SUMMER SEMINAR #2

Derived Differential Stacks

DEFINITION A $\boxed{\text{derived differential stack}}$ is a functor

$$\boxed{DCart^{op} \xrightarrow{\mathcal{F}} sSet}$$

simplicial sets

derived cartesian spaces

such that:

(i) $\mathcal{F}$ takes values in Kan complexes (e.g. nerves of groupoids, n-groupoids, etc.)

(ii) $\mathcal{F}$ satisfies descent with respect to open covers of derived cartesian spaces

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dr] & \\ \mathbb{R}^m \arrow[ur] \arrow[rr, dashed] \arrow[rr, phantom, "\#", shift left=3] & & X \end{tikzcd}$$

$\mathcal{F}(\mathbb{R}^n) = \{``\mathbb{R}^n \to X"\}$

"weak equivalence"

|||

"simplicial homotopy equivalence"

(iii) If $f: U \xrightarrow{\sim} V$ is a weak equivalence of derived cartesian spaces

Then $\mathcal{F}(f)$

$(\mathcal{F}(f): \mathcal{F}(V) \xrightarrow{\sim} \mathcal{F}(U))$

is a weak equivalence of simplicial sets

DEFINITION DCart [(relative category) & (site)]

category derived cartesian spaces

finite-dimensional graded vector space

TEGVect$_{\mathbb{R}}$

$\dim \bigoplus_{n \geq 0} T_n < \infty$

Objects derived cartesian spaces

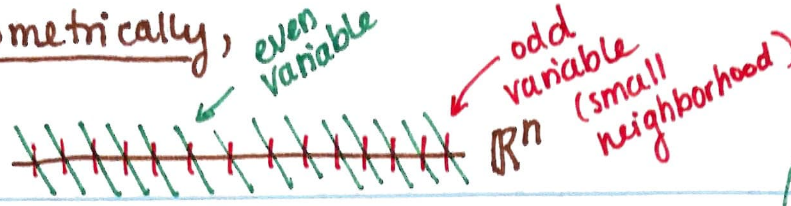
~~(free algebra~~ C^∞-$CDGA_{\mathbb{R}}$) (degree 0)

semi-free algebra

underlying $CGA_{\mathbb{R}}$ is $C^\infty(\mathbb{R}^n)[0] \otimes \mathrm{Sym}(T)$

(degree n) $\mathrm{Sym}(T) = \bigotimes_{n>0,\text{ even}} \mathrm{Sym}(T_n^*) \otimes \bigotimes_{n>0,\text{ odd}} \Lambda T_n^*$ exterior algebra (anti-commute)

Geometrically,



even degree slightly larger neighborhood

(Think: $\bigotimes_{n>0,\text{even}} \mathrm{Sym}(T_n^*)$: polynomials algebra on T_n)

odd degree

$\mathbb{R}[\varepsilon]$

ε has degree 1

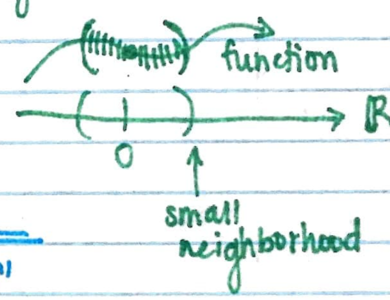
$\varepsilon^2 = 0$

$\mathbb{R}[\varepsilon] = \{ a + b\cdot\varepsilon \mid a, b \in \mathbb{R} \}$

even degree

$\mathbb{R}[t] = \{ a_0 + a_1 t + a_2 t^2 + \dots \mid a_i \in \mathbb{R} \}$

t has degree 2.

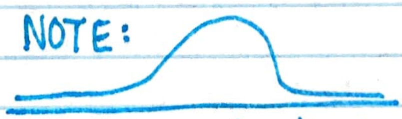


Morphisms

Morphisms of CDGA's$_{\mathbb{R}}$

$$\hom(C^\infty(\mathbb{R}^m), C^\infty(\mathbb{R}^n))_{C^\infty\text{-ring}}$$
$$\cong \downarrow \quad \text{vs.}$$
$$\hom(C^\infty(\mathbb{R}^m), C^\infty(\mathbb{R}^n))_{\mathrm{CAlg}_{\mathbb{R}}}$$

NOTE:



"bump function"

Any morphisms of C^∞-rings is a morphism of algebras
(Milnor's exercise, w/ arbitrary smooth manifolds)

This is actually an isomorphism!

Weak Equivalences

Weak equivalences are quasi-isomorphisms

Open Cover

Recall, usual open cover $U_i \xhookrightarrow{f_i} V$ (open embedding)

Recall, $V = \bigcup_i f_i(U_i)$ * $\quad i \in I$

$\Rightarrow$ generalize the usual notion using the geometric idea!

derived differentiable stacks (continued...)

In our case (derived cartesian spaces)

$\boxed{\text{Covering Families for DCart:}}$

$X \in \text{DCart}$ symmetric algebra of T

$X = \text{spec}(C^\infty(\mathbb{R}^n) \otimes \text{Sym}T)$

Pick any good cover $\{U_i\}_{i\in I}$ of $\mathbb{R}^n$... $\{U_i \hookrightarrow \mathbb{R}^n\}_{i\in I}$

$$\boxed{\{\text{Spec}(C^\infty(U_i) \otimes \text{Sym}(T)) \longrightarrow \text{spec}(C^\infty(\mathbb{R}^n) \otimes \text{Sym}(T))\}_{i\in I}}$$

Recall, map between specs is homomorphism between algebras in the opposite direction

(so, the map above is restriction along inclusion $U_i \hookrightarrow \mathbb{R}^n$)

here,
($U_i \cong \mathbb{R}^n$)

(2 min break)

[NOTE: $\text{DCart}^{op} \longrightarrow \text{sSet}$ "simplicial presheaves"]

This is a very "long" 2 minute break! (general discussion)

$\boxed{\text{DEFINITION}}$ Vector Bundles on Derived Differentiable Stacks (DDS)

$\boxed{\text{EXAMPLE}}$

(a) dimension (d), $\forall d \geq 0$ define $\text{VBun}_d \in \text{DDS}$

(b) $X \in \text{DDS}$: $\text{Hom}(X, \text{VBun}_d)$ category of vector bundles over derived differentiable stack, X

VBun_d is NOT a smooth manifold!

To define VBun_d...

→ Recall; (non derived setting...)

$VBun_d : DCart^{op} \longrightarrow sSet$

$\boxed{\text{DEFINING } VBUN_d}$

$VBun_d : Cart^{op} \longrightarrow sSet$

(ordinary, non-derived stack)

We will use this ~~Deut~~ in the Derived case

define in groupoids first:

$$\begin{tikzcd} VBun_d : Cart^{op} \arrow[rr] \arrow[dr] & & sSet \\ & Grpd \arrow[ur, "N" ', "\text{nerve}"] & \end{tikzcd}$$

$U \in Cart$

$VBun_d(U) =$ groupoid of vector bundles over U

CONSTRUCTION: presheaf valued in groupoids: take the nerve of the groupoid to get a value for VBund on U

<u>Ob</u> $* =$ trivial bundle

$$\begin{array}{c} \mathbb{R}^d \times U \\ \downarrow \text{projection} \\ U \end{array}$$

<u>Mor</u> automorphisms

$$Mor(*,*) = C^\infty(U, \underbrace{Hom^{\times}(\mathbb{R}^d, \mathbb{R}^d)}_{\text{invertible morphisms } GL(\mathbb{R}^d)})$$

Composition is pointwise multiplication

$\boxed{\text{DEFINITION}}$ DELOOPING (category) $B(GL(\mathbb{R}^d))$ groupoid w/ one object and morphisms are automorphisms

Also Recall, $M \in Man$

$$\mathbb{R}Hom(M, VBun_d) = Hom(\check{C}_U M, VBun_d)$$

$\uparrow$ Čech nerve

cocycle description... We will recover it in our setting...

Repeat this construction in the derived setting!

Derived setting: $\mathrm{VBun}_d : \mathrm{DCart}^{op} \longrightarrow \mathrm{sSet}$

$\mathcal{U} \in \mathrm{DCart}$

$\mathrm{VBun}_d(\mathcal{U}) = B(GL(\mathbb{R}^d))$

(w/ different meaning than before!)

$= B(\mathrm{End}(\mathbb{R}^d))$

$\mathrm{End}(\mathbb{R}^d)$
$d \times d$ matrices,
vector space $\mathbb{R}^{d^2}$
$\mathrm{Mat}_{d \times d}$

Take $\mathrm{End}(\mathbb{R}^d) \otimes \mathcal{O}(\mathcal{U})$

algebra of functions on $\mathcal{U}$ valued in $d \times d$ matrices

totally formal curly O

analog of the previous

$\mathrm{Mor}(*,*) = C^\infty(\mathcal{U}, \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^d))$

How do we place them as morphisms in a category? (Just like $\mathrm{Mor}(*,*)$ are morphisms of the previous category)

Recall, DOLD-KAN

next page $\longrightarrow$

$\mathrm{sSet} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

$\mathrm{sAb} = \mathrm{Fun}(\Delta^{op}, \mathrm{Ab})$

$\mathrm{Ch}_{\geq 0}$ chain complexes over ~~abelian groups Mod_R~~ abelian groups

Recall,

DOLD-KAN
Correspondence

$$\begin{tikzcd} sSet \arrow[d, shift right] \arrow[d, phantom, "\dashv"] \\ sMod_R \arrow[u, shift right] \arrow[d, shift right, "N"'] \arrow[d, phantom, "\cong"] \\ Ch_{\geq 0}^{Mod_R} \arrow[u, shift right, "\Gamma"'] \end{tikzcd}$$

$sSet = Fun(\Delta^{op}, set)$

adjoint functors

$sMod_R = Fun(\Delta^{op}, Mod_R)$

Γ (Dold Kan functor)

N (normalized chains)

$Ch_{\geq 0}^{Mod_R}$: chain complexes over modules over R

$$U \longmapsto End(\mathbb{R}^d) \otimes \mathcal{O}(U) \xmapsto{\Gamma} \Gamma(End(\mathbb{R}^d) \otimes \mathcal{O}(U))$$

$$\Gamma(End(\mathbb{R}^d) \otimes \mathcal{O}(U)) \in Mon(sSet)$$

a monoid

... and there is a map $Mon(sSet) \xrightarrow{B} sSet$
(the "delooping")

In Conclusion,
we defined Vector Bundles on Derived Differentiable Stacks!

Prerequisite: define on Dcart FIRST!

Strategy:
1) Define for Dcart
2) Prove descent
3) ???
4) Prove It!

$\boxed{\text{Open Problems:}}$

Problem #1:
Write this down as a paper

Problem #2:
define Ω (diff forms) in this setting

Problem #3: define $VBun^{\nabla}$ ← with connection!

Problem #4: define Chern-Weil homomorphism