Higher Courant groupoids

A homotopical reminder: n-images I

- $f : X \to Y$ - morphism in $\mathcal{A}n$

We have a Postnikov tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \arrow[ddr, bend left, hook] & \\
 & \mathrm{im}_2(f) \arrow[d] \arrow[dr, bend left, hook] & \\
X \arrow[uur] \arrow[ur] \arrow[r, two heads] \arrow[rr, bend right] & \mathrm{im}_1(f) \arrow[r, hook] & Y
\end{tikzcd}$$

A homotopical reminder:

n-images II

- for $n \geq 1$ $X \twoheadrightarrow \mathrm{im}_n(f)$ is an <u>epi</u> on π_0 and an <u>iso</u> on π_i $(0 < i < n-1)$
- $\mathrm{im}_n(f) \hookrightarrow Y$ is a <u>mono</u> on π_{n-1} and an <u>iso</u> on π_i $(i > n-1)$
- $X \xrightarrow[]{\underline{\underline{n\text{-epi}}}} \mathrm{im}_n(f)$; $\mathrm{im}_n(f) \xhookrightarrow[\underline{\underline{n\text{-mono}}}]{} Y$

An example (2.9.2)

- Consider $\mathcal{G} = [G_1 \rightrightarrows G_0]$: Lie Gpd

$G_0 \twoheadrightarrow \mathcal{G}$ is a 1-epimorphism

$$\begin{array}{c} G_0 \\ \| \\ [G_0 \overset{id}{\underset{id}{\rightrightarrows}} G_0] \end{array}$$

$$\begin{tikzcd} G_0 \arrow[r, two heads] \arrow[rr, bend right, "f"'] & B\mathcal{G} \arrow[r] & * \end{tikzcd}$$

$B\mathcal{G} = \mathrm{im}_1(f)$

Another example
(Atiyah groupoid)

- $\mathrm{At}(\nabla) :\equiv \mathrm{im}_1(\nabla)$ where

$$\begin{tikzcd}
 & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[d] \\
X \arrow[ur, "\nabla"] \arrow[r, "\nabla_0"'] & \mathbf{B}^n U(1)
\end{tikzcd}$$

$\leftarrow$ forgetful map

- $\mathrm{At}(\nabla) \longrightarrow \mathrm{At}(\nabla^0)$ is the induced forgetful map

$BU(1)_{conn}$ (via DK)

- Consider $\left(C^\infty(-;U(1)) \xrightarrow{\frac{d\log}{2\pi i}} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n\right)$

Deligne complex $\mathcal{D}^n$

$B^{n-k}\mathcal{D}^k$

$DK(\mathcal{D}^n) = B^n U(1)_{conn}$

-

$$\begin{tikzcd}
C^\infty(-;U(1)) \arrow[r] \arrow[d, "id"] & \Omega^1 \arrow[r] \arrow[d, "id"'] & \cdots \arrow[r] & \Omega^k \arrow[r] \arrow[d, "id"'] & \Omega^{k+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^n \arrow[d] \\
C^\infty(-;U(1)) \arrow[r] & \Omega^1 \arrow[r] & \cdots \arrow[r] & \Omega^k \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

Towers of $BU(1)$

- By previous construction $\mathbb{B}^n U(1)_{conn} \to \mathbb{B}^{n-k}\mathbb{B}^k U(1)_{conn}$
- We can form a towers:

$$At(\nabla) \to At(\nabla^{n-1}) \to \dots \to At(\nabla^k) \to \dots \to At(\nabla^0)$$

$At(-)$

$$\begin{tikzcd}
 & \mathbb{B}^n U(1)_{conn} \arrow[d] \\
 & B(\mathbb{B}^{n-1} U(1)_{conn}) \arrow[d] \\
 & B^2(\mathbb{B}^{n-2} U(1)_{conn}) \arrow[d] \\
 & \vdots \arrow[d] \\
X \arrow[uuuur, "\nabla"] \arrow[uuur, "\nabla^{n-1}"] \arrow[uur, "\nabla^{n-2}"] \arrow[r, "\nabla^0"] & \mathbb{B}^n U(1)
\end{tikzcd}$$

Preamble to § 3.2.2

- Everything I will tell next is useless
- I can just define:

$$CG(\nabla^{n-1}) := At(\nabla^{n-1})$$

- What follows is my feeble attempt to motivate this

Reminders: Lie algebroids I

- Morally: $\mathcal{G}$: Lie Gpd $\rightsquigarrow$ $T\mathcal{G}$: LieAld
- V.B. w/ anchor: M: Man_0, then a <u>Lie algebroid over</u> M is:

1) $E \to X$: $\mathrm{VBun}(X)_0$

2) $[-,-] : \Gamma(E)^{\wedge 2} \longrightarrow \Gamma(E)$ - Lie bracket

3) $\exists\ \rho : E \to TX$ a morphism of Lie Alg $\Gamma(E) \xrightarrow{d\rho} \mathfrak{X}(M)$

Reminder: Lie algebroids II

A) $X, Y : \Gamma(E)$, $\forall f : C^\infty(M)$ we have

$$[X, fY] = f[X,Y] + \rho(X)(f)\cdot Y$$

CE construction:

$$[-,-] : \Gamma(E)^{\wedge 2} \to \Gamma(E) \rightsquigarrow \underbrace{\Gamma(E)^\vee}_{\text{over } C^\infty(M)} \xrightarrow[{[-,-]^\vee = d}]{} (\Gamma(E)^\vee)^{\wedge 2}$$

We get $(\Lambda^\bullet \Gamma(E)^\vee[1], d) : \mathrm{CDGA}_{\mathbb{R}}$

Higher Lie algebroids

- $L_\infty \mathrm{Ald} \hookrightarrow \mathrm{dgCAlg}_{\mathbb{R}}^{op}$

s.t.

α) $\mathcal{L} : L_\infty \mathrm{Ald}$ has $\mathcal{L}_{<0} = 0$

β) $\mathcal{L}_0 = C^\infty(M)$

γ) $\mathcal{L}$ is semifree i.e. $\mathcal{L} = \mathrm{S}(V)$ over $C^\infty(M)$

δ) $\mathrm{rk}\, V < \omega_0$

Courant Bracket

- Courant bracket: take $\Gamma(TM \oplus T^*M)$:

$$[\xi + a, \eta + b]_C = \overbrace{[\xi, \eta]}^{\text{vector field}} + \underbrace{\mathcal{L}_\xi b - \mathcal{L}_\eta b - \frac{1}{2} d\iota_\xi b + \frac{1}{2} d\iota_\eta a}_{1\text{-form}}$$

Courant Bracket II

- $[-,-]_e$ is <u>not</u> a Lie bracket

$$\mathrm{Jac}(A,B,C) = [[A,B]_e,C]_e + [[B,C]_e,A]_e + [[C,A]_e,B]_e$$

Nijenhuis tensor

some metric

- $\mathrm{Jac}(A,B,C) = d(\mathrm{Nij}(A,B,C))$

$$\mathrm{Nij}(A,B,C) = \frac{1}{3}\left(\langle [A,B]_e, C\rangle + \langle [B,C]_e, A\rangle + \langle [C,A]_e, B\rangle\right)$$

Courant algebroids

- The previous discussion shows that $S(\Gamma(TM \oplus T^*M)[1])$ has a structure of CDGA given by $[-,-]_{\mathcal{C}}$

- This structure is called <u>Courant Lie 2-algebroid</u>

Local description of Courant (2-)algebroids

Ref: Collier § 12 "Infinitesimal symmetries of Dixmier-Douady gerbes"

- Let $(\{U_i\}, \{A_{ij}\}, \{g_{ijk}\})$ be a Čech Deligne cocycle (just a cycle in $\mathcal{D}^2(M)$)

then we have $A_{jk} - A_{ik} + A_{ij} = \frac{d\log(g_{ijk})}{2\pi i}$

- Consider $\begin{pmatrix} 1 & 0 \\ -dA_{ij} & 1 \end{pmatrix} : TU_{ij} \oplus T^*U_{ij} \circlearrowleft$

Local description of Courant (2-)algebroids II

- $\begin{pmatrix} 1 & 0 \\ -dA_{ij} & 1 \end{pmatrix}$ defines an extension

$$0 \to T^*M \to E \to TM \to 0$$

$dA_{ij} \curvearrowright TU_{ij}$ by $\xi \mapsto \iota_\xi dA_{ij}$

- I.e. $(\xi, \{a_i\}) : \Gamma(E)$ such that $a_i : i\Omega^1(U_i)$ $\quad a_j - a_i = -\iota_\xi dA_{ij}$.

Local description of Courant (2-)algebroids $\boxed{II}$

- We get a Courant-type bracket given by the formula:

$$[\xi + a, \eta + b]_E = [\xi, \eta] + \underline{\mathcal{L}_\xi b - \mathcal{L}_\eta b -} \underbrace{- \tfrac{1}{2} d \iota_\xi b + \tfrac{1}{2} d \iota_\eta a}_{1\text{-form}}$$

- $dA_{ij} : \Omega^2_{cl} \Rightarrow [-,-]_E$ is well-defined

Example (3.2.6) $At(\nabla^{n-1})$

- Let $n=2$. $X : \mathcal{M}an_0$. $X \xrightarrow[\nabla^1]{} \mathbb{B}(\mathbb{B}U(1)_{conn})$

- $B : Set(At(\nabla)) = Aut_{\mathbb{H}}(\nabla)_0 =$

$\{(\phi, \eta) \mid \phi : Aut_{\mathbb{H}}(X)_0, \eta : \phi^* \nabla' \xrightarrow{\simeq} \nabla'\}$

Diffs of X

$(\phi, \eta) \sim (\phi', \eta') \iff \phi = \phi', \varkappa : \eta \xrightarrow{\simeq} \eta'$

Example (3.2.6) $At(\nabla^{n-1})$ II

- Thus $At(\nabla')(U)$ consists of smooth U-parametrized collections of diffs of X + compatible bundle gerbe transformations.

- Now by comparing this w/ the local description of Courant algebroids of Collier we see that $\pi At(\nabla') = Cour(\nabla')$

Higher Courant groupoids

Def: $\nabla^{n-1} : X \longrightarrow \mathbb{B}(\mathbb{B}^{n-1} U(1)_{conn})$

higher Courant groupoid is given by $At(\nabla^{n-1})$

i.e. $CGpd(\nabla^{n-1}) = im_1(\nabla^{n-1})$

Example:

- Assume
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{B}^n U(1) \\ & * \arrow[ur] & \end{tikzcd}$$

- $\mathbb{B}\,\mathrm{i}\,\mathrm{Set}(\mathrm{At}(\nabla^{n-1}))_0 = \{(\phi, H) \mid$
$\phi : \mathrm{Aut}_H(X),\ H : \Omega^{n-1}(X)\}$

- $\mathrm{CAld}(X) = \mathbb{B}(\Gamma(TX \oplus \Lambda^{n-1} T^* X))$

Concluding remark:

- Given a <u>sylleptic</u> (i.e. 2-fold deloopable) G in $\mathbb{H}$
- One defines $\mathbb{B}^2 G_{\mathrm{conn}} \to \mathbb{B}(\mathbb{B}G_{\mathrm{conn}}) \to \mathbb{B}^2 G$.
- We get $\nabla^{\bullet-1} : X \to \mathbb{B}(\mathbb{B}G_{\mathrm{conn}})$

which is G-p.rin.-conn. "without top deg. conn. data"

Thank you!!!