The geometric cobordism hypothesis
Lecture 3: Locality

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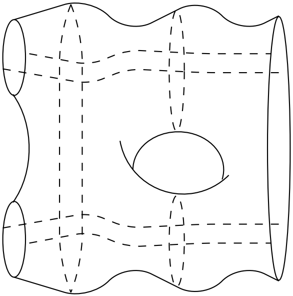
These slides: https://dmitripavlov.org/lecture-3.pdf



Overview

- Yesterday: definitions
- Today: locality and how to use it to prove one half of the GCH
- Tomorrow: the framed GCH (the other half)

Review of smooth symmetric monoidal (∞, d)-categories

- Cart is the site of cartesian spaces and smooth maps (controls smoothness);
- Γ is the opposite category of pointed finite sets (controls monoidal products);
- $\Delta^{\times d}$ is the d-fold product of categories of nonempty ordered finite sets (controls compositions in d directions);

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A smooth symmetric monoidal (∞, d)-category is a functor

$$\mathcal{V}\colon (\mathsf{Cart} \times \Gamma \times \Delta^{\times d})^{\mathrm{op}} \to \mathrm{sSet}.$$

- The injective fibrancy condition;
- The sheaf condition for Cart (ensures gluing of smooth families of objects and morphisms);
- The Segal condition for Γ (ensures multiplication of objects can be performed in a unique way);

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- A Segal condition for every factor of Δ (ensures composition);
- A completeness condition for every factor of Δ (eliminates a redundancy in the encoding of invertible morphisms);

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- A completeness condition for Δ (invertible morphisms);
- A globularity condition for every factor of Δ with its subsequent factors (eliminates a redundancy in the encoding of noninvertible morphisms);

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- A Segal condition for every factor of Δ (ensures composition);
- A completeness condition for Δ (invertible morphisms);
- A globularity condition for Δ (eliminates a redundancy in the encoding of noninvertible morphisms);
- A dualizability condition for Γ and every factor of Δ except the last one (explained in Lecture 4).

Review of geometric structures and bordism categories

- FEmb_d is the site of smooth families of d-manifolds and fiberwise open embeddings;
- Geometric structures $\mathcal{S}$ are simplicial presheaves on FEmb_d;
- $\mathfrak{Bord}_d^{\mathcal{S}}$ is the smooth symmetric monoidal (∞, d)-category of bordisms with geometric structure $\mathcal{S}$;
  - Bordisms come in smooth families over Cart, can be pulled back and glued;
  - Monoidal product: disjoint union of bordisms;
  - Composition: gluing of bordisms along germs;
  - Cuts can be moved using higher invertible morphisms;
  - Higher gauge transformations implemented using higher invertible morphisms.
- $\mathcal{V}$: smooth symmetric monoidal (∞, d)-category of values;
- $\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V})$.

Review of statements

Theorem (G.–P.)

Given $\mathcal{V}$ and $d \geq 0$, the functor $\mathsf{FFT}_{d,\mathcal{V}}$

$$\mathrm{sPSh}(\mathsf{FEmb}_d)^{\mathrm{op}}_{\check{\mathsf{C}}\text{-inj}} \to \mathrm{sSet}, \quad \mathcal{S} \mapsto \mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V})$$

is an $(\infty, 1)$-sheaf, i.e., preserves homotopy limits.

This follows from the following result.

Theorem (G.–P.)

Given $d \geq 0$, the functor

$$\mathrm{sPSh}(\mathsf{FEmb}_d)_{\check{\mathsf{C}}\text{-inj}} \to \mathrm{sPSh}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d})_{\mathsf{loc}}, \qquad \mathcal{S} \mapsto \mathfrak{Bord}_d^{\mathcal{S}}$$

is an $(\infty, 1)$-cosheaf, i.e., preserves homotopy colimits.

The geometric cobordism hypothesis: Part I

Theorem

Given $d \geq 0$, a geometric structure $\mathcal{S}$, and a smooth symmetric monoidal (∞, d)-category $\mathcal{V}$, we have

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathrm{Map}(\mathcal{S}, R_d(\mathcal{V})),$$

$$R_d(\mathcal{V})(W \to U) = \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{W \to U}, \mathcal{V}).$$

where

$$R_d \colon \mathrm{sPSh}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d}) \to \mathrm{sPSh}(\mathsf{FEmb}_d)$$

is the right adjoint of $\mathfrak{Bord}_d$:

$$R_d(\mathcal{V})(W \to U) = \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{W \to U}, \mathcal{V}) = \mathsf{FFT}_{d,\mathcal{V}}(W \to U).$$

The geometric cobordism hypothesis: Part I and II

Part II of GCH (Lecture 4): $R_d(\mathcal{V}) \xrightarrow{\sim} \mathcal{V}^{\vee,\times}$, write $\mathcal{V}_d^\times = R_d(\mathcal{V})$.

Theorem (GCH, Part I and II)

Given $d \geq 0$, a geometric structure $\mathcal{S}$, and a smooth symmetric monoidal (∞, d)-category $\mathcal{V}$, we have (Part I)

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times),$$

where (Part II) $\mathcal{V}_d^\times$ is the smooth ∞-groupoid of fully dualizable objects in $\mathcal{V}$ equipped with an action of the ∞-group $\mathrm{O}(d)$ (implemented as a simplicial presheaf on FEmb_d).

Application: Classifying spaces of FFTs

Theorem (G.–P.)

Given $d \geq 0$, $\mathcal{V} \in \mathrm{C}^\infty\mathrm{Cat}^{\otimes}_{\infty,d}$, and an ∞-cosheaf $F\colon \mathsf{Man} \to \mathrm{sPSh}(\mathsf{FEmb}_d)$ (example: $F(M) = M \times \mathrm{Riem}$), set

$$\mathsf{FFT}_{d,\mathcal{V},F}\colon \mathsf{Man}^{\mathrm{op}} \to \mathrm{sSet}, \qquad M \mapsto \mathsf{FFT}_{d,\mathcal{V}}(F(M)),$$

$$(\mathrm{B}_\int \mathsf{FFT}_{d,\mathcal{V},F})(M) = \operatorname*{hocolim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} \mathsf{FFT}_{d,\mathcal{V}}(\mathbf{\Delta}^n \times M).$$

Then

$$(\mathrm{B}_\int \mathsf{FFT}_{d,\mathcal{V},F})(M) \xrightarrow{\simeq} \mathbf{R}\,\mathrm{Map}(M, (\mathrm{B}_\int \mathsf{FFT}_{d,\mathcal{V},F})(\mathbf{R}^0)).$$

$$\mathsf{FFT}_{d,\mathcal{V},F}[M] \cong [M, (\mathrm{B}_\int \mathsf{FFT}_{d,\mathcal{V},F})(\mathbf{R}^0)].$$

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$$\mathsf{FFT}_{d,\mathcal{V},F}[M] \cong [M, (\mathrm{B}_\int \mathsf{FFT}_{d,\mathcal{V},F})(\mathbf{R}^0)].$$

Proof: Combine Locality and the following result.

Application: Classifying spaces of FFTs

Proof: Combine Locality and the following result.

Theorem (Berwick-Evans–Boavida de Brito–P.)

Given

$$F\colon \mathsf{Man}^{\mathrm{op}} \to \mathrm{sSet},$$

set

$$(\mathrm{B}_{\int} F)(M) = \operatorname*{hocolim}_{n \in \boldsymbol{\Delta}^{\mathrm{op}}} F(\boldsymbol{\Delta}^n \times M).$$

If F is an ∞-sheaf, then so is $\mathrm{B}_{\int} F$ and

$$(\mathrm{B}_{\int} F)(M) \xrightarrow{\simeq} \mathbf{R}\,\mathrm{Map}(M, (\mathrm{B}_{\int} F)(\mathbf{R}^0)).$$

Can replace sSet by any algebraic $(\infty, 1)$-category (e.g., connective ring spectra, connective chain complexes, etc.).

The structure of the proof

Theorem

The left derived functor of a left Quillen functor preserves homotopy colimits.

Theorem (G.–P.)

Given $d \geq 0$, the functor

$$\mathrm{sPSh}(\mathsf{FEmb}_d)_{\check{\mathsf{C}}\text{-inj}} \to \mathrm{sPSh}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d})_{\mathsf{loc}}, \qquad \mathcal{S} \mapsto \mathfrak{Bord}_d^{\mathcal{S}}$$

is a left Quillen functor. In our case: preserves monomorphisms and local weak equivalences.

Review of reflective localizations

Input data:

- *P*: a category of presheaves: $P = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$;
- $\check{C}$: Čech sieves of covering families

Output data and properties:

- $P_{\check{C}}$: $X \in P$ is $\check{C}$-local if $\mathrm{Map}(g, X)$ is an iso for all $g \in \check{C}$;
- S: $f \in P^{\rightarrow}$ is $\check{C}$-local if $\mathrm{Map}(f, X)$ is an iso for all $X \in P_{\check{C}}$;
  - $a\colon P \to P[S^{-1}]$ has a fully faithful right adjoint ι;
  - $P_{\check{C}}$ is the essential image of ι;
  - $P[S^{-1}]$: same objects as P, more isomorphisms;
  - $\mathrm{Ladj}(P[S^{-1}], Q) = \{F \in \mathrm{Ladj}(P, Q) \mid F(\check{C}) \subset \text{isos in } Q\}$;
  - colimits (and limits) in $P[S^{-1}]$ computed objectwise.

Review of left Bousfield localizations

Input data:

P: relative category of simplicial presheaves: $P = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{sSet})$;

$\check{C}$: Čech nerves of covering families

Output data and properties:

$P_{\check{C}}$: $X \in P$ is $\check{C}$-local if $\mathbf{R}\,\mathrm{Map}(g, X)$ is a weak eq for all $g \in \check{C}$;

S: $f \in P^{\to}$ is $\check{C}$-local if $\mathbf{R}\,\mathrm{Map}(f, X)$ is a weak eq for all $X \in P_{\check{C}}$;

- $a\colon P \to \mathcal{L}_S P$ has a homotopically f-f right Quillen adjoint ι;
- $P_{\check{C}}$ is the essential image of $\mathbf{R}\iota$.
- $\mathcal{L}_S P$: same category as P, more weak equivalences.
- $\mathrm{LQF}(\mathcal{L}_S P, Q) = \{F \in \mathrm{LQF}(P, Q) \mid \mathbf{L}F(\check{C}) \subset W_Q\}$.
- homotopy colimits (and limits) in $\mathcal{L}_S P$ computed objectwise.

Specialization to $\mathfrak{Bord}_d$

- $P = \mathrm{sPSh}(\mathsf{FEmb}_d)_{\mathsf{inj}}$, $\mathcal{L}_S P = \mathrm{sPSh}(\mathsf{FEmb}_d)_{\check{\mathsf{C}}\text{-}\mathsf{inj}}$;
- $\check{C}$: Čech nerves of open covers in FEmb_d;
- $Q = \mathrm{sPSh}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d})_{\mathsf{loc}}$;
- $\mathfrak{Bord}_d \colon \mathrm{sPSh}(\mathsf{FEmb}_d)_{\check{\mathsf{C}}\text{-}\mathsf{inj}} \to \mathrm{sPSh}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d})_{\mathsf{loc}}$.

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Proposition (G.–P.)

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Theorem (G.–P.)

Given $d \geq 0$, the left derived functor of the left Quillen functor

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sends Čech nerves of open covers in FEmb_d *to weak equivalences.*

The formal component

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Proof: a formal observation on the construction of $\mathfrak{Bord}_d^{\mathcal{S}}$.

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- $\mathfrak{Bord}_d$ preserves small colimits, hence is a left adjoint;

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- $\mathfrak{Bord}_d$ preserves monomorphisms;
- $\mathfrak{Bord}_d$ preserves objectwise weak equivalences.

The codescent property

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sends the Čech nerve of an open cover $\{W_a \to U_a\}_{a \in A}$ of $(W \to U) \in \mathsf{FEmb}_d$ to a weak equivalence:

$$\operatorname*{hocolim}_{n \in \Delta^{\mathrm{op}}} \coprod_{\alpha : [n] \to A} \mathfrak{Bord}_d^{W_\alpha \to U_\alpha} \xrightarrow{\sim} \mathfrak{Bord}_d^{W \to U},$$

where $W_\alpha = W_{\alpha_0} \cap \cdots \cap W_{\alpha_n}$.

The codescent property: main steps

$$\operatorname*{hocolim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha} \xrightarrow{\sim} \mathfrak{Bord}_d^{W\to U}$$

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The codescent property: main steps

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Step 1 Replace hocolim by colim (use Reedy cofibrancy of the diagram):

$$\operatorname*{hocolim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha} \xrightarrow{\sim} \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha}$$

The codescent property: main steps

$$\operatorname*{hocolim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha} \xrightarrow{\sim} \mathfrak{Bord}_d^{W\to U}$$

Step 1 Replace $\mathrm{hocolim}$ by colim

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Step 3 Introduce a filtration (on n-dimensional stalks)

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha} \to B_0 \to \cdots \to B_d \to \mathfrak{Bord}_d^{W\to U}.$$

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Step 4 Prove all maps in the filtration are weak equivalences.

The codescent property: filtration

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha} \to B_0 \to \cdots \to B_d \to \mathfrak{Bord}_d^{W\to U}.$$

Definition

Given $d \geq 0$ and $(W = \mathbf{R}^d \times U \to U) \in \mathrm{FEmb}_d^{\mathrm{op}}$, the set $\mathfrak{Bord}_d^{\mathbf{R}^d\times U\to U}(V, \langle\ell\rangle, \mathbf{m})_n$ has elements:

- a smooth manifold M;
- a V-family of embeddings $M \to \mathbf{R}^d$;
- a $V \times \mathbf{\Delta}^n$-family of cut tuples with $m_1 \times \cdots \times m_d$ cells;
- $P\colon M \to \langle\ell\rangle$;
- smooth map $V \to U$;

The codescent property: filtration

Definition

We define $B_i(\langle \ell \rangle, \mathbf{m}) \subset \mathfrak{Bord}_d^{W \to U}(\langle \ell \rangle, \mathbf{m})$ as follows.

- An n-simplex is in B_i if for every $t \in \Delta^n$ the corresponding bordism over t satisfies the conditions given below.
- $x \in B_0(\mathbf{m}, \langle \ell \rangle)$ is given by a germ $f \colon M \Rightarrow W$ around $\mathrm{core}[0, \mathbf{m}]$ that maps every connected component of the germ into some $W_a \subset W$.
- $i > 0$: $x \in B_i(\mathbf{m}, \langle \ell \rangle)$ if it admits a cut tuple $\widetilde{C}$ that contains the cut tuple of x (in the ith direction) such that for each $0 \leq j < m_i$, the bordism with the same data as x, but with cut tuple in the ith direction given by two successive cuts $\widetilde{C}_j$ and $\widetilde{C}_{j+1}$, belongs to B_{i-1}.

Filtration: Step 0

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$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha\to U_\alpha} \to B_0 \to \cdots \to B_d \to \mathfrak{Bord}_d^{W\to U}.$$

- An n-simplex is in $B_0(\mathbf{m},\langle\ell\rangle)$ if it is given by a germ $f\colon M \Rightarrow W$ around $\mathrm{core}[0,\mathbf{m}]$ that maps every connected component of the germ into some $W_a \subset W$.
- colim: Same, but f maps the entire core into some $W_a \subset W$.

Filtration: Step 0

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{\alpha:[n]\to A} \mathfrak{Bord}_d^{W_\alpha \to U_\alpha} \to B_0 \to \cdots \to B_d \to \mathfrak{Bord}_d^{W\to U}.$$

- B_0: every connected component of the bordism factors through some $W_a \subset W$.
- colim: the entire bordism factors through some $W_a \subset W$.

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Proposition

The map $\mathrm{colim} \to B_0$ *is a weak equivalence in* $\mathrm{sPSh}(\Gamma \times \Delta^{\times d})_{\mathrm{loc}}$.

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- Evaluate on an arbitrary object of $\Delta^{\times d}$, obtaining a map in $\mathrm{sPSh}(\Gamma)$;

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Proof.

- Evaluate on an arbitrary object of $\Delta^{\times d}$, obtaining a map in $\mathrm{sPSh}(\Gamma)$;
- Introduce a filtration on B_0: B_0^k is the union of B_0^{k-1} and the part of B_0 whose bordisms have at most k connected components;

Filtration: Step 0

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Proof.

- Evaluate on an arbitrary object of $\Delta^{\times d}$, obtaining a map in $\mathrm{sPSh}(\Gamma)$;
- Introduce a filtration on B_0: B_0^k is the union of B_0^{k-1} and the part of B_0 whose bordisms have at most k connected components;
- Present every map $B_0^{k-1} \to B_0^k$ as a transfinite composition of cobase changes of generating acyclic cofibrations of Γ-objects in simplicial sets.

□

Filtration: Step 1

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- B_0: every connected component of the bordism factors through some $W_a \subset W$.
- B_i: bordisms that can be chopped in the ith direction so that every piece belongs to B_{i-1}.

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Proposition

The map $B_{i-1} \to B_i$ is a weak equivalence in $\mathrm{sPSh}(\Gamma \times \Delta^{\times d})_{\mathsf{loc}}$ for every $i > 0$.

Filtration: Step 1

Proposition

The map $B_{i-1} \to B_i$ *is a weak equivalence in* $\mathrm{sPSh}(\Gamma \times \Delta^{\times d})_{\mathsf{loc}}$ *for every* $i > 0$.

Proof.

Filtration: Step 1

Proposition

The map $B_{i-1} \to B_i$ *is a weak equivalence in* $\mathrm{sPSh}(\Gamma \times \Delta^{\times d})_{\mathrm{loc}}$ *for every* $i > 0$.

Proof.

- Evaluate $B_{i-1} \to B_i$ on an arbitrary object X of $\Gamma \times \Delta^{\times(d-1)}$, obtaining a map $B_{i-1}(X) \to B_i(X)$ in $\mathrm{sPSh}(\Delta)$;

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- Extract the kth simplicial degree (for some $k \geq 0$), obtaining a map in $\mathrm{PSh}(\Delta) = \mathrm{sSet}$;
- The resulting simplicial set has
  - vertices: germs of cuts (embedded in W);
  - edges: bordisms between cuts (embedded in W);
  - 2-simplices: composition of bordisms;
  - everything is in smooth families indexed by Δ^k;
  - bordisms must belong to B_{i-1} respectively B_i.

Want to show: $B_{i-1} \to B_i$ is a categorical weak equivalence in the Joyal model structure on simplicial sets. □

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- Answer: Dugger–Spivak necklace categories.

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- Observation: the ambient composed bordism never changes $\Longrightarrow$ can fix it in advance.

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- This implies $B_{i-1} \to B_i$ is a weak equivalence.

The big picture

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- How does this help us to show contractibility of necklace categories?

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- At the final step, all cuts have been collapsed to the source cut of M.