$\forall i = 1, \dots, m$

Quantum Homotopy Seminar (July 1st, 2021)

James [Maurer-Cartan stacks for exceptional generalized geometry]

Leibniz algebroids

bilinear pairing $\underbrace{E = TM \otimes T^*M}_{\text{generalized tangent bundle}}$

Lie bracket

Lie derivative

due to Courant 1990's

$$\langle x + \xi, y + \eta \rangle := [x, y] + \mathcal{L}_x \eta - \mathcal{L}_y d\xi - \tfrac{1}{2} d(\iota_x \eta - \iota_y \xi)$$

This is the "Courant Bracket" (taken sectionally)

Maximal, isotropic subbundles are called "Dirac structures," and closure under the Courant Bracket is the "integrability ~~theorem~~ condition"

e.g.: $\underset{\text{Dirac structures}}{O(d,d)} \hookrightarrow \underset{\text{doubled Riemannian structures}}{GL(2d, \mathbb{R})}$

Application: $G_2 \times G_2$-structures, $SU(n) \times SU(n)$ structures (generalized Calabi-Yau manifolds)

$J : \mathcal{E} \longrightarrow \mathcal{E}$; $J^2 = -\mathrm{id}_E$ generalized complex structures

compared to

So... $K : TM \to TM$; $K^2 = \mathrm{id}_{TM}$, M even-dimensional eigenvalues of K

<u>Given</u> M: 11-manifold, w/metric g and 3-form C_3, the field strength $F_4 = dC_3$ ... we have the equation

$$d(*F_4) + \frac{1}{2} F_4 \wedge F_4 = 0$$

Hodge star, a 7-form
$*F_4 = F_7$

And this identity is a "Bianchi-type identity" for F_7 (and can be found in Sati and Schreiber's works)

Rewriting, LOCALLY...

$$d(F_7 + \frac{1}{2} C_3 \wedge F_4) = 0$$

and, again, LOCALLY...
dual 6-form potentially

$$dC_6 = F_7 + \frac{1}{2} C_3 \wedge F_4$$

(we choose a C_6 such that dC_6 satisfies this condition)

The equation $d(*F_4) + \frac{1}{2} F_4 \wedge F_4 = 0$ is an equation of motion. Locally, we make a choice of 6-form, and obtain the equation of motion above.

Gauge invariants of F_3, F_7, but we make choices of C_3, C_6. This requires us to solve:

$$F_4 = dC_3$$
$$F_7 = dC_6 - \frac{1}{2} C_3 \wedge dC_3$$

and choosing another closed 3-form Z_3 and 6-form Z_6:

$$C_3' = C_3 + Z_3$$
$$C_6' = C_6 + Z_6 + \frac{1}{2} C_3 \wedge Z_3$$

we define a group action of closed 3-forms and closed 6-forms on potentials:

$$(Z_3, Z_6)(Z_3', Z_6') = (Z_3 + Z_3', Z_6 + Z_6' - \frac{1}{2} Z_3 \wedge Z_3')$$

...and...

... Lie algebra of closed 3- and 6-forms:

define Lie bracket $[Z_3, Z_3'] = - Z_3 \wedge Z_3'$

and we also define:

$$W := TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M$$

The actions identified on W define a bracket:

$$\{X + \underset{\text{2-form}}{a_2} + \underset{\text{5-form}}{a_5},\ Y + \underset{\text{2-form}}{b_2} + \underset{\text{5-form}}{b_5}\} := [X,Y] + \mathcal{L}_X b_2 - \mathcal{L}_Y a_2 \underbrace{- \iota_Y da_2 - \iota_Y da_5 + da_2 \wedge b_2}_{\text{much confusion surrounding how to define this bracket from the action on } W}$$

This bracket is supposed to define a Leibniz Algebroid ... so it has certain properties.

[References: Gualtieri, Baraglia "Leibniz algebroids", ~~"Twisted~~ "Twistings and exceptional generalizations"]

Twist this bracket $\{-,-\}$ by F_4, F_7:

should be a_2?

$$\{X + a_2 + a_5,\ Y + b_2 + b_5\}_{F_4,F_7} := [X,Y] + \mathcal{L}_X b_2 - \iota_Y da_5 + \iota_X \iota_Y F_4 + \mathcal{L}_X b_5 - \iota_Y da_5 + da_2 \wedge b_2 + \iota_X F_4 \wedge b_2 + \iota_X \iota_Y F_7$$

twisted bracket

and $\{-,-\}_{F_4,F_7}$ satisfies the Leibniz identity iff F_4, F_7 satisfy the identity

$$dF_4 = 0$$
$$dF_7 + \frac{1}{2} F_4 \wedge F_4 = 0, \quad \text{etc.}$$

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a Leibniz algebroid is... $\longrightarrow$

On a manifold M, a "Leibniz algebroid" is a vector bundle V with a map $\rho : V \longrightarrow TM$ and a billinear pairing
(anchor map)
on sections of V, $\Gamma(V) \otimes \Gamma(V) \xrightarrow[\{-,-\}]{} \Gamma(V)$

satifying the properties of a Leibniz Algebra and compatibility condition w/ anchor map.

(i) $\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}$

and for $\mathcal{L} : M \longrightarrow \mathbb{R}$

(ii) $\{a, \mathcal{L} b\} = \rho(a)(\mathcal{L}) b + \mathcal{L}_0 \{a, b\}$

Anti-symmetry of $\{-,-\} \Longrightarrow$ (i) becomes Jacobi identity and $(V, \{-,-\}, \rho)$ becomes a Lie algebroid.

MORAL:
supergravity Equations of motion allow you to define a Leibniz algebra structure

should also be a Lie Algebroid? Produces a "better" version of supergravity?