Two dimensional Perturbative Scalar Field Theory and Atiyah-Segal Gluing

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I) Path Integral Quantization and Atiyah-Segal Axiomatics of QFT

d-dimensional classical field theory

Σ a d-dimensional compact manifold

- $\mathcal{F}(\Sigma)$ = space of fields
- $S : \mathcal{F}(\Sigma) \to \mathbb{R}$, action functional

Example: Σ Riem mfld
$\mathcal{F}(\Sigma) = C^\infty(\Sigma) \ni \phi$

$$S(\phi) = \int_\Sigma \tfrac{1}{2}(d\phi \wedge * d\phi) + m^2\phi^2 \, dVol + p(\phi)\, dVol$$

- $dVol$ = volume form on Σ
- $p(\phi) = \sum_{n \geq 3} \frac{p_n}{n!}\phi^n$ $(n \geq 3)$
- $p_n \in \mathbb{R}$
- $m > 0$ "mass"

Path Integral Quantization:

Define $Z_\Sigma = $ "$\displaystyle\int_{\mathcal{F}(\Sigma)} e^{-S(\phi)} \mathcal{D}\phi$" called the partition function

When Σ has a boundary Y, we can consider

$$Z_\Sigma(\eta) = \int_{\{\phi \in \mathcal{F}(\Sigma) \mid \phi|_Y = \eta\}} e^{-S(\phi)} \mathcal{D}\phi$$

This suggests that $Z(\Sigma) \in \mathcal{H}_Y = \{$functionals on $\mathcal{F}(Y)\}$

Also, these path integrals, heuristically, satisfy formal Fubini's theorem which motivates Atiyah-Segal Axiomatics of QFT.

There are many names such as Kontsevich, Witten, ...

Atiyah-Segal axiomatics of QFT:

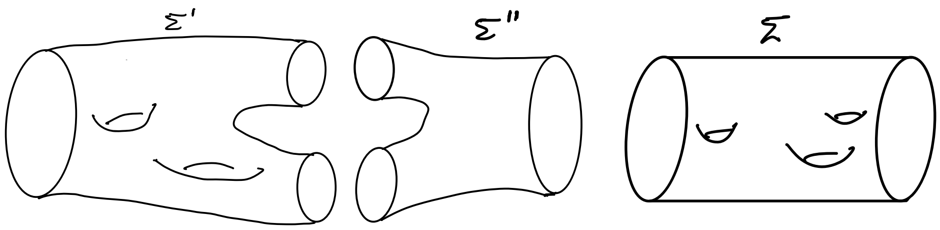
Data:

- $(d-1)$ closed manifold $Y \longrightarrow \mathcal{H}_Y$ "space of states"
- d-manifold Σ with boundary $\longrightarrow Z_\Sigma \in \mathcal{H}_{\partial\Sigma}$ "partition function"

Axioms:

- $\mathcal{H}_{Y_1 \sqcup Y_2} \longrightarrow \mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2}$
- Gluing: if $\partial\Sigma' = Y_1 \sqcup Y_2$, $\partial\Sigma'' = Y_2 \sqcup Y_3$ and $\Sigma = \Sigma' \cup_{Y_2} \Sigma''$ then

$$Z_\Sigma = \langle Z_{\Sigma'}, Z_{\Sigma''} \rangle_{\mathcal{H}_{Y_2}}$$



Categorical Formulation:

$(\mathcal{H}, Z) : d\text{-Bord} \xrightarrow{\text{Gcom}} \text{Vect}^{\text{extra str.}}$

Obj: closed $(d-1)$ mfld $Y \longmapsto$ vector space $\mathcal{H}_Y$

Mor: compact d-mfld $Y_1 \xrightarrow{\Sigma} Y_2 \longmapsto Z_\Sigma : \mathcal{H}_{Y_1} \to \mathcal{H}_{Y_2}$ linear maps

$\circ$: Gluing $\longmapsto$ composition of maps

$\otimes$: disjoint union $\sqcup$ $\longmapsto$ $\otimes$

<u>Questions:</u> Construct theories that satisfy Atiyah-Segal axioms in geometric setting (non topological theories)

- Can theories considered in Physics be turned into a Atiyah-Segal type theories?

Goal: Starting with 2d scalar field theory, construct a class of examples of Atiyah-Segal type theories

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2)

Classical scalar field theory on a surface with polynomial interaction:

$\Sigma =$ Oriented 2d Riemannian manifold

$$S_\Sigma(\phi) = \int \tfrac{1}{2}(d\phi \wedge * d\phi) + \tfrac{1}{2} m^2 \phi^2 \, d\mathrm{vol} + p(\phi)\, d\mathrm{vol}$$

$$p(\phi) = \sum_{n \geq 3} \frac{p_n}{n!} \phi^n$$ ← Polynomial interaction

We want to quantize this theory.

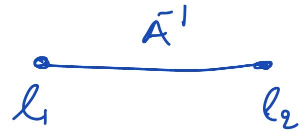
3) Digression (Feynmann diagrams: finite dimension)

- $V = \mathbb{R}^n$
- $S(x) = \frac{1}{2}\langle Ax, x\rangle$ where A is a positive definite matrix

Then

- $$\int_V e^{-S(x)}\,dx = \left(\tfrac{A}{2\pi}\right)^{-1/2}$$

Wick's lemma



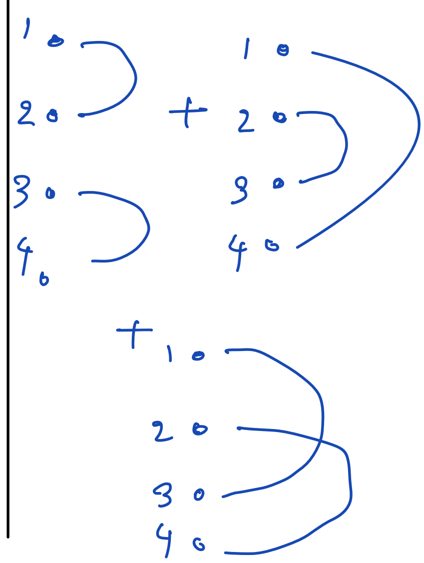
- $$\int_V l_1(x)\, l_2(x)\, e^{-S(x)}\,dx$$

$$= \det\left(\tfrac{A}{2\pi}\right)^{-1/2} \langle A^{-1} l_1, l_2\rangle$$

A^{-1} = "propagator"

- $$\int_V l_1(x) l_2(x) l_3(x) l_4(x)\, e^{-S(x)}\,dx$$

$$= \det\left(\tfrac{A}{2\pi}\right)^{1/2}\Big[\langle A^{-1} l_1, l_2\rangle \cdot \langle A^{-1} l_3, l_4\rangle + \langle A^{-1} l_1, l_3\rangle \langle A^{-1} l_2, l_4\rangle + \langle A^{-1} l_2, l_3\rangle \cdot \langle A^{-1} l_1, l_4\rangle\Big]$$



4) Quantization on a closed surface: (Perturbative path integral)

Let Σ be a closed oriented 2-dim Reimann surface.

$$\int_V e^{-S_{free}(x) + p(x)}$$

$$= \int_V e^{-p(x)}\, e^{-S_{free}(x)}$$

$=$

$$Z_\Sigma = \text{``}\int_{C^\infty(\Sigma)} \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S_\Sigma(\phi)}\text{''}$$

$$:= \det{}^{-1/2}(\Delta_\Sigma + m^2) \sum_\Gamma \frac{\hbar^{E-V}}{|Aut(\Gamma)|} \Phi_\Gamma$$

where $\underbrace{\log \det(\Delta_\Sigma + m^2)}_{\text{zeta regularized determinant}} = -\zeta'(0)$

- $\hbar$ is a formal parameter
- $\Gamma(\mathcal{E}, \mathcal{V}) \to$ Feynman graph (combinatorial graph)
- $E = |\mathcal{E}|$, $V = |\mathcal{V}|$
- $$\Phi_\Gamma := \int_{Conf_V(\Sigma)} \left(\prod_{v \in \mathcal{V}} (-P_{val(v)}) \cdot \prod_{e=(u,v) \in \mathcal{E}} G(x_u, x_v) \right) d^2x_1 \cdots d^2x_V$$

Where $G(x,y)$ is the Green's function of $(\Delta + m^2)$.

Fact: $G(x,y) \sim -\frac{1}{2\pi} \log(d(x,y))$ for $x \to y$.

Thus Φ_Γ is convergent as long as Γ does not contain a short loop ( ).

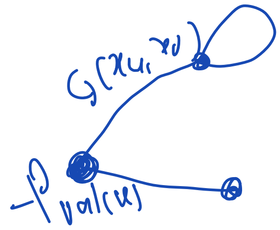
- The naive approach, excluding short loops is not compatible with gluing.

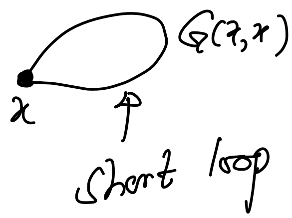
- $S_\Sigma = S_{free} + S_{Int}$.
- det comes from the free part

$\zeta(s) = \sum_\lambda \lambda^{-s}$ zeta function of $\Delta_\Sigma + m^2$

e.g. $\det(\Delta_{S^1_R} + m^2) = 4 \sinh^2 \pi m R$

$val(v)$ = valency of v





- Glimm + Jaffe constructed a non-perturbative path integral rigourously excluding short loops.

$$\mathcal{H}_{\partial\Sigma} = \bigoplus_{n\geq 0} \mathcal{H}^n_{\partial\Sigma}$$

$$\mathcal{H}^n_{\partial\Sigma} = \left\{ \Psi \;\middle|\; \Psi(\eta) = \int_{\mathrm{Conf}_n(\partial\Sigma)} \psi(\gamma_1, \dots, \gamma_n)\, \eta(\gamma_1) \cdots \eta(\gamma_n)\, d\gamma_1 \cdots d\gamma_n \right\}$$

with gluing.

- Our input: Consider a new datum $\tau(x) =$ " $G(x,x)$ ". We call it tadpole function.

One model for $\tau(x)$: $\tau_{\mathrm{split}}(x) = \lim_{y \to x} \left[G(x,y) + \frac{1}{2\pi} \log(d(x,y)) \right]$

$\Rightarrow$ Φ_Γ is convergent.

It appears (no tadpoles) to work for non-perturbative theory (Pickrell 2007)

$\tau(x)$

Tadpole

5) Quantization of a surface with boundary:

State space:

$$\mathcal{H}_{\partial\Sigma} = \left\{ \Psi(\eta) = \sum_{n \geq 0} \int_{\mathrm{Conf}_n(\partial\Sigma)} \psi_n(y_1, \dots, y_n)\, \eta(y_1) \cdots \eta(y_n)\, dy_1 \cdots dy_n \right\}$$

- $\eta \in C^\infty(\partial\Sigma)$
- $\psi_n \in C^\infty(\mathrm{Conf}_n(\partial\Sigma))^{S_n}[[\hbar^{1/2}]]$

"n-particle wave function"

ψ_n must satisfy some regularity condition near diagonals.

Perturbative partition function:

Let η be a boundary field. Then

$$Z_\Sigma(\eta) = \text{“} \int_{\{\phi : \phi|_{\partial\Sigma} = \sqrt{\hbar}\,\eta\}} \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_\Sigma(\phi)} \text{”}$$

$S_{free}(\phi)$

$= S_{free}(\hat{\phi}) + S(\phi_\eta) + S_{int}$

$$:= e^{-\frac{1}{2}\int_{\partial\Sigma} \eta\, D_\Sigma\, \eta}\, \det{}^{-\frac{1}{2}}(\Delta_D + m^2).$$

$$Z_\Sigma(\eta) = \int_{\{\phi : \phi|_{\partial\Sigma} = \hbar^{1/2}\eta\}} e^{-\frac{1}{\hbar} S_\Sigma(\phi)}$$

$\Rightarrow$ $\mathcal{H}_{\partial\Sigma} =$ "Fun" $(C^\infty(\partial\Sigma))$

Note: We have not defined a pairing on $\mathcal{H}_{\partial\Sigma}$ yet.

e.g.

$\psi_n = O(\log(d(x_i, x_j)))$ if $x_i \to x_j$

- $\psi_n = O(\epsilon^{2-k})$ if k-points come together at distance ϵ

$\phi = \hat{\phi} + \phi_\eta$ $\quad$ $\hat{\phi}|_{\partial\Sigma}$

ϕ_η is extension of η

$\Delta_D + m^2 \to$ Dirichlet boundary condition

$$\sum_{\Gamma} \frac{\hbar^{E-V-n/2}}{|\mathrm{Aut}(\Gamma)|}\Phi_\Gamma(\eta)$$

- Γ is a graph that may have 1-valent boundary vertex but no boundary-to-boundary edges,
  - n = number of boundary vertices on Γ
  - V = number of bulk vertices on Γ

- $\Phi_\Gamma(\eta)$:
  - Bulk vertices $\to -p_{val(v)}$
  - Boundary vertices $\to \eta(y_i)$
  - Bulk-bulk edges $\to G(x_\alpha, x_\beta)$
  - Bulk-boundary edges $\to \frac{\partial}{\partial\nu(y_i)} G(x_\alpha, y_i)$

  Take products and integrate with respect to bulk variables x_α and boundary variables y_i i.e.

$$\Phi_\Gamma(\eta) = \int_{\mathrm{Conf}_n(\partial\Sigma)} dy_1\cdots dy_n \prod_i \eta(y_i) \cdot \int_{\mathrm{Conf}_V(\Sigma)} dx_1\cdots dx_V \prod_\alpha (-p_{val(v_\alpha)}) \cdot \prod_{(\alpha,\beta)\in E} G(x_\alpha, x_\beta) \prod_{(i,\alpha)\in E} \frac{\partial}{\partial\nu(y_i)} G(x_\alpha, y_i)$$

Unfortunately, $Z_\Sigma(\eta) \notin \mathcal{H}_{\partial\Sigma}$

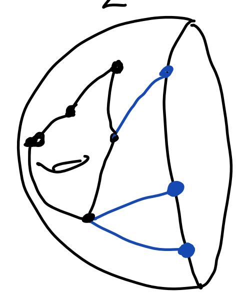
But $\hat{Z}_\Sigma(\eta) := e^{\frac{i}{2}\int \eta D_\Sigma \eta} \cdot Z_\Sigma(\eta) \in \mathcal{H}_{\partial\Sigma}$

D_Σ = Dirichlet-to-Neumann operator

$D_\Sigma(\eta) = \frac{\partial \phi_\eta}{\partial \nu}\Big|_{\partial\Sigma}$

ϕ_η solves $(\Delta + m^2)\phi = 0$, $\phi|_{\partial\Sigma} = \eta$

Σ



Γ

6) Gluing: Assume Σ is closed and
$$\Sigma = \Sigma_1 \cup_Y \Sigma_2$$

Then we want: $Z_\Sigma =$ "$\int_{C^\infty(Y)} D\eta \, Z_{\Sigma_1}(\eta) Z_{\Sigma_2}(\eta)$"

We need to define RHS. More generally we want to define pairing of states.

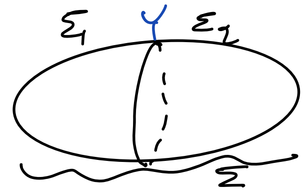
<u>Pairing</u>: $\langle \Psi_1, \Psi_2 \rangle_D := \det(D)^{-1/2} \sum_{\xi} \int_{\mathrm{Conf}_{m+n}(Y)} \psi_1(y_1, \cdots, y_n) \, \psi_2(y_{n+1}, \cdots, y_{n+m}) \prod_{(i,j)\in\xi} D^{-1}(y_i, y_j)$

$\Psi_1 \in \mathcal{H}_Y^{(n)}, \Psi_2 \in \mathcal{H}_Y^{(m)}$

where
- $D = D_{\Sigma_1} + D_{\Sigma_2}$ depends on Σ_1 and Σ_2
- ξ runs over perfect matching on a set of $m+n$ elements

Define $\hat{Z}_{\Sigma_i} = e^{\frac{1}{2}\int_Y \eta D_{\Sigma_i} \eta} Z_{\Sigma_i} \in \mathcal{H}_{\partial\Sigma_i}$

<u>Theorem</u> $Z_\Sigma = \langle \hat{Z}_{\Sigma_1}(\eta), \hat{Z}_{\Sigma_2}(\eta) \rangle_D$



$Z_\Sigma =$ "$\int_{C(Y)} D\eta \, e^{-\frac{1}{2}\int \eta (D_{\Sigma_1} + D_{\Sigma_2})\eta} \hat{Z}_{\Sigma_1}(\eta) \hat{Z}_{\Sigma_2}(\eta)$"

Key Ingredients to prove this Theorem.

- BFK Gluing formula for determinants
$$\det(\Delta_\Sigma + m^2) = \frac{\det(\Delta_{\Sigma_2} + m^2)\det(\Delta_{\Sigma_1} + m^2)}{\det(D)}$$

← Dirichlet boundary conditions on Σ_i
(This can be predicted using Free theory)

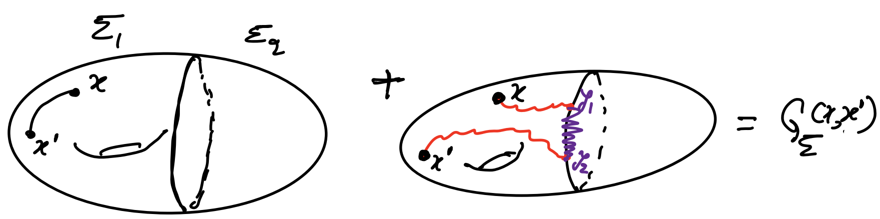
- Gluing relation for Green's function:

If $x, x' \in \Sigma_1$ then

$$G_\Sigma(x,x') = G_{\Sigma_1}(x,x') + \int_Y\int_Y \frac{\partial}{\partial\nu(y_1)} G_{\Sigma_1}(x,y_1)\, D^{-1}(y_1,y_2)\, \frac{\partial}{\partial\nu(y_2)} G_{\Sigma_1}(y_2,x')\, dy_1\, dy_2$$

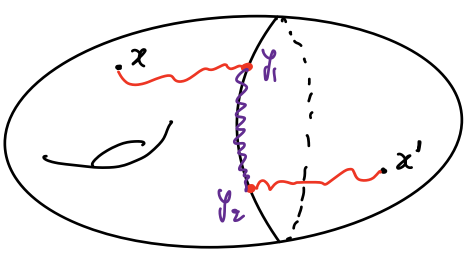
This can be predicted using two point functions and it can be proved mathematically

Giles Carron



If $x \in \Sigma_1$ and $x' \in \Sigma_2$ then

$$G_\Sigma(x,x') = \int_Y\int_Y \frac{\partial}{\partial\nu(y_1)} G_{\Sigma_1}(x,y_1)\, D^{-1}(y_1,y_2)\, \frac{\partial}{\partial\nu(y_2)} G_{\Sigma_1}(y_2,x')\, dy_1\, dy_2$$

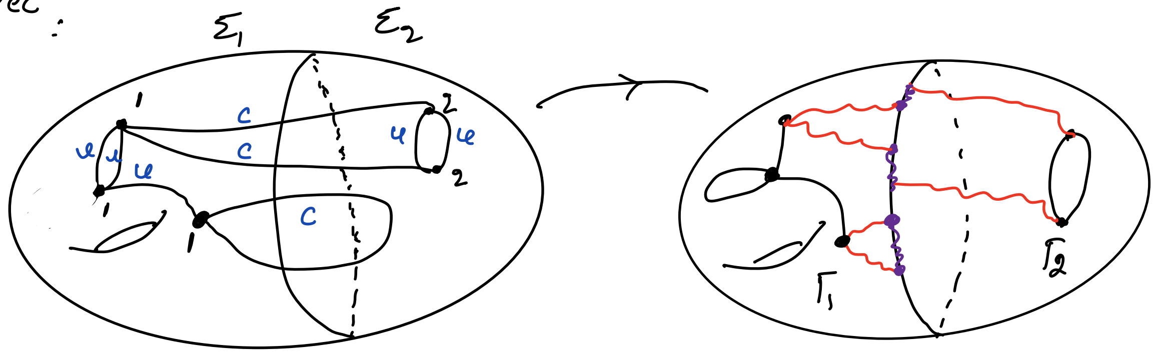


- Cutting and Gluing Feynman diagram

$$\Phi_\Gamma^\Sigma = \sum_{\Gamma^{dec}} \Phi^\Sigma_{\Gamma^{dec}} = \sum_{\Gamma_1,\Gamma_2} \langle \Phi_{\Gamma_1}^{\Sigma_1}, \Phi_{\Gamma_2}^{\Sigma_2} \rangle_D$$

Idea: Insert the gluing formula for Green's function and analyze that

Γ^{dec}:



$\Phi^{\Sigma}_{\Gamma^{dec}}$

- u, C "uncut" and "cut" comes from gluing formula

- Tadpole functions for short loops

$$\Upsilon(x) = \lim_{s \to 1} \left[\int_0^\infty dt\, t^{s-1} K(t,x,x) - \frac{1}{4\pi(s-1)} \right]$$

$$\Upsilon_{split}(x) = \lim_{y \to x} \left[G(x,y) + \frac{1}{2\pi} \log d(x,y) \right]$$

- u place G_{Σ_1} or G_{Σ_2}
- c place G_{Σ}

Fact: $\Upsilon(x) = \Upsilon_{split}(x) - \frac{\log 2 - \gamma}{2\pi}$

$\Upsilon(x)$ is a better choice because it is "consistent" with zeta-regularized determinant.

- Functoriality:

The pairing $\langle\, ,\, \rangle_D$ is not functorial as it depends on Σ_1 and Σ_2.

Idea: Use something that only depends on the geometry of Y and yet it is "comparable" to D.

Let $\mathcal{K} = \sqrt{\Delta_Y + m^2}$ ← Square root of Helmholtz operator on Y

Fact: $D_{\Sigma_i} - \sqrt{\Delta_Y + m^2}$ is a pseudodifferential operator of order ≤ -2

Define
$$\overline{Z}_\Sigma(\eta) = e^{\frac{1}{2}\int_{\partial\Sigma} \eta \mathcal{K} \eta}\, Z_\Sigma(\eta)$$

Then:

Prop: $\overline{Z}_\Sigma = \langle \overline{Z}_{\Sigma_1}, \overline{Z}_{\Sigma_2} \rangle_{2\mathcal{K}}$

Theorem: $(\mathcal{H}, \overline{Z})$ is a functor.

$$\mathrm{Cob}_2^{\mathrm{Riem}} \longrightarrow \mathcal{H}ilb$$

collared Riem 1-mflds $\longrightarrow$ real Hilbert spaces $\otimes\, \mathbb{R}[[t^{1/2}]]$

$\Sigma \longrightarrow$ Hilbert-Schmidt operators

gluing $\longrightarrow$ composition

$\amalg \longrightarrow \hat{\otimes}$

Recall
$$D = D_{\Sigma_1} + D_{\Sigma_2}$$

Examples

Σ	$\lambda_n - \omega_n$
Cylinder	$O(n^{-\infty})$
Hemisphere	$O(n^{-3})$
disk	$O(n^{-2})$

Rmk: $\mathcal{K}$ also appears in the construction of state space in constructive QFT

Rmk: $2\mathcal{K}$ induces on a Gaussian measure $\mu_{\mathcal{K}}$ "$\mathcal{E}^0(Y)$"

and D induces another Gaussian measure on "$\mathcal{E}^0(Y)$"

$D - 2\mathcal{K}$ "small" means we can take Radon-Nikodym derivative $\mu_D/\mu_{\mathcal{K}}$.

- Decorate Γ
- 1, 2 restrict integration to Σ_1, Σ_2
- u "uncut"

$$G_{\Sigma_1}(z, \underset{\substack{\uparrow \\ u}}{p}) \quad \text{or} \quad G_{\Sigma_2}(z, \underset{\substack{\uparrow \\ u}}{p})$$

$$C \longrightarrow \iint \text{ second term}$$