Partial Solutions to Homework 5

1. Let $s \in \mathbb{C}$. Find a holomorphic function b_s on $B_1(0)$ which satisfies the differential equation

$$b_s'(z) = \frac{s}{1+z} b_s(z), \quad b_s(0) = 1.$$

a) Find a power series with center 0 ("binomial series") satisfying the differential equation. You might want to define binomial coefficients $\binom{s}{k}$, where $s \in \mathbb{C}$, $k \in \mathbb{N}$. Identify the power series in the case $s \in \mathbb{N}$ with a function you know, which has convergence radius ∞. Show that if $s \notin \mathbb{N}$, then the convergence radius is 1.

b) Show that

$$b_s(z) = exp(s\lambda(z)),$$

where $\lambda(z) = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$.
Instructions: Show that $exp(s\lambda(z))$ satisfies the differential equation, and does not have zeros. Then show that if f is any solution of the differential equation, then the derivative of $f(z)/exp(s\lambda(z))$ is zero, so $f(z) = exp(s\lambda(z))$.

c) Show that $b_s(z)b_t(z) = b_{s+t}$ for $s, t \in \mathbb{C}$ using what you have proven in part b: there is exactly one solution to the differential equation.

d) Show that $B_s(z)B_t(z) = B_{s+t}$ for $s, t \in \mathbb{C}$ using the multiplication formula for power series.

Solution:

a) If $f : B_1(0) \to \mathbb{C}$ is represented by a power series $\sum_{n=0}^{\infty} a_n z^n$ with positive convergence radius $R \geq 1$, then $f'(z) = \frac{s}{1+z} f(z) \Leftrightarrow (1+z)f'(z) = sf(z) \Leftrightarrow \sum_{n=0}^{\infty} (na_n + (n+1)a_{n+1})z^n = \sum_{n=0}^{\infty} sa_n z^n$. So we should try to find coefficients a_n s.t. $a_0 = 1$ and for $n \geq 0$: $na_n + (n+1)a_{n+1} = sa_n \Leftrightarrow a_{n+1} = \frac{s-n}{n+1} a_n$ This leads to $s_0 = 1, s_1 = \frac{s}{1}, s_2 = \frac{s(s-1)}{2 \cdot 1}, s_3 = \frac{s(s-1)(s-2)}{3 \cdot 2 \cdot 1}, \ldots$, i.e. $a_n = \binom{s}{n}$, where we define the binomial coefficient for $n = 0$ as $\binom{s}{n} = 1$ and for $n > 0$ as

$$\binom{s}{n} = \frac{s(s-1)\ldots(s-n+1)}{n!}.$$

The series $\sum_{n=0}^{\infty} \binom{s}{n} z^n$ now formally solves the differential equation, we only have to pay attention that this is a series with convergence radius at least 1. But for $s \notin \mathbb{N}$, we get convergence radius 1 by a ratio test. For $s \in \mathbb{N}$ the coefficient a_s is the highest non-zero coefficient. We get a polynomial of convergence radius ∞, in fact we get $(1+z)^s$. For $s \in -\mathbb{N}$ one can use the last part of the problem to see that we also get $(1+z)^s$, which is now a rational function.

b) We know that $exp' = exp$ and we compute that $\lambda'(z) = \frac{1}{1+z}$. Now apply the chain rule. Since exp has no zeros, we see that $exp(s\lambda(z))$ has no zeros. So we can divide by this function and still get a differentiable function. The derivative of $f(z)/exp(s\lambda(z))$ is computed to be zero, so the function $f(z)/exp(s\lambda(z))$ is constant. Since the values at 0 agree, we have $f(z) = exp(s\lambda(z))$, in particular for $f = b_s$.

c) Show that $B_s(z)B_t(z)$ satisfies the differential equation for b_{s+t}.

d) You have to prove that

$$\sum_{j=0}^{n} \binom{s}{j} \binom{t}{n-j} = \binom{s+t}{n}.$$

Use induction on n to do this.

2. (Do this exercise after Friday's lecture.)
We define the (complex) hyperbolic functions $\cosh, \sinh : \mathbb{C} \to \mathbb{C}$ as in the real case by

$$cosh(z) = \frac{exp(z) + exp(-z)}{2}, \quad sinh(z) = \frac{exp(z) - exp(-z)}{2}.$$

Prove that for all $w, z \in \mathbb{C}$:

a) $\sinh z = -i \sin(iz)$, $\cosh z = \cos(iz)$,

b) $\sinh(w + z) = \sinh w \cosh z + \cosh w \sinh z$, $\cosh(w + z) = \cosh w \cosh z + \sinh w \sinh z$,

c) $\cosh^2 z - \sinh^2 z = 1$ (this is why the functions are called hyperbolic: the points $(\cosh z, \sinh z)$ lie on the hyperbola given by the equation $z_1^2 - z_2^2 = 1$),

d) $cosh'(z) = sinh(z)$ and $sinh'(z) = cosh(z)$,

e) $\sinh(z + 2\pi i) = \sinh(z)$ and $\cosh(z + 2\pi i) = \cosh(z)$.

f) The zeros of $\sinh z$ are exactly the elements of $\{n\pi i \mid n \in \mathbb{Z}\}$, and the zeros of $\cosh z$ are exactly the elements of $\{(n + \frac{1}{2})\pi i \mid n \in \mathbb{Z}\}$.

g) Find power series expressions for $\cosh z$ and $\sinh z$.

Solution:
a) follows from the definitions of $\sin, \cos$, b),c),e) from the addition theorem for exp, d) from $exp' = exp$.
For f) you have to solve $exp(z) = exp(-z)$ and $exp(z) = -exp(-z)$.
Now $exp(z) = exp(-z) \Leftrightarrow exp(2z) = 1 \Leftrightarrow 2z \in \{2n\pi i \mid n \in \mathbb{Z}\}$
and $exp(z) = -exp(-z) \Leftrightarrow exp(2z) = -1 \Leftrightarrow 2z \in \{(2n + 1)\pi i \mid n \in \mathbb{Z}\}$.
g) Power series expressions follow easily from the one for exp.