From whence do differential forms come?

Reference:

Moerdijk & Reyes: Models for smooth infinitesimal analysis
1991

Lavendhomme (1996) Basic Concepts of synthetic differential geometry

Kock: 1) Synthetic differential geometry (1981, 2006)
2) Synthetic geometry of manifolds (2009)

From whence do differential forms come?

Recall the definition of singular cohomology $\mathbb{R}$

M smooth manifold

$\cancel{C_n = \hom(\Delta^n, M)}$

$$C_n = \mathbb{R}^{\hom_{Top}(\Delta^n, M)}$$

$$C_n \to C_{n+1}$$

$$c \longmapsto \Big(f : \Delta^{n+1} \to M \longmapsto \sum_{0 \le i \le n+1} c(f \circ d^i)\Big)$$

$d^i : \Delta^n \to \Delta^{n+1}$ ith face.

$H_n(C) :=$ the singular cohomology w/ coefficients in $\mathbb{R}$

C_n is <u>huge</u>. Canonical example:

~~Possible ways~~ integration: $f : \Delta^n \to M \longmapsto \int_{\Delta^n} G \circ f$

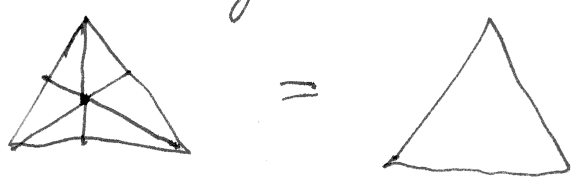
$$M \xrightarrow{G} \mathbb{R}$$

Possible ways to cut down on size of C_n:

1) Require maps $\Delta^n \longrightarrow M$ to be smooth (not just continuous)

2) Require maps $\hom_{C^\infty}(\Delta^n \to M) \longrightarrow \mathbb{R}$ to be smooth, not arbitrary

3) Require additivity under barycentric subdivision



Theorem (P., Stolz, Teichner)

Under (1)–(3), there is an <u>isomorphism</u> (<u>not</u> just quasi-iso)

$$C \longrightarrow \Omega.$$

—

Explanation: (1), (2) enable to construct a differential form by differentiation

(3): the value on a large simplex can be recovered from values on arbitrarily small simplices

<u>Thus</u>, we can cut down even further by passing to a <u>neighborhood</u> of the diagonal

$$\cancel{\Delta} \qquad \cancel{M^{n+1} \to}$$

$$M \longrightarrow \hom_{C^\infty}(\Delta^n, M)$$

$$m \longmapsto (p \mapsto m).$$

Which neighborhood can we take?

1) germ
2) formal: (meaning we remember <u>all</u> derivatives at $M \hookrightarrow \hom_{C^\infty}(\Delta^n, M)$).
3) first-order: (meaning we remember the first derivative at $M \hookrightarrow \hom_{C^\infty}(\Delta^n, M)$).

What is the geometric meaning of (3)?

e.g. $M = \mathbb{R}^n$ or $M = \mathbb{R}$

A simplex $\sigma: \Delta^n \to M$ "belongs" to (3)
if $\forall i,j$ $0 \le i \le j \le n$: $\sigma(i) - \sigma(j)$
are infinitesimally close to first order,
i.e., $(\sigma(i) - \sigma(j))^2 = 0$.

Problem: ~~there are no real numbers r such that r~~
If $r^2 = 0$, then $r = 0$.

Resolution: allow rings of functions with
nilpotent elements.

Good notion of ring for smooth/differential geometry
C^∞-ring

Def ordinary commutative $\mathbb{R}$-algebra A:
for any real polynomial p in variables
we have an operation $A^n \xrightarrow{p} A$
associativity:
$$\begin{tikzcd} A^{m_1+\cdots+m_k} \arrow[r, "q_1, \dots, q_k"] \arrow[rr, bend right, "p(q_1, \dots, q_k)"'] & A^k \arrow[r, "p"] & A \end{tikzcd}$$

unitality: $A^1 \xrightarrow{p=x} A = \mathrm{id}_A$

Def C^∞-ring : replace polynomials by smooth functions. 4

In particular, C^∞-rings are comm $\mathbb{R}$-alg.

Example: $C^\infty(M)$ smooth manifold M.

Example: A fin dim local $\mathbb{R}$-alg.

$\forall a \in A \ \exists!\ u, n$ $\quad u \in \mathbb{R}$, n nilpotent

$a = u + n$

$$f(u_1 + n_1, \ldots, u_k + n_k) = \sum \partial_{\vec{k}} f(\vec{u}) \cdot \vec{n}^{\vec{k}}$$

finite sum.

Manifolds

$$\begin{tikzcd} \downarrow \\ C^\infty\text{-loci} \arrow[r, phantom, ":="] \arrow[r, bend right, "C^\infty"'] & (C^\infty\text{-ring})^{op} \arrow[l, bend right, "\mathrm{Spec}"'] \end{tikzcd}$$

Def

Interpret these geometrically.

Example: 1) $\mathrm{Spec}(\mathbb{R}) = pt$ $\quad pt \xrightarrow{a} M \leftrightarrow C^\infty M \xrightarrow{ev_a} \mathbb{R}$

$\mathrm{Spec}(\mathbb{R}^n) = n \times pt$

2) $\mathrm{Spec}(\mathbb{R}[\varepsilon]/(\varepsilon^2))$

$= \mathrm{Spec}(\mathbb{R} + \varepsilon \cdot \mathbb{R}) = L$

What is L?

$\mathbb{R}[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon = 0} \mathbb{R}$

$pt \longrightarrow L$

What is a map $L \to M$, $M \in$ Manfold?

$$C^\infty(M) \xrightarrow[\gamma]{} \mathbb{R}[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon=0} \mathbb{R}$$

$$f \longmapsto \alpha(f) + \beta(f)\cdot\varepsilon$$

α is a homomorphism of algebras $C^\infty(M) \xrightarrow[ev_a]{} \mathbb{R}$

$$\gamma(fg) = \alpha(fg) + \beta(fg)\cdot\varepsilon$$

$$\parallel$$

$$\gamma(f)\gamma(g) = (\alpha(f) + \beta(f)\cdot\varepsilon)\,(\alpha(g) + \beta(g)\cdot\varepsilon)$$

$$= \alpha(f)\alpha(g) + (\beta(f)\cdot\alpha(g) + \alpha(f)\beta(g))\cdot\varepsilon$$

$$\beta(fg) = \beta(f)\,\alpha(g) + \alpha(f)\,\beta(g).$$

i.e., $\beta : C^\infty(M) \to \mathbb{R}$

is a derivation with respect to α.

Recall: Want to define

$$IS_n \xrightarrow{\cancel{a}} \hom_{C^\infty}(\Delta^n, M)$$

$\cap$ C^∞ Locus

consisting of infinitesimal simplices

Easiest way to define IS_n

is to define maps $S \longrightarrow IS_n$

$\forall$ C^∞-locus S.

Infinitesimal simplices

Def $M \in$ Manifold, $S \in C^\infty$ Loc

An S-indexed family of infinitesimal n-simplices is a collection of $(n+1)$-morphisms

$$\alpha_i : S \to M \quad ; \text{ i.e., homomorphisms}$$

$0 \le i \le n$, $h: M \to \mathbb{R}$, of $\mathbb{R}$-algebras, i.e.,

$$C^\infty(M) \xrightarrow[C^\infty(\alpha_i)]{} C^\infty(S),$$

such that $\forall h: M \to \mathbb{R}$ $\quad C^\infty(\mathbb{R}) \xrightarrow{C^\infty h} C^\infty(M)$

the compositions $C^\infty \mathbb{R} \xrightarrow{C^\infty h} C^\infty M \xrightarrow{C^\infty \alpha_i} C^\infty S$

give elements $\beta_i \in C^\infty S$

such that $(\beta_i - \beta_j)^2 = 0. \quad \forall i, j.$

Def The <u>de Rham complex</u> of a smooth manifold M is the ~~a~~ infinitesimal smooth singular cochain complex of M.

$$\Omega^n M := C^\infty(IS_n(M), \mathbb{R})$$

that vanishes on $DIS_n(M) \subset IS_n(M)$ where two consecutive points coincide.

$$\Omega^n M \xrightarrow{d} \Omega^{n+1} M \quad \omega \mapsto \sum_{0 \le i \le n+1} (-1)^i \omega(\alpha_0, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)$$

~~induced by~~

$$\Delta^n \xrightarrow{C^\infty} M$$
$$\hom_{C^\infty}(\Delta^n, M) \xrightarrow{C^\infty} \mathbb{R}$$

<u>Theorem</u>

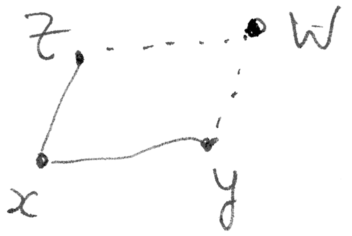
$$\begin{tikzcd} C^{\infty *}(M) \arrow[d, shift left, "\text{restrict to infinitesimal simplices}"] \\ \Omega^*(M) \arrow[u, shift left, "\int"] \end{tikzcd}$$

is a quasi-isomorphism

The image of $\int$ is precisely additive cochains

<u>Other examples</u>

Connection on TM



$\lambda x, y, z \overset{\lambda}{\mapsto} w$

torsion-free

$\lambda(x, y, z) = \lambda(x, z, y)$.

$\lambda(x, y, z) = y - x + z$

torsion: $b_x(y, z) = \lambda(\lambda(x, y, z), y, z)$

$\cancel{W} = \nabla_{y,x}(z)$

flat: $\nabla_{z,x} = \nabla_{z,y} \circ \nabla_{y,x}$