JE) \supset \Omega^n(JE) \to \Omega^n(M \times \Phi) \to C^\infty(\Phi) \ni S$.

Goal for today: Filtration on the space of jets, Euler-Lagrange equations, derive a Hamiltonian field theory picture.

Example: (Riemannian sigma-model.) $M$ and $N$ are Riemannian, $E = M \times N \to M$, $\Phi = C^\infty(M, N)$, $\lambda(\phi) = \|T\phi(x)\|^2 d\mathrm{vol}_M$.

Definition: $J^\infty E = \{(x, \phi) \mid x \in M \wedge \phi \text{ is a local section of } E \text{ near } x\}/\sim$. Here $(x, \phi_1) \sim (x, \phi_2)$ iff $\phi_1$ has $k$-contact with $\phi_2$ at $x$ for all $k$.

Lemma: In coordinates this means that derivatives of $\phi_i$ coincide at point $x$.

So we have a sequence of bundles $M \leftarrow J^0E = E \leftarrow J^1E = TE \leftarrow J^2E \leftarrow \cdots$. $JE = \lim_k J^kE$.

Define $C^\infty(JE) := \operatorname{colim} C^\infty(J^kE)$ and $\Omega^*(JE) := \operatorname{colim} \Omega^*(J^kE)$.

For any bundle $J \to M$ we have sub-dga of $\Omega^*(J)$ constructed as follows. $\Omega^*_H(J) = C^\infty(J) \otimes_{C^\infty M} \pi^*\Omega^*M$. This leads to the Serre spectral sequence for de Rham cohomology. Key geometric fact about JE: it is the differential ideal of "contact form". Jet bundle has a canonical flat connection: $\Omega^1(JE) = \Omega_H(JE) \oplus C^1(JE)$. Here $C^*(JE) = \{\omega \in \Omega^*JE \mid j^*(\omega)(\phi_u) = 0 \text{ for all local sections } \phi_u\}$. Note that this works only for infinite jets.