SUMMER SEMINAR #3

Derived Geometry
sudden structures

GOAL: Define shifted symplectic structures on the derived critical locus of a smooth function

ⓐ smooth functions on derived differentiable stacks

$C^\infty M \in CAlg_{\mathbb{R}}$ where $Man^{op} \xrightarrow{C^\infty} CAlg_{\mathbb{R}}$

C^∞ is a sheaf

Given an open cover $\{U_i\}_{i\in I}$ of $X \in Man$, then $X = \operatorname{colim}_k U_k$

Passage from Man to Man^{op}, colimit → limit

~~colimit~~

colimit in Man,
limit in Man^{op}

$$\underbrace{\lim_k U_k}_{Man^{op}} \longrightarrow \lim_k C^\infty U_k$$

By the Universal property of the categories of sheaves (model category of simplicial presheaves) we need extensions of the functor C^∞...

∞-sheaves/
simplicial
presheaves/
stacks

First, extend to sheaves... we have unique extensions

$$Sh(Man)^{op} \xrightarrow{C^\infty} CAlg_{\mathbb{R}}$$

(continuous functor,
limits are sent to limits)

[or...: colimits of sheaves are sent
to limits of algebras!]

colimits $\longmapsto$ limits

$$sPSh(Man)^{op} \longrightarrow \begin{cases} CAlg_{\mathbb{R}} \\ CDGA_{\mathbb{R}} \end{cases}$$

(homotopy continuous functor)

homotopy colimits are sent to limits of algebras (in $CAlg_{\mathbb{R}}$)

or, homotopy limits of DGA's (in $CDGA_{\mathbb{R}}$)

$$\text{hocolims of sheaves} \longmapsto \begin{cases} \text{limits of alg} \\ \text{holims of } DGA_{\mathbb{R}} \end{cases}$$

Among other things, we recover classical results on infinite dimensional manifolds (see nlab for references)

We want to make sense of

$$C^\infty X = \mathbb{R}Map(X, \mathbb{R}) = \mathbb{R}Hom(X, \mathbb{R})$$

derived hom, right-derived

[in the way that $C^\infty M = Hom(M, \mathbb{R})$]

recovering the algebra structure, etc.

Consider presheaves on manifolds w/ values in CDGA's

$$PSh(Man, CDGA_{\mathbb{R}})$$

We have a hom functor: $Hom : PSh(Man, sSet)^{op} \times PSh(Man, CDGA_{\mathbb{R}}) \longrightarrow PSh(Man, CDGA_{\mathbb{R}})$

simplicial presheaves $\times$ presheaves of CDGA's $\longrightarrow$ presheaves of CDGA's

Universal property of presheaves, it is enough to define this hom functor on representables

This functor is a Left-Quillen bifunctor (so it can be derived)...

$$Hom \xrightarrow[\text{derive!}]{} \mathbb{R}Hom$$

This is the functor we wanted! $\longrightarrow$

In the derived setting...
we have the unique extension:

$$sPSh(DCart)^{op} \longrightarrow CDGA_{\mathbb{R}}$$

derived cartesian spaces replaces Manifolds

(homotopy continuous functor, sends hocolims of sheaves to holims of DGA's)

Then, similarly, $C^\infty X = \mathbb{R}Hom(X, \mathbb{R})$

defined on $PSh(DCart, CDGA_{\mathbb{R}})$

$$Hom: PSh(DCart, sSet)^{op} \times PSh(DCart, CDGA_{\mathbb{R}}) \to PSh(DCart, CDGA_{\mathbb{R}})$$

How to define smooth functions on derived cartesian spaces? We consider DCart as an algebra...

$$\begin{array}{c} DCart = \text{semi-free } CDGA_{\mathbb{R}}^{op} \\ \Psi \\ X \end{array}$$

Given X, $C^\infty X$ is the same object but taken as a semi-free commutative differential graded algebra in the opposite category

$$C^\infty X = X \in \text{semi-free } CDGA_{\mathbb{R}}^{op}$$

[Smooth functions on derived critical locus can be computed also...]

$\longrightarrow$

DIFFERENTIAL FORMS:
(on DDS)

Traditionally, on smooth manifolds, there is an algebraic viewpoint and a geometric viewpoint:

traditional algebraic viewpoint of differential forms on M

(ALGEBRAIC) $\Omega M :=$ the free C^∞CDGA$_\mathbb{R}$ on $C^\infty M$

Given $f \in C^\infty M = \Omega^0 M$ (a zero form)
$df \in \Omega^1 M$ (a one form)

Recall,
where $\Omega^0 M \xrightarrow{d} \Omega^1 M$ is a "C^∞-derivation;"
which is to say that

$$d(g(f_1, \dots, f_n)) = \sum_i \frac{\partial g}{\partial x_i}(f_1, \dots, f_n)$$

When g: polynomial, we recover Kähler differentials (which satisfy the Leibniz rule), but w/ arbitrary g, we have a new notion (part of the defn of C^∞CDGA's)

traditional geometric viewpoint of differential forms on M

(GEOMETRIC) $\Omega M = C^\infty(\mathrm{Hom}(\mathrm{spec}\,\mathbb{R}[\varepsilon]/\varepsilon^2, M))$

smooth functions on internal hom from spec $\mathbb{R}[\varepsilon]/\varepsilon^2$ to the manifold M

(tangent bundle on M shifted in degree 1, take smooth functions)

where ε has degree 1

NOTE: in DDS: $\mathrm{Spec}\,\mathbb{R}[\varepsilon]/\varepsilon^2 \longrightarrow M = \mathrm{Spec}\,C^\infty M$

passing to CDGA's;
we consider maps:

$C^\infty M \longrightarrow \mathbb{R}[\varepsilon]/\varepsilon^2$ } $TM[-1]$ shifted tangent bundle

takes smooth function and assigns real #

NOTE continued...

$$C^\infty(TM[-1]) = \mathrm{sym}(\underset{\text{odd}}{TM[-1]}) = \Lambda T^* M = \Omega M$$

[Dan's suggestion: Try defining degreewise in chain complexes and extend]

IN THE DERIVED CASE:

ALGEBRAIC (derived) $\Omega X :=$ the derived free $C^\infty DGA_{\mathbb{R}}$ on $C^\infty M$

(here, $C^\infty M \in C^\infty CDGA_{\mathbb{R}}$, we already have a grading, and this construction adds another grading... so differential forms here are bigraded! Take totalization of gradings)

bigrading $\rightsquigarrow$ totalization

Dr. Weinberg NOTE about the use of spectral sequences!

hom degree

differential maps (top to bottom)

$$\begin{tikzcd}
2 \arrow[d] & \Omega^0_2 \arrow[r] \arrow[d] & \Omega^1_2 \arrow[r] \arrow[d] & \Omega^2_2 \arrow[r] \arrow[d] & \Omega^3_2 \arrow[d] \\
1 \arrow[d] & \Omega^0_1 \arrow[r] \arrow[d] & \Omega^1_1 \arrow[r] \arrow[d] & \Omega^2_1 \arrow[r] \arrow[d] & \Omega^3_1 \arrow[d] \\
0 & \Omega^0_0 \arrow[r] & \Omega^1_0 \arrow[r] & \Omega^2_0 \arrow[r] & \Omega^3_0 \\
& 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \text{differential maps (left to right)}
\end{tikzcd}$$

0-forms   1-forms   2-forms   3-forms

Problem: Show algebraic forms are quasi-isomorphic to geometric forms for DDS $\Omega_{alg} \cong \Omega_{geom}$

Problem: de Rham theorem for DDS

previously, closed form was form whose differential $= 0$

here, homotopy class $= 0$
differential

The notion of "closed"

closed, nondegenerate 2-form — symplectic form

~~THEOREM~~ **Problem** Spectral sequences for DDS.

Problem Jets and differential operations for DDS

NOTE: (semi-free DGA polynomial algebra)

Problem Formalize all of Calculus of variation

Problem Elliptic Regularity for DDS and Index theorem for DDS

GEOMETRIC (derived)

Ω_{geom}

$$\Omega X = C^\infty(\mathbb{R}\mathrm{Hom}(\mathrm{spec}\, \mathbb{R}[\varepsilon]/\varepsilon^2, X))$$

in DDS: $\mathrm{Spec}\, \mathbb{R}[\varepsilon]/\varepsilon^2 \longrightarrow X$

in CDGA: $C^\infty X \longrightarrow \mathbb{R}[\varepsilon]/\varepsilon^2$

Show both send colimits to limits... only show on representables, you can compute both explicitly

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SHIFTED SYMPLECTIC STRUCTURES ON DERIVED CRITICAL LOCI:

$X \in DDS$ $\qquad$ $\alpha \in \Omega^1_{closed} X$

closed one-form, think like deRham differential of some smooth function on X

zero locus:

$$\begin{tikzcd} Z^h\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & X \arrow[d, "0"] \\ X \arrow[r, "\alpha"] & T^*X \end{tikzcd}$$

homotopy pull back

Loop spaces, Ben Zvi, does part of this computation (algebraic part)

We want to construct a non-trivial 2-form on $Z^h\alpha$ $\longrightarrow$ $\omega_0 \in \Omega^2_{closed}$

$$\begin{tikzcd} Z^h\alpha \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, "q"] \arrow[d, phantom, shift left=2ex, "\dashv h"] & X \arrow[d, "0"] \\ X \arrow[r, "\alpha"'] & T^*X \end{tikzcd}$$

we will accomplish this by...

Pullback cotangent bundles of spaces to $Z^h\alpha$...
we get the map "q"

this is an exact sequence

$$q^* T^*_{T^*X} \longrightarrow j^* T^*_X \oplus j^* T^*_X \longrightarrow T^*_{Z^h\alpha}$$

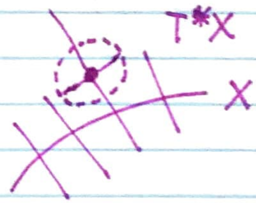
NOTE: $T^*_{T^*X}$ is the cotangent bundle of T^*X (the cotangent bundle of X)
where $T^*X \in DDS$ (considered as a space)
and $T^*_{T^*X} \in VBun_{DDS}$ (as a vector bundle)

Recall, $X \in Man$

$\Theta \in \Omega^1(T^*X)$

$$\Theta(h, v) = v(h)$$

$h \in TX$ (tangent bundle), $v \in T^*X$ (cotangent bundle)



$T^*_{T^*X}$ consists of a porition in TX and a portion in T^*X (take the direct sum)

$d\Theta \in \Omega^2_{closed}$ nondegenerate

What does it mean for a 2-form to be nondegenerate

$$\Omega^0 \xrightarrow{\omega} \Omega^1 \wedge \Omega^1$$

$$T \xrightarrow{\cong} \Omega^1$$

isomorphism (nondegenerate)

What it means for a 2-form to be nondegenerate (continued...)

where the isomorphism is given by

$$T \longrightarrow \Omega^1$$

$$\overset{\omega}{v} \longmapsto \underbrace{(u \longmapsto \omega(u,v))}_{= \, \iota_v \omega}$$

where ω is the given 2-form.

we define: $\omega : \mathcal{O}_{T^*X} \longrightarrow \Omega^2_{T^*X}$

canonical symplectic form (2-form) on T^*X

$C^\infty(T^*X)$ smooth functions

Apply q^* to ω:

$$q^*\omega : q^*\mathcal{O}_{T^*X} \xrightarrow{q^*\omega} q^*\Omega^2_{T^*X}$$

$$q^*\mathcal{O}_{T^*X} = \mathcal{O}_{Z^h_\alpha}$$

pullback of functions is again functions

We will compute the second exterior power of the sequence:

$$q^*T^*_{T^*X} \longrightarrow j^*T^*_X \oplus j^*T^*_X \longrightarrow T^*_{Z^h_\alpha}$$

which produces...

$$q^*\Omega^2_{T^*X} \longrightarrow \Lambda^2(j^*T^*_X \oplus j^*T^*_X) \longrightarrow \Omega^2_{Z^h_\alpha}$$

$$\|$$

$$q^*\mathcal{O}_{T^*X} \xrightarrow{q^*\omega} q^*\Omega^2_{T^*X}$$

CLAIM: This composition VANISHES $= 0$

CLAIM (discussion):

$$\begin{tikzcd}
q^*\mathcal{O}_{T^*X} \arrow[d, "q^*\omega"'] \arrow[drr, "0"] \arrow[dr, phantom, "\#"] & & \\
q^*\Omega^2_{T^*X} \arrow[r] & \Lambda^2(j^*T^*_X \oplus j^*T^*_X) \arrow[r] & \Omega^2_{Z^h_\alpha}
\end{tikzcd}$$

0 (homotopic to zero, 0 as a cohomology class)

This is because the sections we are restricting along are Lagrangian:

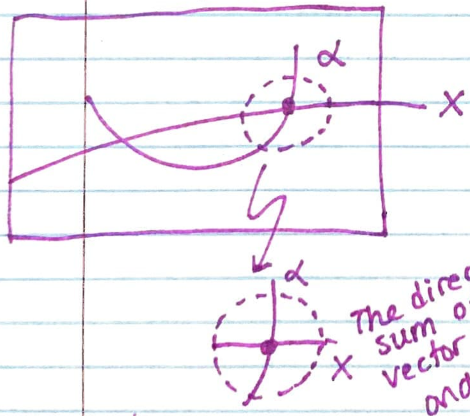
$$\begin{tikzcd}
Z^h_\alpha \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, "q"] \arrow[dr, phantom, very near start, "\lrcorner h"] & X \arrow[d, "0"] \\
X \arrow[r, "\alpha"'] & T^*X
\end{tikzcd}$$

0 (Lagrangian)

α (Lagrangian)

Geometrically,

T^*X



The direct sum of the vector space from α and the vector space from X is the entire space T^*X

Geometric prequantization of DDS

PROBLEM

DEFINING THE 1-FORM:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_1\Omega^2_{Z^h_\alpha} \arrow[dlll, dashed] \\
H_0 q^*\Omega^2_{T^*X} \arrow[r] & H_0\Lambda^2(j^*T^*_X \oplus j^*T^*_X) \arrow[r] & H_0\Omega^2_{Z^h_\alpha} \arrow[d, dashed] & \\
 & & \cdots &
\end{tikzcd}$$

This sequence is exact

[Reference: shifted symplectic structures]

(Pantev – Toën – Vaquié – Vezzo)