# Math 54 - Midterm #2 Solutions

GSI: Santiago Canez
July 29, 2005

1. (10 points) Find bases for the row space and column space of the following matrix $A$, and determine $\text{rank}(A)$ and $\dim NS(A)$.

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ -2 & -1 & -2 & -7 & -11 \\ 2 & -2 & 4 & 12 & 14 \\ 4 & 11 & -2 & 7 & 17 \end{pmatrix}$$

*Solution.* First we put the matrix into echelon form. We have

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ -2 & -1 & -2 & -7 & -11 \\ 2 & -2 & 4 & 12 & 14 \\ 4 & 11 & -2 & 7 & 17 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 3 & -2 & -1 & -1 \\ 0 & -6 & 4 & 6 & 4 \\ 0 & 3 & -2 & -5 & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 3 & -2 & -1 & -1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & -4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 3 & -2 & -1 & -1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this, we can say that a basis for $\text{RS}(A)$ is

$$(1,2,0,3,5), (0,3,-2,-1,-1), (0,0,0,4,2)$$

(i.e. the nonzero rows of the echelon form) and basis for $\text{CS}(A)$ is

$$\begin{pmatrix} 1 \\ -2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \\ 11 \end{pmatrix}, \begin{pmatrix} 3 \\ -7 \\ 12 \\ 7 \end{pmatrix}$$

(i.e. the columns of $A$ that correspond to pivot columns in the echelon form). Hence, $\text{rank}(A) = 3$ and $\dim NS(A) = 5 - \text{rank}(A) = 2$. □

2. (15 points) Determine the matrix that induces the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates the $xz$-plane by $\frac{\pi}{6}$ radians and reflects the $y$-axis.

*Solution.* The matrix that induces this linear transformation is

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{pmatrix}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$. We compute (using a unit circle) that

$$T(\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}\sqrt{3}/2\\0\\1/2\end{pmatrix}$$
$$T(\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}0\\-1\\0\end{pmatrix}$$
$$T(\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}-1/2\\0\\\sqrt{3}/2\end{pmatrix}.$$

Hence the matrix that induces this linear transformation is

$$A = \begin{pmatrix}\sqrt{3}/2 & 0 & -1/2\\0 & -1 & 0\\1/2 & 0 & \sqrt{3}/2\end{pmatrix}.$$

□

3. Let $A$ be the following symmetric matrix.

$$A = \begin{pmatrix}3 & 1 & 1\\1 & 3 & 1\\1 & 1 & 3\end{pmatrix}$$

(a) (10 points) Show that the eigenvalues of $A$ are 2, 2, and 5.
(b) (10 points) Find bases for each eigenspace of $A$.
(c) (10 points) Apply the Gram-Schmidt process to the eigenvectors you found in (b).
(d) (5 points) The resulting vectors in (c) form an orthonormal basis for $\mathbb{R}^3$. Compute the coordinate vector of $(1, 0, 1)$ relative to this basis.

*Solution.* (a) The characteristic polynomial of this matrix is (using a cofactor expansion along the first row)

$$\begin{aligned}
\det(\lambda I - A) &= \begin{vmatrix}\lambda-3 & -1 & -1\\-1 & \lambda-3 & -1\\-1 & -1 & \lambda-3\end{vmatrix}\\
&= (\lambda-3)\begin{vmatrix}\lambda-3 & -1\\-1 & \lambda-3\end{vmatrix} + \begin{vmatrix}-1 & -1\\-1 & \lambda-3\end{vmatrix} - \begin{vmatrix}-1 & \lambda-3\\-1 & -1\end{vmatrix}\\
&= (\lambda-3)(\lambda^2-6\lambda+8) + (-\lambda+2) - (1+\lambda-3)\\
&= (\lambda-3)(\lambda-4)(\lambda-2) - (\lambda-2) - (\lambda-2)\\
&= (\lambda-2)((\lambda-3)(\lambda-4) - 1 - 1)\\
&= (\lambda-2)(\lambda^2-7\lambda+10)\\
&= (\lambda-2)(\lambda-2)(\lambda-5).
\end{aligned}$$

Hence the eigenvalues of $A$ are 2 (with multiplicity 2) and 5.

(b) First, for $\lambda = 2$ we have

$$2I - A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finding a basis for the null space of this matrix we find that a basis for $W_2$ is

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda = 5$ we have

$$5I - A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finding a basis for the null space of this matrix we find that a basis for $W_5$ is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(c) We first apply the Gram-Schmidt process to the eigenvectors corresponding to 2. We have

$$\begin{aligned}
\mathbf{b}_1' &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\
\mathbf{b}_2' &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \\
\mathbf{b}_1 &= \frac{\mathbf{b}_1'}{\|\mathbf{b}_1'\|} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \\
\mathbf{b}_2 &= \frac{\mathbf{b}_2'}{\|\mathbf{b}_2'\|} = \begin{pmatrix} -1/\sqrt{6} \\ \sqrt{2/3} \\ -1/\sqrt{6} \end{pmatrix}.
\end{aligned}$$

Now, we divide the eigenvector we found corresponding to 5 by its length to get

$$\mathbf{b}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.$$

(d) Let $\mathbf{x} = (1, 0, 1)$. The orthonormal basis found in part (c) is

$$\mathbf{b}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} -1/\sqrt{6} \\ \sqrt{2/3} \\ -1/\sqrt{6} \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.$$

Since this basis is orthonormal, there is an easy way to compute coordinate vectors — namely

$$[\mathbf{x}] = \begin{pmatrix} \mathbf{x} \cdot \mathbf{b}_1 \\ \mathbf{x} \cdot \mathbf{b}_2 \\ \mathbf{x} \cdot \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2/\sqrt{6} \\ 2/\sqrt{3} \end{pmatrix}.$$

□

4. (15 points) Prove that $\lambda$ is an eigenvalue of a matrix $A$ if and only if $\det(\lambda I - A) = 0$.

*Solution.* By definition, $\lambda$ is an eigenvalue of $A$ if and only if there is some nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Rearranging this equation, this is true if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, which is true if and only if $(\lambda I - A)$ is not invertible. Finally, this is true if and only if $\det(\lambda I - A) = 0$. □

5. (a) (10 points) Show that $e^x \cos x$ and $e^x \sin x$ are solutions of the differential equation

$$y'' - 2y' + 2y = 0.$$

(b) (10 points) Show that $e^x \cos x$ and $e^x \sin x$ are linearly indepedent.

(c) (5 points) What can you conclude about the general solution of the above differential equation?

*Solution.* (a) Let $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$. Then

$$\begin{aligned} y_1' &= -e^x \sin x + e^x \cos x \\ y_1'' &= -e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x = -2e^x \sin x \\ y_2' &= e^x \cos x + e^x \sin x \\ y_2'' &= -e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x = 2e^x \cos x. \end{aligned}$$

Thus

$$y_1'' - 2y_1' + 2y_1 = (-2e^x \sin x) - 2(-e^x \sin x + e^x \cos x) + 2(e^x \cos x) = 0$$

showing that $y_1$ is a solution and

$$y_2'' - 2y_2' + 2y_2 = (2e^x \cos x) - 2(e^x \cos x + e^x \sin x) + 2(e^x \sin x) = 0$$

showing that $y_2$ is a solution.

(b) We can calculate the Wronskian by brute force, or we can use Abel's theorem. By Abel's theorem, the Wronskian is

$$W(e^x \cos x, e^x \sin x)(x) = ce^{-\int -2\,dx} = ce^{2x}$$

for some constant $c$. To find $c$, we can explicitly calculate the Wronskian at 0:

$$W(e^x \cos x, e^x \sin x)(0) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

This implies that $c = 1$ so the Wronskian of $y_1$ and $y_2$ is

$$W(e^x \cos x, e^x \sin x)(x) = ce^{-\int -2\,dx} = e^{2x}.$$

Note: You get the same result calculating the Wronskian by brute force. Since the Wronskian is nonzero, we conclude that $y_1$ and $y_2$ are linearly independent.

(c) Since $y_1$ and $y_2$ are linearly independent, they form a fundamental solution set so any solution of the differential equation can be written as a linear combination of $y_1$ and $y_2$. Thus the general solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x = e^x(c_1 \cos x + c_2 \sin x).$$

□

6. (0 points) Draw a picture portraying your love/hate relationship with linear algebra.