Higher quantomorphisms & Heisenberg groupoids I (§ 3.2.1)

Intro

(An alternative title pre^2-quantum geometry = Symplectic geometry = = Classical mechanics)

Def: A symplectic str-re on a manifold M is $\omega : \Omega^2_{cl}(M)$ s.t.

ω is <u>non-degenerate</u> (not presymplectic)

R/k: (immediate consequences of the def)

α) M <u>must</u> be of even dimension

β) ω defines an isomorphism
$T^*M \cong TM$

E.g. $p : M$ we have

$$\omega : T_pM \to T_p^*M$$
$$v \longmapsto \omega_p(v, -)$$

Moreover $\Gamma(M;TM) \xrightarrow{\simeq} \Gamma(M;T^*M)$

$$\mathfrak{X}(M) \xrightarrow{\simeq} \Omega^1(M)$$

v.f.

Ex:

1) $\forall$ $V : \mathrm{Vect}_{\mathbb{R}}$ s.t. $\dim V = 2n$

we have a symp. str-re

$\omega : V \times V \to \mathbb{R}$

$\{v_1, \dots v_n\}$ - basis

$$\omega(v_i, v_j) = \begin{cases} 1 & j-i=n \quad 1 \le i \le n \\ -1 & i-j=n \quad 1 \le j \le n \\ 0 & \text{otherwise} \end{cases}$$

$$\Omega = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

Choice of W is the polarization of V.

2) Given M : Mfld

T^*M has canonical symp. str-re

$\underset{\text{open}}{U \subseteq M}$ $\quad (q^1 \dots q^n, \xi_1, \dots \xi_n)$

$T^*U \cong \mathbb{R}^n$

$\mathbb{R}^n \oplus (\mathbb{R}^n)^\vee$

$(q^1 \cdots q^n, p_1, \cdots p_n)$

we can construct a ~~un~~ 1-form

the Liouville-Poincaré taut 1-fm

$$\theta = \sum_{i=1}^{n} p_i \, dq^i$$

Canonical symp. form is given by

$$\omega = d\theta = dq^i \wedge dp_i$$

We should further examine

$$\omega : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$$

Def: A v.f. $X : \mathfrak{X}(M)$ is

a) symplectic if $\iota_X \omega : \Omega^1_{cl}(M)$

b) hamiltonian if $\iota_X \omega : \Omega^1_{ex}(M)$

Observations: 1) By Cartan's magic formula $X : \mathfrak{X}_{symp}(M)$

$\mathcal{L}_X \omega = 0$, i.e. if

$F_X : \mathbb{R} \times M \longrightarrow \mathbb{R}$ is the <u>flow</u> of $X \Rightarrow F_X(t, m)^* \omega = \omega$

(2) By def. $\iota_X \omega = df$ for Hamiltonian v.f. $f : C^\infty(M)$.

Given $f : C^\infty(M)$ $\iota_X \omega = df$ determines a $X : \mathfrak{X}_{Ham}(M)$ <u>uniquely</u>.

The converse holds up to add. of $c : H^0(M; \mathbb{R})$.

We have a short exact sequence:

$$0 \to H^0(M; \mathbb{R}) \to C^\infty(M) \xrightarrow[\omega^{-1} d]{} \mathfrak{X}_{Ham}(M) \to 0$$

7)

$$\mathfrak{X}_{Ham, Symp}(M) \hookrightarrow \mathfrak{X}(M)$$

Lie Alg

$$\mathfrak{X}_{Ham} \triangleleft \mathfrak{X}_{Symp} \hookrightarrow \mathfrak{X}$$

Δ) We can define a Poisson algebra str-re on $C^\infty(M)$

Def: $\{\cdot,\cdot\}: C^\infty(M)^{\wedge 2} \to C^\infty(M)$

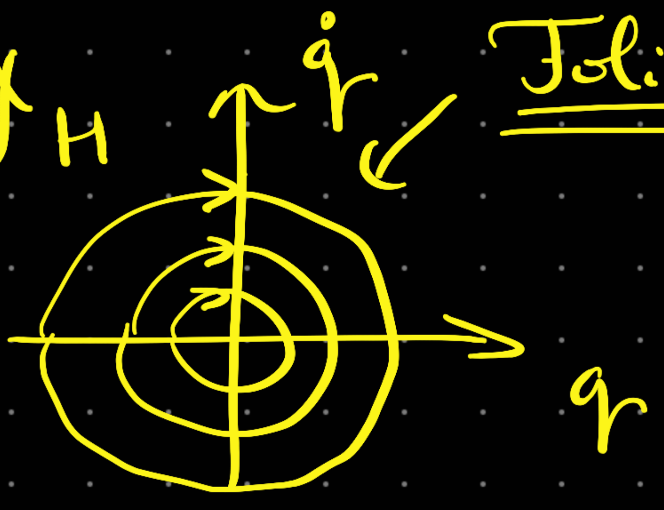
$$f,g \mapsto \omega(X_f, X_g)$$

$$= \iota_{X_g}(\iota_{X_f}\omega) = \iota_{X_g}(df) =$$

$$= df(X_g) = X_g(f) = \mathcal{L}_{X_g} f$$

Symp. geom	Physics
(M,ω)	Phase space
$H: C^\infty(M)$	Observable
$X_H: \mathfrak{X}_{Ham}(M)$	Phase sp. d-n
	Foliation

Int. curve to X_H



$\gamma: \mathbb{R} \to M \quad \gamma(0) = m$

$\gamma(t) = (q(t), p(t))$

$\dot\gamma(t) = X_H(\gamma(t))$

$$\begin{cases} \dot q = \dfrac{\partial H}{\partial p} \\ \dot p = \dfrac{\partial H}{\partial q} \end{cases}$$

Sol. to eqns of motion.

More S. G.

$f: C^\infty(\mathbb{R}\times M)$

A choice of $H : C^\infty(M)$

the Hamiltonian

$\{f, H\} = -X_H(f) = X_f(H)$

$= \dfrac{\partial f}{\partial p_i}\dfrac{\partial H}{\partial q^i} - \dfrac{\partial f}{\partial q^i}\dfrac{\partial H}{\partial p_i}$

$= \dfrac{\partial f}{\partial q^i}\dfrac{\partial p_i}{\partial t} - \dfrac{\partial f}{\partial p_i}\dfrac{d q^i}{dt} =$

$= \dfrac{d}{dt}(f).$

More Phys

Time evol/ Dynamics.

Poisson br. w/ H

S.G.	Phys
$\mathrm{Symp}(M,\omega)$	Can. coord. trans.
???	"Sym. of the system"

Naive guess:

G: Lie Grp

$G \to \mathrm{Symp}(M)$

Not the corr. notion

Q: What sym. of T^*M come from sym. of M.