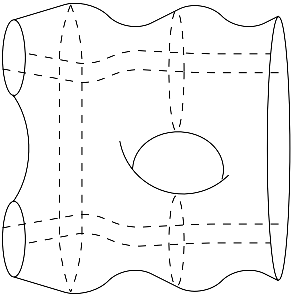
The geometric cobordism hypothesis
Lecture 2: Definitions

Daniel Grady, Dmitri Pavlov (Texas Tech University, Lubbock, TX)

These slides: `https://dmitripavlov.org/lecture-2.pdf`



Outline

1. (∞, d)-categories via d-fold complete Segal spaces.

Outline

1. (∞, d)-categories via d-fold complete Segal spaces.
2. Symmetric monoidal structure via Γ-spaces.

Outline

1. (∞, d)-categories via d-fold complete Segal spaces.
2. Symmetric monoidal structure via Γ-spaces.
3. Smooth structure via cartesian spaces.

Outline

1. (∞, d)-categories via d-fold complete Segal spaces.
2. Symmetric monoidal structure via Γ-spaces.
3. Smooth structure via cartesian spaces.
4. The bordism category.

Outline

1. (∞, d)-categories via d-fold complete Segal spaces.
2. Symmetric monoidal structure via Γ-spaces.
3. Smooth structure via cartesian spaces.
4. The bordism category.
5. Geometric structures on bordisms.

Outline

1. (∞, d)-categories via d-fold complete Segal spaces.
2. Symmetric monoidal structure via Γ-spaces.
3. Smooth structure via cartesian spaces.
4. The bordism category.
5. Geometric structures on bordisms.
6. Descent for field theories (locality).

Models: We use model categories for the above gadgets.

Outline

1. (∞, d)-categories via d-fold complete Segal spaces.
2. Symmetric monoidal structure via Γ-spaces.
3. Smooth structure via cartesian spaces.
4. The bordism category.
5. Geometric structures on bordisms.
6. Descent for field theories (locality).

Models: We use model categories for the above gadgets. Model structures will always be given by a left Bousfield localization of some category of presheaves on a small category C.

Higher categories

Higher categories

- Everyone knows what a category is.

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms"

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms"

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even…

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise.

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise. Categories internal to categories (i.e. a double category).

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise. Categories internal to categories (i.e. a double category). Categories enriched in categories (i.e., a strict 2-category).

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise. Categories internal to categories (i.e. a double category). Categories enriched in categories (i.e., a strict 2-category). By induction, one defines n-categories.

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise. Categories internal to categories (i.e. a double category). Categories enriched in categories (i.e., a strict 2-category). By induction, one defines n-categories.
- These notions are not good enough in practice!

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise. Categories internal to categories (i.e. a double category). Categories enriched in categories (i.e., a strict 2-category). By induction, one defines n-categories.
- These notions are not good enough in practice! Many naturally occurring examples are not strict (e.g., fundamental 2-groupoid).

Higher categories

- Everyone knows what a category is.
- Higher categories seek to generalize a category (or 1-category) to include "morphisms between morphisms" and even "morphisms between morphisms between morphisms" and even...
- There are different ways of making this precise. Categories internal to categories (i.e. a double category). Categories enriched in categories (i.e., a strict 2-category). By induction, one defines n-categories.
- These notions are not good enough in practice! Many naturally occurring examples are not strict (e.g., fundamental 2-groupoid).
- Keeping track of the coherence data is notoriously annoying (see Todd Trimble's weak 4-category!).

What is an (∞, d)-category?

What is an (∞, d)-category?

We use the d-fold Segal space formalism (Segal, Rezk, Barwick).

What is an (∞, d)-category?

We use the d-fold Segal space formalism (Segal, Rezk, Barwick).
Let's start with $(\infty, 1)$-categories (i.e. $d = 1$).

What is an (∞, d)-category?

We use the d-fold Segal space formalism (Segal, Rezk, Barwick).
Let's start with $(\infty, 1)$-categories (i.e. $d = 1$).

Definition

A Segal space is a simplicial object in sSet, i.e. a functor $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, such that

What is an (∞, d)-category?

We use the d-fold Segal space formalism (Segal, Rezk, Barwick).
Let's start with $(\infty, 1)$-categories (i.e. $d = 1$).

Definition

A Segal space is a simplicial object in sSet, i.e. a functor $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, such that

1. It is fibrant in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{inj}} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{Reedy}}$;

What is an (∞, d)-category?

We use the d-fold Segal space formalism (Segal, Rezk, Barwick).
Let's start with $(\infty, 1)$-categories (i.e. $d = 1$).

Definition

A Segal space is a simplicial object in sSet, i.e. a functor $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, such that

1. It is fibrant in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{inj}} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{Reedy}}$;
2. (Segal's special Δ condition) it is local with respect to the maps

$$\Delta^n \amalg_{\Delta^0} \Delta^m \to \Delta^{n+m},$$

What is an (∞, d)-category?

We use the d-fold Segal space formalism (Segal, Rezk, Barwick).
Let's start with $(\infty, 1)$-categories (i.e. $d = 1$).

Definition

A Segal space is a simplicial object in sSet, i.e. a functor $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$, such that

1. It is fibrant in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{inj}} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{Reedy}}$;
2. (Segal's special Δ condition) it is local with respect to the maps
$$\Delta^n \amalg_{\Delta^0} \Delta^m \to \Delta^{n+m},$$

What does this mean, morally?

- Segal's Δ condition (along with fibrancy) means that for each $n, m \in \mathbb{N}$, the square

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{p_{0,\ldots,m}} & X_m \\ {\scriptstyle p_{m,\ldots,m+n}}\downarrow & & \downarrow{\scriptstyle p_m} \\ X_n & \xrightarrow[p_0]{} & X_0 \end{array}$$

is a homotopy pullback square.

- Segal's Δ condition (along with fibrancy) means that for each $n, m \in \mathbb{N}$, the square

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{p_{0,\dots,m}} & X_m \\ \downarrow{\scriptstyle p_{m,\dots,m+n}} & & \downarrow{\scriptstyle p_m} \\ X_n & \xrightarrow[p_0]{} & X_0 \end{array}$$

  is a homotopy pullback square.

Morally, the space X_n is the space of composable n-chains of morphisms in X_1.

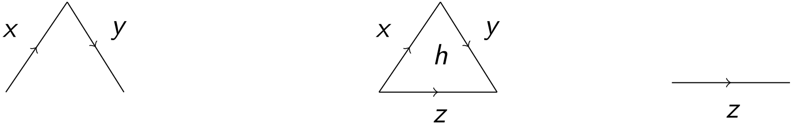
- Segal's Δ condition (along with fibrancy) means that for each $n, m \in \mathbb{N}$, the square

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{p_{0,\dots,m}} & X_m \\ {\scriptstyle p_{m,\dots,m+n}}\downarrow & & \downarrow{\scriptstyle p_m} \\ X_n & \xrightarrow[p_0]{} & X_0 \end{array}$$

  is a homotopy pullback square.

Morally, the space X_n is the space of composable n-chains of morphisms in X_1. For example, if $n = 2$:

$$X_1 \times_{X_0} X_1 \longleftarrow X_2 \longrightarrow X_1$$



Completeness

Completeness

- Let E be the nerve of the groupoid

$$\boxed{\mathrm{Ob} := \{x, y\} \quad \mathrm{Mor} := \{f\colon x \to y, f^{-1}\colon y \to x, \mathrm{id}_x, \mathrm{id}_y\}}$$

Completeness

- Let E be the nerve of the groupoid

$$\boxed{\mathrm{Ob} := \{x, y\} \quad \mathrm{Mor} := \{f\colon x \to y, f^{-1}\colon y \to x, \mathrm{id}_x, \mathrm{id}_y\}}$$

Definition

Completeness

- Let E be the nerve of the groupoid

$$\boxed{\mathrm{Ob} := \{x, y\} \quad \mathrm{Mor} := \{f : x \to y, f^{-1} : y \to x, \mathrm{id}_x, \mathrm{id}_y\}}$$

Definition

A complete Segal space is a Segal space that is local with respect to the canonical map $E \to \Delta^0$.

Completeness

- Let E be the nerve of the groupoid

$$\boxed{\mathrm{Ob}:=\{x,y\}\quad \mathrm{Mor}:=\{f\colon x\to y, f^{-1}\colon y\to x, \mathrm{id}_x, \mathrm{id}_y\}}$$

Definition

A complete Segal space is a Segal space that is local with respect to the canonical map $E\to\Delta^0$.

This amounts to forcing X_0 to be equivalent to the invertible part of $X_\bullet$.

Completeness

- Let E be the nerve of the groupoid

$$\boxed{\mathrm{Ob}:=\{x,y\} \quad \mathrm{Mor}:=\{f\colon x\to y, f^{-1}\colon y\to x, \mathrm{id}_x, \mathrm{id}_y\}}$$

Definition

A complete Segal space is a Segal space that is local with respect to the canonical map $E \to \Delta^0$.

This amounts to forcing X_0 to be equivalent to the invertible part of $X_\bullet$.

Definition

An $(\infty, 1)$-category is a complete Segal space.

An example

- Let C be a small category.

An example

- Let C be a small category. Let $C^{\times}$ denote the maximal subgroupoid in C, i.e., the core.

An example

- Let C be a small category. Let $C^\times$ denote the maximal subgroupoid in C, i.e., the core.
- Define the functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ by $X_n = N((C^{[n]})^\times)$, for example,

$$X_0 = N(C^\times), \quad X_1 = N((C^\to)^\times).$$

An example

- Let C be a small category. Let $\mathsf{C}^\times$ denote the maximal subgroupoid in C, i.e., the core.
- Define the functor $X\colon \Delta^{\mathrm{op}} \to \mathsf{sSet}$ by $X_n = N((\mathsf{C}^{[n]})^\times)$, for example,

$$X_0 = N(\mathsf{C}^\times), \quad X_1 = N((\mathsf{C}^\to)^\times).$$

- One can check this gives a Segal space.

An example

- Let C be a small category. Let $\mathsf{C}^\times$ denote the maximal subgroupoid in C, i.e., the core.
- Define the functor $X\colon \Delta^{\mathrm{op}} \to \mathsf{sSet}$ by $X_n = N((\mathsf{C}^{[n]})^\times)$, for example,

$$X_0 = N(\mathsf{C}^\times), \quad X_1 = N((\mathsf{C}^\to)^\times).$$

- One can check this gives a Segal space. It is complete, since the functor $C^\times \to (C^\to)^\times$ sending

$$\begin{array}{c} x \\ \downarrow{\scriptstyle\cong} \\ y \end{array} \quad \mapsto \quad \begin{array}{ccc} x & \xrightarrow{\mathrm{id}} & x \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ y & \xrightarrow{\mathrm{id}} & y \end{array}$$

is fully faithful and essentially surjective onto the full subgroupoid on invertible morphisms in C.

An example

- Let C be a small category. Let $C^\times$ denote the maximal subgroupoid in C, i.e., the core.
- Define the functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ by $X_n = N((C^{[n]})^\times)$, for example,

$$X_0 = N(C^\times), \quad X_1 = N((C^{\to})^\times).$$

- One can check this gives a Segal space. It is complete, since the functor $C^\times \to (C^{\to})^\times$ sending

$$\begin{array}{c} x \\ \downarrow \cong \\ y \end{array} \quad \mapsto \quad \begin{array}{ccc} x & \xrightarrow{\mathrm{id}} & x \\ \downarrow \cong & & \downarrow \cong \\ y & \xrightarrow{\mathrm{id}} & y \end{array}$$

is fully faithful and essentially surjective onto the full subgroupoid on invertible morphisms in C.
- Note that the naive thing: $X_n = N(C)_n$ is not complete!

d-fold complete Segal spaces

d-fold complete Segal spaces

Definition

A d-fold complete Segal space is a d-fold simplicial object $X\colon (\Delta^{\mathrm{op}})^{\times d} \to \mathrm{sSet}$ such that

d-fold complete Segal spaces

Definition

A d-fold complete Segal space is a d-fold simplicial object $X\colon (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}((\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$

d-fold complete Segal spaces

Definition

A d-fold complete Segal space is a d-fold simplicial object $X\colon (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}((\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.

d-fold complete Segal spaces

Definition

A d-fold complete Segal space is a d-fold simplicial object $X\colon (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}((\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.

Something is still missing...

Something is still missing...

- We have encoded a d-categorical version of a double category.

Something is still missing...

- We have encoded a d-categorical version of a double category.
- We need a preferred composition direction to get an (∞, d)-category.

Something is still missing...

- We have encoded a d-categorical version of a double category.
- We need a preferred composition direction to get an (∞, d)-category.
- For $\mathbf{m} \in \Delta^{\times d}$, let $\hat{\mathbf{m}}$ be defined by

$$[\hat{m}_j] = \begin{cases} [0]\,, & \text{if there is } i < j \text{ with } m_i = 0 \\ [m_j]\,, & \text{otherwise.} \end{cases}$$

Something is still missing...

- We have encoded a d-categorical version of a double category.
- We need a preferred composition direction to get an (∞, d)-category.
- For $\mathbf{m} \in \Delta^{\times d}$, let $\hat{\mathbf{m}}$ be defined by

$$[\hat{m}_j] = \begin{cases} [0], & \text{if there is } i < j \text{ with } m_i = 0 \\ [m_j], & \text{otherwise.} \end{cases}$$

Definition

An (∞, d)-category is a d-fold complete Segal space that is local with respect to the maps $\mathbf{m} \to \hat{\mathbf{m}}$ (globularity condition).

Something is still missing...

- We have encoded a d-categorical version of a double category.
- We need a preferred composition direction to get an (∞, d)-category.
- For $\mathbf{m} \in \Delta^{\times d}$, let $\hat{\mathbf{m}}$ be defined by

$$[\hat{m}_j] = \begin{cases} [0]\,, & \text{if there is } i < j \text{ with } m_i = 0 \\ [m_j]\,, & \text{otherwise.} \end{cases}$$

Definition

An (∞, d)-category is a d-fold complete Segal space that is local with respect to the maps $\mathbf{m} \to \hat{\mathbf{m}}$ (globularity condition).

- We have an $(\infty, 1)$-category of all (∞, d)-categories

$$\boxed{\mathrm{Cat}_{\infty,d} := \mathrm{Fun}((\Delta^{\mathrm{op}})^{\times d}, \mathrm{sSet})_{\mathrm{inj,loc}}}$$

Globular vs multiple

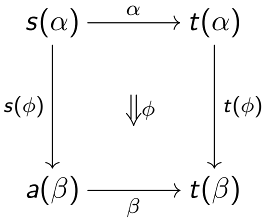
- Let C be a double category, i.e., a category internal to categories.

Globular vs multiple

- Let C be a double category, i.e., a category internal to categories.
- C has a category of objects C_0 and category of morphisms C_1, along with a source and target map $s, t\colon C_1 \to C_0$, an identity morphism functors $e\colon C_0 \to C_1$ and a composition morphism $\circ\colon C_1 \times_{C_0} C_1 \to C_1$.
- One can think of C as encoding a sort of 2-category.

Globular vs multiple

- Let C be a double category, i.e., a category internal to categories.
- C has a category of objects C_0 and category of morphisms C_1, along with a source and target map $s, t \colon C_1 \to C_0$, an identity morphism functors $e \colon C_0 \to C_1$ and a composition morphism $\circ \colon C_1 \times_{C_0} C_1 \to C_1$.
- One can think of C as encoding a sort of 2-category. The 2-morphisms are 1-morphisms $\phi \in \mathrm{Mor}(C_1)$, which can be pictures as cells



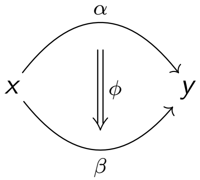
- This can be turned into a bisimplicial space $X_{\bullet\bullet}$.

- This can be turned into a bisimplicial space $X_{\bullet\bullet}$. The above diagram gives a vertex in X_{11}.
- $X_{\bullet\bullet}$ is not globular, since being local w.r.t. the morphism $([0],[1]) \to ([0],[0])$ implies that the vertical morphisms are identities.
- Let C be a strict 2-category.

- This can be turned into a bisimplicial space $X_{\bullet\bullet}$. The above diagram gives a vertex in X_{11}.
- $X_{\bullet\bullet}$ is not globular, since being local w.r.t. the morphism $([0],[1]) \to ([0],[0])$ implies that the vertical morphisms are identities.
- Let C be a strict 2-category. A 2-morphism is a morphism $\phi \in \mathrm{Mor}(\mathsf{C}(x,y))$.

- This can be turned into a bisimplicial space $X_{\bullet\bullet}$. The above diagram gives a vertex in X_{11}.
- $X_{\bullet\bullet}$ is not globular, since being local w.r.t. the morphism $([0],[1]) \to ([0],[0])$ implies that the vertical morphisms are identities.
- Let C be a strict 2-category. A 2-morphism is a morphism $\phi \in \mathrm{Mor}(\mathsf{C}(x, y))$. It can be pictured as a 2-cell

- This can be turned into a bisimplicial space $X_{\bullet\bullet}$. The above diagram gives a vertex in X_{11}.
- $X_{\bullet\bullet}$ is not globular, since being local w.r.t. the morphism $([0],[1]) \to ([0],[0])$ implies that the vertical morphisms are identities.
- Let C be a strict 2-category. A 2-morphism is a morphism $\phi \in \mathrm{Mor}(\mathsf{C}(x,y))$. It can be pictured as a 2-cell



- This can be turned into a globular bisimplicial space.

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Definition

The category Γ is the opposite category of the following category.

- Objects are of the form $\langle \ell \rangle = \{*, 1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$.

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Definition

The category Γ is the opposite category of the following category.

- Objects are of the form $\langle \ell \rangle = \{*, 1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$.
- Morphisms are functions $f \colon \langle \ell \rangle \to \langle k \rangle$ sending $*$ to $*$.

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Definition

The category Γ is the opposite category of the following category.

- Objects are of the form $\langle \ell \rangle = \{*, 1, 2, \dots, \ell\}$, $\ell \in \mathbb{N}$.
- Morphisms are functions $f \colon \langle \ell \rangle \to \langle k \rangle$ sending $*$ to $*$.

Similar to Segal spaces, we define Γ-spaces as follows:

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Definition

The category Γ is the opposite category of the following category.

- Objects are of the form $\langle \ell \rangle = \{*, 1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$.
- Morphisms are functions $f \colon \langle \ell \rangle \to \langle k \rangle$ sending $*$ to $*$.

Similar to Segal spaces, we define Γ-spaces as follows:

Definition

A Γ-space is a functor $X \colon \Gamma^{\mathrm{op}} \to \mathrm{sSet}$ such that

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Definition

The category Γ is the opposite category of the following category.

- Objects are of the form $\langle \ell \rangle = \{*, 1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$.
- Morphisms are functions $f\colon \langle \ell \rangle \to \langle k \rangle$ sending $*$ to $*$.

Similar to Segal spaces, we define Γ-spaces as follows:

Definition

A Γ-space is a functor $X\colon \Gamma^{\mathrm{op}} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathsf{sSet})_{\mathrm{inj}}$;

Adding symmetric monoidal structure

We use Segal's Γ-spaces.

Definition

The category Γ is the opposite category of the following category.

- Objects are of the form $\langle \ell \rangle = \{*, 1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$.
- Morphisms are functions $f \colon \langle \ell \rangle \to \langle k \rangle$ sending $*$ to $*$.

Similar to Segal spaces, we define Γ-spaces as follows:

Definition

A Γ-space is a functor $X \colon \Gamma^{\mathrm{op}} \to \mathrm{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathrm{sSet})_{\mathrm{inj}}$;
2. (Segal's special Γ condition) it is local with respect to the maps

$$\langle \ell \rangle \amalg_{\langle * \rangle} \langle k \rangle \to \langle \ell + k \rangle, \quad \emptyset \to \langle * \rangle$$

An example

How does this encode symmetric monoidal structure?

An example

How does this encode symmetric monoidal structure?

- Let M be a commutative monoid.

An example

How does this encode symmetric monoidal structure?

- Let M be a commutative monoid. We can turn M into a Γ-set by sending
$$\langle \ell \rangle \mapsto M^{\times \ell}.$$

An example

How does this encode symmetric monoidal structure?

- Let M be a commutative monoid. We can turn M into a Γ-set by sending
$$\langle \ell \rangle \mapsto M^{\times \ell}.$$

- A morphism $f\colon \langle \ell \rangle \to \langle k \rangle$ is sent to a map $f_*\colon M^{\times \ell} \to M^{\times k}$ whose i-th component is

An example

How does this encode symmetric monoidal structure?

- Let M be a commutative monoid. We can turn M into a Γ-set by sending
$$\langle \ell \rangle \mapsto M^{\times \ell}.$$
- A morphism $f\colon \langle \ell \rangle \to \langle k \rangle$ is sent to a map $f_*\colon M^{\times \ell} \to M^{\times k}$ whose i-th component is
$$f_*(x)_i = \begin{cases} \sum_{j \in f^{-1}(i) \setminus \{*\}} x_j & 1 \leq i \leq k \\ * & i = * \end{cases}$$

An example

How does this encode symmetric monoidal structure?

- Let M be a commutative monoid. We can turn M into a Γ-set by sending
$$\langle \ell \rangle \mapsto M^{\times \ell}.$$
- A morphism $f \colon \langle \ell \rangle \to \langle k \rangle$ is sent to a map $f_* \colon M^{\times \ell} \to M^{\times k}$ whose i-th component is
$$f_*(x)_i = \begin{cases} \sum_{j \in f^{-1}(i) \setminus \{*\}} x_j & 1 \leq i \leq k \\ * & i = * \end{cases}$$

Can encode the nerve of a permutative category $(\mathsf{C}, \oplus)$ as a Γ-space by assigning $X(\langle \ell \rangle) = N(\mathsf{C})^{\times \ell}$.

An example

How does this encode symmetric monoidal structure?

- Let M be a commutative monoid. We can turn M into a Γ-set by sending
$$\langle \ell \rangle \mapsto M^{\times \ell}.$$
- A morphism $f \colon \langle \ell \rangle \to \langle k \rangle$ is sent to a map $f_* \colon M^{\times \ell} \to M^{\times k}$ whose i-th component is
$$f_*(x)_i = \begin{cases} \sum_{j \in f^{-1}(i) \setminus \{*\}} x_j & 1 \leq i \leq k \\ * & i = * \end{cases}$$

Can encode the nerve of a permutative category $(\mathsf{C}, \oplus)$ as a Γ-space by assigning $X(\langle \ell \rangle) = N(\mathsf{C})^{\times \ell}$. Structure maps use the symmetric monoidal structure.

Definition

A symmetric monoidal (∞, d)-category is a functor $X\colon \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathrm{sSet}$ such that

Definition

A symmetric monoidal (∞, d)-category is a functor $X\colon \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$

Definition

A symmetric monoidal (∞, d)-category is a functor $X\colon \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.

Definition

A symmetric monoidal (∞, d)-category is a functor $X\colon \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.

Definition

A symmetric monoidal (∞, d)-category is a functor $X\colon \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.
4. It satisfies the globular condition (optional).

Definition

A symmetric monoidal (∞, d)-category is a functor $X\colon \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathrm{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathrm{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.
4. It satisfies the globular condition (optional).
5. Is satisfies Segal's special Γ-condition.

Adding smooth structure

Definition

The category Cart is the category whose objects are open subsets of $\mathbb{R}^n$, for some $n \in \mathbb{N}$, that are diffeomorphic to $\mathbb{R}^n$. Morphisms are smooth maps.

Definition

A smooth space is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}}, \mathsf{sSet})_{\mathrm{inj}}$
2. (Descent condition) it is local with respect to Čech covers

$$c^{\{U_\alpha\}} \to U,$$

Here, $\cdots \Rrightarrow \coprod_{\alpha\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha \xrightarrow{\text{hocolim}} c^{\{U_\alpha\}} \longrightarrow U$

An example

How does this encode smooth structure?

- Let X be a smooth manifold.

An example

How does this encode smooth structure?

- Let X be a smooth manifold. Then we can turn X into a smooth space by assigning

$$U \mapsto C^\infty(U, X), \quad U \in \mathrm{Cart}.$$

An example

How does this encode smooth structure?

- Let X be a smooth manifold. Then we can turn X into a smooth space by assigning

$$U \mapsto C^\infty(U, X), \quad U \in \mathrm{Cart}.$$

- A morphism $f\colon U \to V$ is sent to the map $g \mapsto g \circ f$, $g \in C^\infty(V, X)$.

An example

How does this encode smooth structure?

- Let X be a smooth manifold. Then we can turn X into a smooth space by assigning

$$U \mapsto C^\infty(U, X), \quad U \in \mathrm{Cart}.$$

- A morphism $f\colon U \to V$ is sent to the map $g \mapsto g \circ f$, $g \in C^\infty(V, X)$. Being local with respect to the Čech morphisms just says that X is a sheaf:

$$C^\infty(U, X) \cong \lim \left\{ \prod_\alpha C^\infty(U_\alpha, X) \rightrightarrows \prod_{\alpha\beta} C^\infty(U_{\alpha\beta}, X) \right\}$$

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.
4. It satisfies the globular condition (optional).

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.
4. It satisfies the globular condition (optional).
5. It satisfies Segal's special Γ-condition.

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.
4. It satisfies the globular condition (optional).
5. It satisfies Segal's special Γ-condition.
6. It satisfies descent.

Definition

A smooth symmetric monoidal (∞, d)-category is a functor $X\colon \mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d} \to \mathsf{sSet}$ such that

1. It is fibrant in $\mathrm{Fun}(\mathsf{Cart}^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \times (\Delta^{\mathrm{op}})^{\times d}, \mathsf{sSet})_{\mathrm{inj}}$
2. It is a Segal space in all directions.
3. It is complete in all directions.
4. It satisfies the globular condition (optional).
5. It satisfies Segal's special Γ-condition.
6. It satisfies descent.

The $(\infty, 1)$-category of all smooth symmetric monoidal (∞, d)-categories is presented by a big left Bousfield localization

$$\boxed{\mathrm{C}^{\infty}\mathsf{Cat}^{\otimes}_{\infty,d} := \mathsf{PSh}_{\Delta}(\mathsf{Cart} \times \Gamma \times \Delta^{\times d})_{\mathrm{inj,loc}}}$$

The bordism category

The bordism category

To say what a bordism is, we need the notion of a cut tuple

The bordism category

To say what a bordism is, we need the notion of a cut tuple

- A cut for a submersion $p\colon M \to U$ (with d-dimensional fibers) is a triple $(C_<, C_=, C >)$ of subsets of M such that

The bordism category

To say what a bordism is, we need the notion of a cut tuple

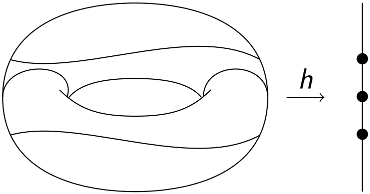
- A cut for a submersion $p\colon M \to U$ (with d-dimensional fibers) is a triple $(C_<, C_=, C >)$ of subsets of M such that
- There is smooth $h\colon M \to \mathbb{R}$ with 0 a (fiberwise) regular value, $C_= = h^{-1}(0), C_< = h^{-1}(-\infty, 0), C_> = h^{-1}(0, \infty)$.

The bordism category

To say what a bordism is, we need the notion of a cut tuple

- A cut for a submersion $p\colon M \to U$ (with d-dimensional fibers) is a triple $(C_<, C_=, C >)$ of subsets of M such that
- There is smooth $h\colon M \to \mathbb{R}$ with 0 a (fiberwise) regular value, $C_= = h^{-1}(0), C_< = h^{-1}(-\infty, 0), C_> = h^{-1}(0, \infty)$.
- A cut $[m]$-tuple is a collection of cuts $C_j = (C_{j<}, C_{j=}, C_{j>})$, $j \in [m]$, such that

$$C_{\leq 0} \subset C_{\leq 1} \subset \ldots \subset C_{\leq m}$$



Definition

For fixed $\mathbf{m} \in \Delta^{\times d}$, $\langle \ell \rangle \in \Gamma$ and $U \in \mathsf{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$$

as the nerve of the following groupoid:

Definition

For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathsf{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$$

as the nerve of the following groupoid:

- Objects:
  1. A d-dimensional manifold M.

Definition

For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathrm{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$$

as the nerve of the following groupoid:

- Objects:
  1. A d-dimensional manifold M.
  2. For each $1 \leq i \leq d$, a cut $[m_i]$-tuple C^i for the projection $p\colon M \times U \to U$.

Definition

For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathrm{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$$

as the nerve of the following groupoid:

- Objects:
  1. A d-dimensional manifold M.
  2. For each $1 \leq i \leq d$, a cut $[m_i]$-tuple C^i for the projection $p\colon M \times U \to U$.
  3. A choice of map $P\colon M \times U \to \langle \ell \rangle$, partitioning the set of connected components into ℓ disjoint subsets.

Definition

For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathrm{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$$

as the nerve of the following groupoid:

- Objects:
  1. A d-dimensional manifold M.
  2. For each $1 \leq i \leq d$, a cut $[m_i]$-tuple C^i for the projection $p\colon M \times U \to U$.
  3. A choice of map $P\colon M \times U \to \langle \ell \rangle$, partitioning the set of connected components into ℓ disjoint subsets.

  such that $\pitchfork$

Definition

For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \text{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d(U, \langle \ell \rangle, \mathbf{m})$$

as the nerve of the following groupoid:

- Objects:
  1. A d-dimensional manifold M.
  2. For each $1 \leq i \leq d$, a cut $[m_i]$-tuple C^i for the projection $p\colon M \times U \to U$.
  3. A choice of map $P\colon M \times U \to \langle \ell \rangle$, partitioning the set of connected components into ℓ disjoint subsets.

  such that $\pitchfork$
- Morphisms: cut respecting diffeomorphisms.

Geometric structures

Geometric structures

We encode geometric structures on bordisms via sheaves on a certain category.

Geometric structures

We encode geometric structures on bordisms via sheaves on a certain category.

- Let FEmb_d be the category whose objects are submersions $p\colon M \to U$ with d-dimensional fibers, $U \in \mathsf{Cart}$.

Geometric structures

We encode geometric structures on bordisms via sheaves on a certain category.

- Let FEmb_d be the category whose objects are submersions $p\colon M \to U$ with d-dimensional fibers, $U \in \mathsf{Cart}$.
- Morphisms are fiberwise open embeddings (over U).

Geometric structures

We encode geometric structures on bordisms via sheaves on a certain category.

- Let FEmb_d be the category whose objects are submersions $p\colon M \to U$ with d-dimensional fibers, $U \in \mathrm{Cart}$.
- Morphisms are fiberwise open embeddings (over U).
- We topologize FEmb_d by taking covering families to be $\{p_\alpha\colon M_\alpha \to U_\alpha\}$ such that $\{M_\alpha\}$ is an open cover of M.

Definition

A fiberwise d-dimensional geometric structure is a simplicial presheaf on FEmb_d.

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

- Let $\tau\colon (p\colon M \to U) \to \mathbf{B}\mathrm{GL}(d)$ be the fiberwise tangent bundle.

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

- Let $\tau\colon (p\colon M \to U) \to \mathbf{B}\mathrm{GL}(d)$ be the fiberwise tangent bundle.
- Let $Y \to \mathbf{B}\mathrm{GL}(d)$ be any morphism of simplicial presheaves.

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

- Let $\tau\colon (p\colon M \to U) \to \mathbf{B}\mathrm{GL}(d)$ be the fiberwise tangent bundle.
- Let $Y \to \mathbf{B}\mathrm{GL}(d)$ be any morphism of simplicial presheaves.
- Define the simplicial presheaf

$$\tau_Y(p\colon M \to U) := \Gamma(M, M \times_{\mathbf{B}\mathrm{GL}(d)} Y).$$

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

- Let $\tau\colon (p\colon M \to U) \to \mathbf{B}\mathrm{GL}(d)$ be the fiberwise tangent bundle.
- Let $Y \to \mathbf{B}\mathrm{GL}(d)$ be any morphism of simplicial presheaves.
- Define the simplicial presheaf

$$\tau_Y(p\colon M \to U) := \Gamma(M, M \times_{\mathbf{B}\mathrm{GL}(d)} Y).$$

- Such simplicial presheaves encode tangential structures.

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

- Let $\tau\colon (p\colon M \to U) \to \mathbf{B}\mathrm{GL}(d)$ be the fiberwise tangent bundle.
- Let $Y \to \mathbf{B}\mathrm{GL}(d)$ be any morphism of simplicial presheaves.
- Define the simplicial presheaf

$$\tau_Y(p\colon M \to U) := \Gamma(M, M \times_{\mathbf{B}\mathrm{GL}(d)} Y).$$

- Such simplicial presheaves encode tangential structures. , e.g. $Y = \mathbf{B}\mathrm{G}, \mathbf{B}\mathrm{String(d)}, \mathbf{B}\mathrm{String(d)}_\nabla, \mathbf{B}\mathrm{Fivebrane(d)}.$

Tangential structures

- Let $\mathbf{B}\mathrm{GL}(d)$ be the simplicial presheaf on FEmb_d defined by

$$(p\colon M \to U) \mapsto \mathrm{Vect}_d(M).$$

- Let $\tau\colon (p\colon M \to U) \to \mathbf{B}\mathrm{GL}(d)$ be the fiberwise tangent bundle.
- Let $Y \to \mathbf{B}\mathrm{GL}(d)$ be any morphism of simplicial presheaves.
- Define the simplicial presheaf

$$\tau_Y(p\colon M \to U) := \Gamma(M, M \times_{\mathbf{B}\mathrm{GL}(d)} Y).$$

- Such simplicial presheaves encode tangential structures. , e.g. $Y = \mathbf{B}\mathrm{G}, \mathbf{B}\mathrm{String(d)}, \mathbf{B}\mathrm{String(d)}_\nabla, \mathbf{B}\mathrm{Fivebrane(d)}$.

Examples of simplicial presheaves on FEmb_d include conformal structures, Riemannian metrics, pseudo-Riemannian metrics, maps to a fixed manifold, or combinations of these.

Bordisms with geometric structure

Definition

Let $\mathcal{S} \in \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)$.

Bordisms with geometric structure

Definition

Let $\mathcal{S} \in \mathsf{PSh}_{\Delta}(\mathsf{FEmb}_d)$. For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathsf{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d^{\mathcal{S}}(U, \langle \ell \rangle, \mathbf{m})$$

as the diagonal of the nerve of the following simplicial groupoid:

Bordisms with geometric structure

Definition

Let $\mathcal{S} \in \mathsf{PSh}_{\Delta}(\mathsf{FEmb}_d)$. For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathsf{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d^{\mathcal{S}}(U, \langle \ell \rangle, \mathbf{m})$$

as the diagonal of the nerve of the following simplicial groupoid:

- Objects: $\coprod_{(M,P)} \mathcal{S}(M \times U \to U) \times \mathrm{Cut}(M \times U)$

Bordisms with geometric structure

Definition

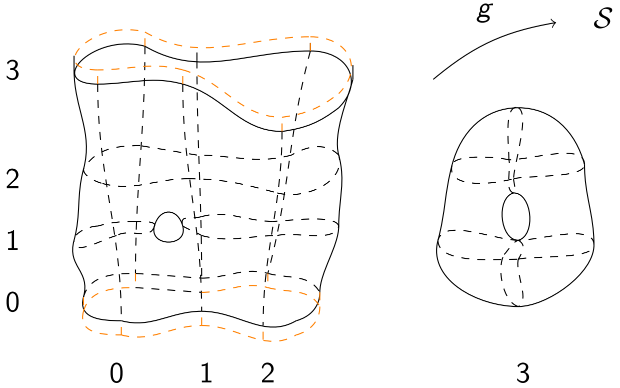
Let $\mathcal{S} \in \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)$. For fixed $\mathbf{m} \in \Delta^{\times d}, \langle \ell \rangle \in \Gamma$ and $U \in \mathsf{Cart}$, we define the simplicial set

$$\mathrm{Bord}_d^{\mathcal{S}}(U, \langle \ell \rangle, \mathbf{m})$$

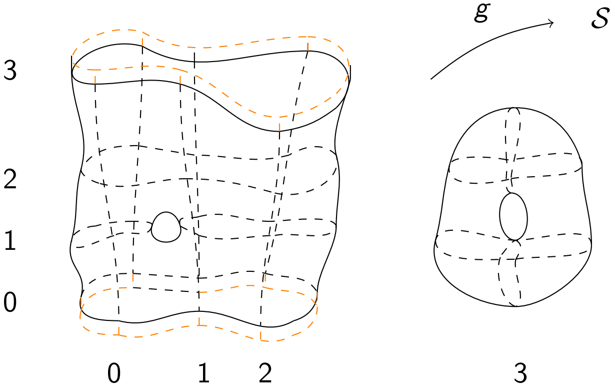
as the diagonal of the nerve of the following simplicial groupoid:

- Objects: $\coprod_{(M,P)} \mathcal{S}(M \times U \to U) \times \mathrm{Cut}(M \times U)$
- Morphisms: $\coprod_{(M,P) \to (\widetilde{M}, \widetilde{P})} \mathcal{S}(\widetilde{M} \times U \to U) \times \mathrm{Cut}(\widetilde{M} \times U)$

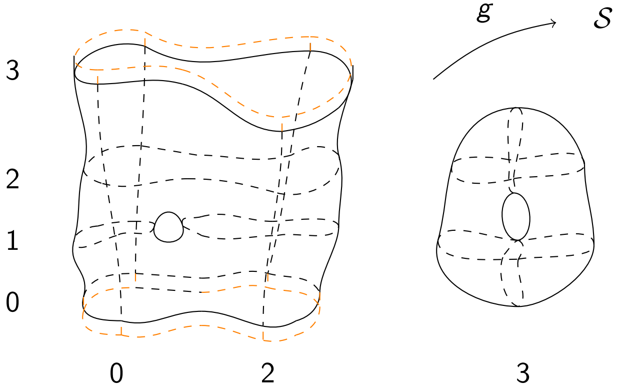
- The simplicial set $\mathrm{Cut}(M \times U)$ has l-simplices given by a Δ^l-family of cut $\mathbf{m}$-tuples on $M \times U$.



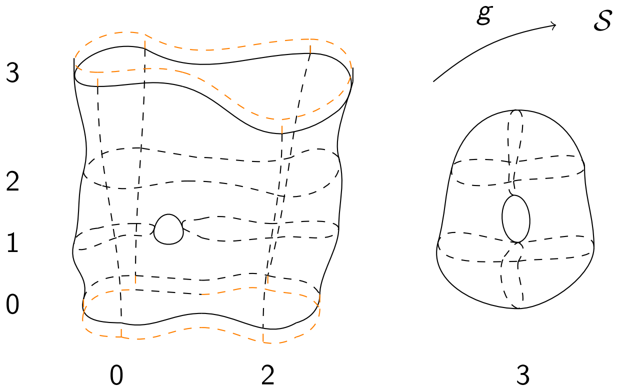
3
2
1
0
0
1
2
g
S
3



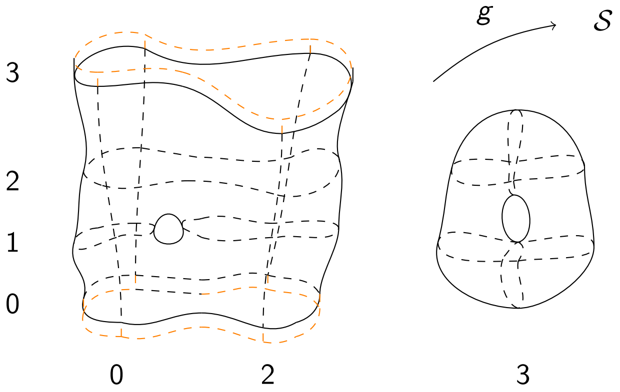
- Δ structure map



- Δ structure map



- Δ structure map
- Γ structure map repartitions set of connected components.



- Δ structure map
- Γ structure map repartitions set of connected components.
- Cart structure map pulls back bundles of bordisms along a smooth map $f\colon U \to V$.

The geometrically framed bordism category

- The geometric structure is a representable presheaf of the form $\mathbb{R}^d \times U \to U$, for some cartesian space U.

The geometrically framed bordism category

- The geometric structure is a representable presheaf of the form $\mathbb{R}^d \times U \to U$, for some cartesian space U.

$$(\mathbb{R}^d \times U \to U)(M \to V) = \left\{ \begin{array}{ccc} M & \xrightarrow{f} & \mathbb{R}^d \times U \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & U \end{array} \right\}$$

- Fix $\mathbf{m} \in \Delta^{\times d}$, $\langle \ell \rangle \in \Gamma$ and $V \in \mathrm{Cart}$.

The geometrically framed bordism category

- The geometric structure is a representable presheaf of the form $\mathbb{R}^d \times U \to U$, for some cartesian space U.

$$(\mathbb{R}^d \times U \to U)(M \to V) = \left\{ \begin{array}{ccc} M & \xrightarrow{f} & \mathbb{R}^d \times U \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & U \end{array} \right\}$$

- Fix $\mathbf{m} \in \Delta^{\times d}$, $\langle \ell \rangle \in \Gamma$ and $V \in \mathrm{Cart}$. A vertex in $\mathrm{Bord}_d^{\mathbb{R}^d \times U \to U}(V, \langle \ell \rangle, \mathbf{m})$ is family of V-family of chopped manifolds, together with a partition of the set of connected components and a fiberwise embedding into $\mathbb{R}^d$.

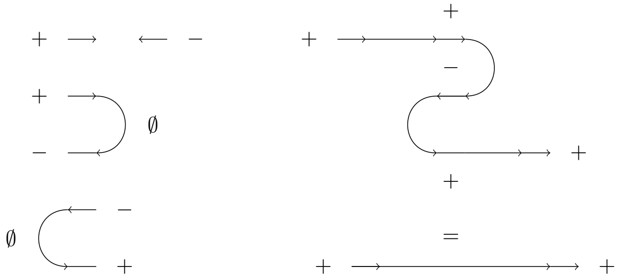
The geometrically framed bordism category

- The geometric structure is a representable presheaf of the form $\mathbb{R}^d \times U \to U$, for some cartesian space U.

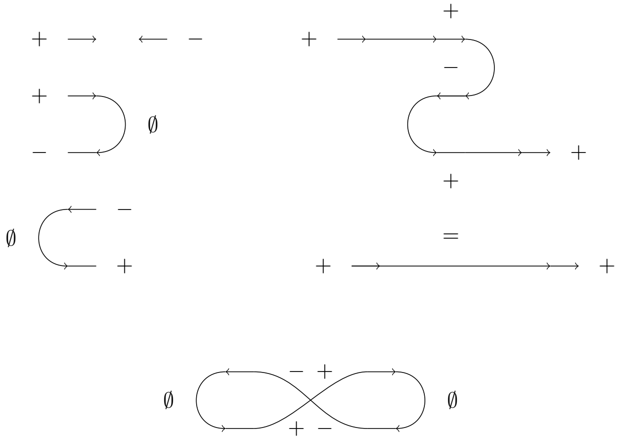
$$(\mathbb{R}^d \times U \to U)(M \to V) = \left\{ \begin{array}{ccc} M & \xrightarrow{f} & \mathbb{R}^d \times U \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & U \end{array} \right\}$$

- Fix $\mathbf{m} \in \Delta^{\times d}$, $\langle \ell \rangle \in \Gamma$ and $V \in \mathrm{Cart}$. A vertex in $\mathrm{Bord}_d^{\mathbb{R}^d \times U \to U}(V, \langle \ell \rangle, \mathbf{m})$ is family of V-family of chopped manifolds, together with a partition of the set of connected components and a fiberwise embedding into $\mathbb{R}^d$.
- Note that we do not have closed d-manifolds as bordisms!

$d=1$



$d=1$



Smooth field theories and locality

Smooth field theories and locality

Fix a target category $T \in \mathrm{C}^\infty\mathsf{Cat}^{\otimes}_{\infty,d}$.

Smooth field theories and locality

Fix a target category $T \in \mathrm{C}^{\infty}\mathsf{Cat}^{\otimes}_{\infty,d}$. Fix a geometric structure $\mathcal{S} \in \mathsf{PSh}_{\Delta}(\mathsf{FEmb}_d)$.

Smooth field theories and locality

Fix a target category $T \in \mathrm{C}^\infty\mathsf{Cat}^{\otimes}_{\infty,d}$. Fix a geometric structure $\mathcal{S} \in \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)$. We define

$$\boxed{\mathrm{FFT}_{d,T}(\mathcal{S}) := \mathrm{Fun}^{\otimes}(\mathsf{Bord}_d^{\mathcal{S}}, T)}$$

Smooth field theories and locality

Fix a target category $T \in \mathrm{C}^\infty\mathsf{Cat}^{\otimes}_{\infty,d}$. Fix a geometric structure $\mathcal{S} \in \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)$. We define

$$\boxed{\mathrm{FFT}_{d,T}(\mathcal{S}) := \mathrm{Fun}^{\otimes}(\mathsf{Bord}^{\mathcal{S}}_d, T)}$$

- $\mathrm{Fun}^{\otimes}(-,-)$ is (∞, d)-category obtained by forgetting structure on the internal $(\infty, 1)$-hom.

Smooth field theories and locality

Fix a target category $T \in \mathrm{C}^\infty\mathsf{Cat}^{\otimes}_{\infty,d}$. Fix a geometric structure $\mathcal{S} \in \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)$. We define

$$\boxed{\mathrm{FFT}_{d,T}(\mathcal{S}) := \mathrm{Fun}^{\otimes}(\mathsf{Bord}^{\mathcal{S}}_d, T)}$$

- $\mathrm{Fun}^{\otimes}(-,-)$ is (∞, d)-category obtained by forgetting structure on the internal $(\infty, 1)$-hom.

Theorem

The functor

$$\mathrm{FFT}_{d,T}\colon \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)^{\mathrm{op}}_{\mathrm{inj,loc}} \to \mathsf{Cat}_{\infty,d}$$

is an ∞-sheaf (i.e. $\mathrm{FFT}_{d,T}$ preserves all homotopy limits).

Relation to the cobordism hypothesis

The sheaf property of $\mathsf{FFT}_{d,T}$ is half the cobordism hypothesis.

Relation to the cobordism hypothesis

The sheaf property of $\mathsf{FFT}_{d,T}$ is half the cobordism hypothesis.

- Define the functor

$$\mathsf{Cat}_{\infty,d} \to \mathsf{PSh}_{\Delta}(\mathsf{FEmb}_d)_{\mathrm{inj,loc}}, \quad T \mapsto \mathrm{FFT}^{\times}_{d,T} =: T^{\times}_d$$

Relation to the cobordism hypothesis

The sheaf property of $\mathsf{FFT}_{d,T}$ is half the cobordism hypothesis.

- Define the functor
$$\mathsf{Cat}_{\infty,d} \to \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)_{\mathrm{inj,loc}}, \quad T \mapsto \mathrm{FFT}^\times_{d,T} =: T^\times_d$$

- The left adjoint is the functor
$$\mathsf{Bord}_d \colon \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)_{\mathrm{inj,loc}} \to \mathsf{Cat}_{\infty,d}, \quad \mathcal{S} \mapsto \mathsf{Bord}^{\mathcal{S}}_d.$$

Relation to the cobordism hypothesis

The sheaf property of $\mathsf{FFT}_{d,T}$ is half the cobordism hypothesis.

- Define the functor

$$\mathsf{Cat}_{\infty,d} \to \mathsf{PSh}_{\Delta}(\mathsf{FEmb}_d)_{\mathrm{inj,loc}}, \quad T \mapsto \mathrm{FFT}^{\times}_{d,T} =: T^{\times}_d$$

- The left adjoint is the functor

$$\mathsf{Bord}_d \colon \mathsf{PSh}_{\Delta}(\mathsf{FEmb}_d)_{\mathrm{inj,loc}} \to \mathsf{Cat}_{\infty,d}, \quad \mathcal{S} \mapsto \mathsf{Bord}^{\mathcal{S}}_d.$$

- The theorem can be rephrased by saying that the above adjunction is Quillen at the level of the Čech local model structure.

Relation to the cobordism hypothesis

The sheaf property of $\mathsf{FFT}_{d,T}$ is half the cobordism hypothesis.

- Define the functor

$$\mathsf{Cat}_{\infty,d} \to \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)_{\mathrm{inj,loc}}, \quad T \mapsto \mathrm{FFT}^\times_{d,T} =: T^\times_d$$

- The left adjoint is the functor

$$\mathsf{Bord}_d \colon \mathsf{PSh}_\Delta(\mathsf{FEmb}_d)_{\mathrm{inj,loc}} \to \mathsf{Cat}_{\infty,d}, \quad \mathcal{S} \mapsto \mathsf{Bord}^{\mathcal{S}}_d.$$

- The theorem can be rephrased by saying that the above adjunction is Quillen at the level of the Čech local model structure.
- By the universal property of the adjunction, we have an equivalence of derived mapping spaces

$$\boxed{\mathrm{Fun}^\otimes(\mathsf{Bord}^{\mathcal{S}}_d, T) \simeq \mathrm{Map}(\mathcal{S}, T^\times_d).}$$

Plan for talks 3 and 4

- In the next talk, Dmitri will sketch the proof of the codescent property.

Plan for talks 3 and 4

- In the next talk, Dmitri will sketch the proof of the codescent property.
- In the final talk, I will sketch the proof of the geometrically framed case.