$\mathcal{C}$ - category $Q: \Delta \to \mathcal{C}$ $Q[n] =: Q^n$

$$\underbrace{\mathcal{C} \longrightarrow sSet}_{Sing^Q_\bullet} \qquad \underline{X} \mapsto Hom_{\mathcal{C}}(Q^\bullet, X)$$

$\mathcal{C} = Top$ $|\Delta^n| = \{ - - - - \}$

$\mathcal{D} \in Cat$ $\Delta \hookrightarrow Cat$

$\{0 \to 1 \to 2 \to \dots \to n\}$ $\mathcal{D}$

$$\underline{Sing^\Delta_\bullet(\mathcal{D}) = N_\bullet(\mathcal{D})} \qquad \bullet \to \bullet \to \bullet \to \bullet \cdots \to$$

N (nerve) — $|-|$ (realization) adjunction.

Proposition: Assume that $\mathcal{C}$ is a category $Q^\bullet: \Delta \to \mathcal{C}$ $Sing^Q_\bullet: \mathcal{C} \to sSet$. $\mathcal{C}$ is cocomplete ($\exists$ all colimits) then there a left adjoint functor $|-|^Q_\bullet$.

P-f:

$S_\bullet$ - sSet is good if the functor is corepresentable.

$$\mathcal{C} \ni C \longmapsto Hom_{sSet}(S_\bullet, Sing^Q_\bullet(C))$$

$$\shortparallel$$

$\exists\ |S_\bullet|^Q \in \mathcal{C}$ s.t. $Hom_{\mathcal{C}}(\underline{|S_\bullet|^Q}, C)$

$\underline{\Delta^n}$ $C \mapsto Hom_{sSet}(\Delta^n, Sing^Q_\bullet(C)) =$

$= Hom_{\mathcal{C}}(Q^n, C)$ $|\Delta^n|^Q \rightsquigarrow \underline{Q^n}$

$\mathcal{C}$ is cocomplete and the set of all good sSets

collection

contains all simplices ⇒ all sSets are good.

$$\underbrace{|\cdot|^{Q} : sSet \to \mathcal{C}.}$$

□

Kan complexes

Def: $S_\bullet \in sSet$ is a Kan complex iff

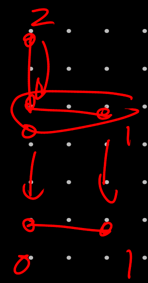
(*) $\forall\ \Lambda^n_i \xrightarrow{f} S_\bullet \quad \exists\ \hat{f} : \Delta^n \to S_\bullet.$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Not an example: Δ^n it is not a Kan complex.

Δ^1 •—•

2-horn $$\begin{tikzcd} \Lambda^2_0 \arrow[d] \\ \Delta^1 \end{tikzcd}$$



$0 \mapsto 0$

$1 \mapsto 1$

$2 \mapsto 0$

Any 1-dim sSet (i.e. a graph) is not a Kan complex.

Proposition: $X \in Top$. $Sing_\bullet(X)$ is a Kan complex.

Pf: Assume we have a map $\sigma_0 : \Lambda^n_i \to Sing_\bullet(X)$

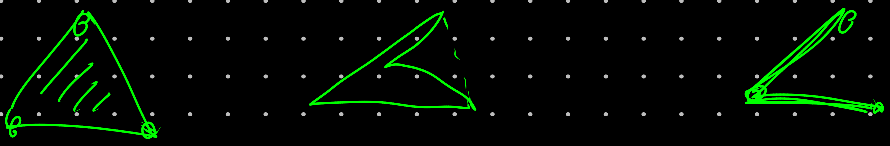
$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d, hook] & Sing_\bullet(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r, hook] \arrow[dr, "|\sigma_0|"'] & |\Delta^n| \arrow[d, dashed, "|\sigma|"] \\ & X \end{tikzcd}$$

$|\sigma| = |\sigma_0| \circ ?$

$$r(t_0 \ldots t_n) = (t_0 - c, \ldots, t_{i-1} - c, t_i + nc, t_{i+1} - c \ldots t_n - c)$$

$$c = \min\{t_0 \ldots t_{i-1}, t_{i+1} \ldots t_n\}$$



□

Proposition: $N_\cdot(\mathcal{C})$ - is a Kan complex $\Rightarrow$ $\mathcal{C}$ is a groupoid.

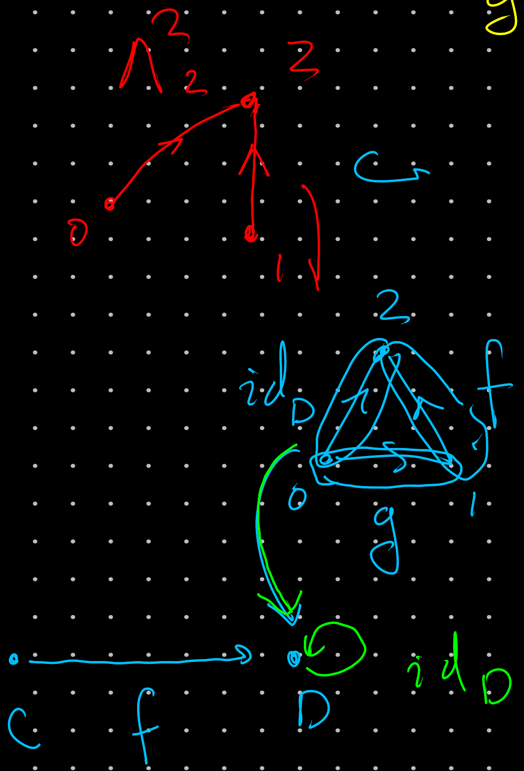
P-f: $\Gamma\; \mathrm{Hom}_{sSet}(\Delta^2, N_\cdot(\mathcal{C})) \xrightarrow{1-1} \mathrm{Hom}_{sSet}(\Lambda^2_2, N_\cdot(\mathcal{C}))$

$\exists\, \sigma \in N_2(\mathcal{C})$ s.t. $\underline{d_0(\sigma)} = f$

f

$d_1(\sigma) = id_D$

$g := d_2(\sigma)$



$f \cdot g = id_D$

$hf = id_C$

$\boxed{h = g}$

□

$\Delta^{op} \to \underline{Set}$

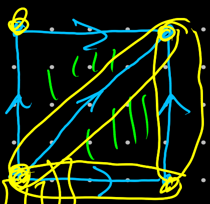
complete and cocomplete

sSets have all limits and colimits

$(\lim(\mathcal{F}))_n = \lim(\mathcal{F})_n$

$\mathcal{F} : \mathcal{J} \to sSet$

$\lim\left(\overset{\Delta^1}{\cdot} \quad \overset{\Delta^1}{\cdot}\right) = \underline{\Delta^1} \times \underline{\Delta^1} = X_\bullet$



$X_0 = \{([0],[0]), ([0],[1]), ([1],[0]), ([1],[1])\}$.

$e, f \in \{[0\ 0], [0\ 1], [1\ 1]\}$

9 1 - simplices in $X_\bullet$

$d_i(x, y) = (d_i x, d_i y)$

$\underbrace{d_0([0,1], [0,1])} = (d_0[0,1], d_0[0,1]) = ([1],[1])$

$\underline{d_1}(\quad) = (0,0)$

2 - simplices 16

14 degenerate of them $s_0 s_0 = s_1 s_0$

2 non - degenerate simplices

3 - simplices and higher are degenerate:

$s_0 s_0 \underset{\substack{\uparrow \\ \text{1-simplex}}}{\underline{\overset{\Delta^1}{e}}}, s_1 s_0 e, \ldots\ldots s_2 s_1 e,$

$(s_1 s_0 e, s_1 s_0 f) = \underline{s_1}(s_0 e, s_0 f)$

⋮

Simplicial homotopies.

$f, g : X \to Y$

$H : \Delta^1 \times X \to Y$

$f = H \circ (d_1 \times X)$

$g = H \circ (d_0 \times X)$

$H : X \to Y^{\Delta^1}$

$f = Y^{d_1} \circ H$

$g = Y^{d_0} \circ H$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$f \circ g \sim id_X$

$g \circ f \sim id_Y$

Y should be Kan.

Kan fibration „=" Serre fibration

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & Y \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

„relative Kan complex"

$X = *$

$$\begin{tikzcd} Y \arrow[d] \\ * \end{tikzcd}$$

Simplicial Whitehead theorem.

$f : X \to Y$

Kan complexes

f is a simp. homotopy equiv. if we have a lifting property for any com. square:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

comm. up to homotopy fixing the boundary

Sketch of proof: → Induction on $sk^n Y$ Construct homotopy inverse simplex-by-simplex

$f_* : \pi_0(X) \xrightarrow{1-1} \pi_0(Y)$

$f_* : \pi_n(X, x) \xrightarrow{\simeq} \pi_n(Y, f(x))$

Homotopy groups

Pointed Kan complex is a Kan complex S + $s \in S_0$

$$\begin{tikzcd} \partial\Delta^n \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d] \\ * \arrow[r] & \underline{\Delta^n/\partial\Delta^n} \end{tikzcd}$$

$\underline{\$^n}$

$\pi_n(S, s) = \hom_{sSet}^{\circledcirc}(\$^n, S) / \sim$

$\boxed{\text{Group str-re on } \pi_{n>0}(S,s) \quad \pi_{n>1}(S,s) \text{ are abelian}}$

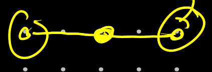
$\boxed{\text{Kerodon } 3.2.3}$

$(X, x) \xrightarrow{f} (S, s)$ – Kan fib. of pointed Kan complexes

$- - - - - - - \to \pi_0(X_s, x) \to \pi_0(X, x) \to \pi_0(S, s)$

Ex-functor

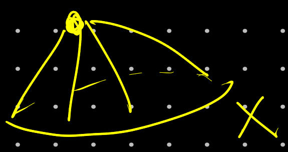
$sd : sSet \to sSet$



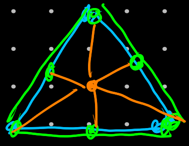
$sd\, \Delta^0 = \Delta^0 \qquad sd\, \Delta^n = C(sd(\partial\Delta^n))$

$C : \Delta \to \Delta \qquad [m] \mapsto [m] \cup \{*\}$

$sSet \to sSet$



$$N_\bullet(P(\Delta^n))$$



$$Ex(X)_n = \hom_{sSet}(sd\,\Delta^n, X)$$

$$\|$$

$$\hom_{sSet}(\Delta^n, Ex(X)_\bullet)$$

Def: $id_{sSet} \Rightarrow Ex : sSet \to sSet$

$X \mapsto Ex(X)$ << last vertex maps >>

$\boxed{sd(X) \to X}$

$sd\,\Delta^0 = \Delta^0 \xrightarrow[id]{} \Delta^0$

$sd\,\Delta^n = C(sd\,\partial\Delta^n) \to \Delta^n$

Concrete description of simplices in Ex

n-simplices $\longleftrightarrow$ $\underbrace{sd\,\Delta^n \to X}_{\text{simp. maps}}$

$X \hookrightarrow Ex\,X$ $\quad \sigma \in X_n \rightsquigarrow sd\,\Delta^n \to \Delta^n \xrightarrow[\sigma]{} X$

$$Ex(X) = Sing^T_\bullet(X) \qquad T([n]) = N_\bullet(P([n]))$$

Def: Ex^∞ functor

$$Ex^\infty(X) = \operatorname{colim}(X \to Ex(X) \to Ex^2(X) \to \dots)$$

$$sSet \xrightarrow{Ex^\infty} sSet$$

Proposition: Ex^∞ preserves finite limits, filtered colimits, monomorphisms of sSets.

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[dr, dashed] & Ex^\infty(X) \\
& Ex^m(X) \arrow[u]
\end{tikzcd}$$

Ex is right adjoint and we take fin. limit $\Rightarrow$ Ex^∞ preserves finite limits. ($sd^n \Delta^n$ is a compact simplicial set

„covered" by finitely many non-degenerate simplices, or

$$\begin{tikzcd}
S \arrow[r] \arrow[rr, dashed, bend right] & \forall\ \operatorname{colim}(X_1 \to X_2 \to \dots \to & X_n \to\ )
\end{tikzcd}$$

$Ex^n(X)$

Proposition: $Ex^\infty(X)$ is a Kan complex

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & Ex^\infty X \\
\Delta^n \arrow[ur, dashed]
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & Ex^m X \arrow[d] \arrow[r] & Ex^\infty X \\
\Delta^n \arrow[r] & Ex^{m+1} X \arrow[ur]
\end{tikzcd}$$

$X' := Ex^{m-1}(X)$

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & Ex X' \arrow[d] \\
\Delta^n \arrow[r] & Ex^2 X'
\end{tikzcd}$$

$$\begin{tikzcd}
& sd \Lambda^n_k \arrow[dr] & \\
sd^2 \Lambda^n_k \arrow[ur] \arrow[dr] & ? & X' \\
& sd^2 \Delta^n \arrow[uu, dashed] \arrow[ur, "e'"] &
\end{tikzcd}$$