

## Partial Solutions to Homework 9

1. Sarason, VII.11, exercise 3. (Compose with another map.)

Solution:

Compose such a function  $f$  with the map  $g : z \mapsto \frac{1-z}{1+z}$ , which maps the right half-plane to the open unit disk. Now Liouville's theorem implies  $g \circ f$  is constant. So  $f$  is constant (since  $g$  is injective).

2. Sarason, VII.13, exercises 1 and 2.

Solution:

1) Assume such a function exists. By continuity, 0 must be a zero of  $f$ . It cannot have infinite order because this would imply  $f = 0$  near 0. Let  $k$  be the order of the zero 0 of  $f$ . Then  $f = z^k g$ , where  $g : B_1(0) \rightarrow \mathbb{C}$  is holomorphic and  $g(0) \neq 0$ . But then  $g(\frac{1}{n}) = 2^{-n} n^k$  which tends to 0 as  $n$  tends to  $\infty$ . So  $g(0) = 0$  by continuity. Contradiction.

2) There is a holomorphic function  $g : G \rightarrow \mathbb{C}$  such that  $f(z) = (z - z_0)^m g(z)$  and  $g(z_0) \neq 0$ . This implies that  $g$  has an  $m$ -th root  $h$  in some neighbourhood of  $z_0$ . Then  $k(z) = (z - z_0)h(z)$  defines an  $m$ -th root of  $f$  in the same neighbourhood of  $z_0$ .

3. Sarason, VII.14, exercises 1 and 2.

Solution:

1) Assume such a function exists. Then  $f(0) = 0$  by continuity, so  $f(z) = z f_1(z)$  for some holomorphic function  $f_1 : B_1(0) \rightarrow \mathbb{C}$ . Also  $f_1(0) = 0$  by continuity, so there is  $f_2 : B_1(0) \rightarrow \mathbb{C}$  holomorphic such that  $f(z) = z^2 f_2(z)$ . Then  $f_2(\frac{1}{2n}) = 1$  for all positive integers  $n$ . By the identity theorem  $f_2$  must be constant, equal to 1. But it isn't. Contradiction.

2) Define  $g : G \rightarrow \mathbb{C}$  by  $g(z) = \overline{f(\bar{z})}$ . Then  $g$  is holomorphic by a previous homework problem, and  $g$  equals  $f$  on the intersection of  $G$  with the real axis. This intersection is non-empty except if  $G = \emptyset$  (since  $G$  is connected, else you could decompose  $G$  into the disjoint open sets which are the intersections of  $G$  with the upper resp. lower half-plane). Since  $G$  is open, the intersection with the real line contains a non-empty interval. By the identity theorem,  $f = g$  on  $G$ , which is exactly what we need to prove.

4. Sarason, VII.16, exercises 1 and 2.

Solution:

1) Suppose  $z_0$  is a local minimum of  $|f|$  which is not a zero of  $f$ . Then there is a small disk around  $z_0$  in which  $f$  does not have a zero. In this disk,  $\frac{1}{f}$  is holomorphic, and  $z_0$  is a local maximum of its absolute value. So  $\frac{1}{f}$  must be constant, i.e.  $f$  is constant in this disk, so it is constant in  $G$  by the identity theorem.

2) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a nonconstant polynomial. Then  $|f|$  tends to  $\infty$  as  $|z|$  tends to  $\infty$ . So  $|f|$  has a local minimum, which must be a zero of  $f$  by the preceding problem. (Or use  $\frac{1}{f}$  and use the maximum modulus principle.)