Stacks & simplicial presheaves

Def: Cart the site of cartesian spaces

Ob: $\mathbb{R}^n$ $(n \geq 0)$

Mor: $\mathrm{Mor}(\mathbb{R}^n, \mathbb{R}^m) = C^\infty(\mathbb{R}^n, \mathbb{R}^m)$

Covering families = open covers

Def: $sPSh(Cart) = PSh_\Delta(Cart) :=$

$:= \mathrm{Fun}(Cart^{op}, sSet)$

Morally, if $S \in Cart$, then given $F \in PSh_\Delta(Cart)$, then

$F(S) \in sSet$ is the simpl. set of smooth maps $S \to F$

Formally, by Yoneda

$$\mathrm{Map}(yS, F) \xrightarrow{\simeq} F(S)$$

$$yS : Cart^{op} \longrightarrow sSet$$

$$T \longmapsto \underbrace{\delta C^\infty(T, S)}_{\text{discrete sSet}}$$

Examples:

α) M : Man

$\mathrm{Cart}^{op} \longrightarrow \mathrm{sSet}$

$T \longrightarrow \delta C^\infty(T, M)$

β) Ω^n : Sh(Cart)

$T \mapsto \Omega^n(T)$

If $n > 0$, then "points" of Ω^n are given by $\Omega^n(\mathbb{R}^0) = \{0\}$

Hence Ω^n has a single "point" (for $n>0$) But $\Omega^n \not\simeq *$ since $\forall M$: Man

$\mathrm{Mor}(M; \Omega^n) \cong \Omega^n(M)$

γ) G: Lie Grp

$\mathbb{B}G : \mathrm{Cart}^{op} \longrightarrow \mathrm{sSet}$

$T \mapsto BC^\infty(T; G)$

H : Grp, $BH :\equiv N(*/\!/G)$

It classifies principal H-bundles (i.e. covering spaces w/ deck transf. given by H)

$\mathbb{R}\mathrm{Map}(M, BG) \simeq N\left(\begin{matrix}\text{the groupoid} \\ \text{of prin. } G\text{-bun.} \\ \text{on } M\end{matrix}\right)$

δ) $\mathbb{B}_\nabla G : \mathrm{Cart}^{op} \longrightarrow s\mathrm{Set}$

$T \longmapsto N(\Omega^1(T;\mathfrak{g}) /\!/ C^\infty(T;G))$

$f \in C^\infty(T;G)$, $\omega \in \Omega^1(T;\mathfrak{g})$

$f \cdot \omega = f^*\theta + \mathrm{Ad}_{f^{-1}}\omega$

For an abelian G $\mathrm{Ad}_{f^{-1}} = \mathrm{Id}$

and we get $f^*\theta + \omega$

$\mathbb{B}_\nabla G(\mathbb{R}^0) = \underline{BG^\delta}$

conn. sSet.

$\mathbb{R}\mathrm{Map}(M;\mathbb{B}_\nabla G) \simeq$

$\simeq N\left\{\begin{matrix}\text{groupoid of} \\ \text{prin. } G\text{-bun.} \\ \text{w/ } \nabla \text{ \& } \\ \text{conn. preserv.} \\ \text{isos}\end{matrix}\right\}$

Def: $s\mathrm{Grp} = \mathrm{Fun}(\Delta^{op}, \mathrm{Grp}) =$

$= \mathrm{Grp}(s\mathrm{Set})$

$\swarrow$ conn.

$s\mathrm{Set}_* \simeq s\mathrm{Set}_0$

$G \longleftarrow$ Quillen equiv.

$$s\mathrm{Grp} \underset{G}{\overset{\overline{W} \leftarrow \text{"}B\text{"}}{\rightleftarrows}} s\mathrm{Set}_0$$

$G \simeq \underbrace{\mathbb{R}\mathrm{Hom}(S^1_*, -)}_{\text{Simplicial loop space}}$ ← "Ω"

$$\mathrm{TopGrp} \underset{\underline{\Omega}}{\overset{B}{\rightleftarrows}} \mathrm{Top}_*$$

Moore loops = loops of arbitrary "length"

Def: $\mathrm{Fun}(\mathrm{Cart}^{op}, s\mathrm{Grp})$ ← Group object in $\mathbb{H} = s\mathrm{PSh}(\mathrm{Cart})$

$$G \uparrow \dashv \downarrow \mathbb{B}$$

$$\mathrm{Fun}(\mathrm{Cart}^{op}, s\mathrm{Set}_0)$$

$G : \mathcal{L}ie\,\mathrm{Grp} \rightsquigarrow T \mapsto C^\infty(T; G)$

Ω^n - coll. of diff. forms

$G : \mathcal{L}ie\,\mathrm{Grp}$

$*/\!/G = \mathbb{B}G : s\mathrm{PSh}(\mathrm{Cart})$

$G : \mathcal{L}ie\,\mathrm{Ab} \hookrightarrow s\mathrm{PSh}(\mathrm{Cart}; \mathrm{Ab})$

$\mathbb{B} : s\mathrm{PSh}(\mathrm{Cart}; \mathrm{Ab}) \to s\mathrm{PSh}(\mathrm{Cart}; \mathrm{Ab})$

SI SI OK

$[1] = \Sigma : \mathrm{PSh}(\mathrm{Cart}; \mathrm{Ch}) \circlearrowleft$

$\mathbb{B}^n G = K(G; n) \leftarrow$ the E.-M. stack.

$\mathbb{B}^n G$ classifies bundle $(n-1)$-gerbes w/ str-re group G.

Gerbes & 2-bundles:

G : Lie Grp $\quad \mathbb{B}G$: Lie 2Gb

$\cap$

$sPSh(Cart; Grp)$

Lie 2Grps $\hookrightarrow sPSh(Cart; Grp) \to sPSh_0(Ct$

$\overline{H} \longrightarrow \mathbb{B}H$

$\mathbb{R}Map(M; \mathbb{B}H) \simeq$

$\simeq N \{$ 2-groupoid of prin. 2-bundles w/ str-re 2-group $H \}$

$\simeq N \{$ 2-groupoid class. of bundle gerbes w/ str-re group $G \}$