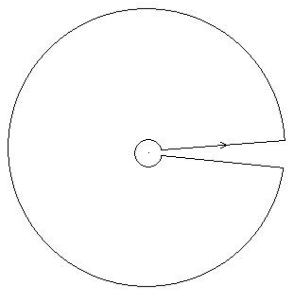
Partial Solutions to Homework 12

1. Sarason, X.10, exercise 5.
Again the question is: Over which curve does one have to integrate? In the case of $\frac{1}{1+x^a}$ we wanted the segment $[\epsilon, R]$ as part of the integration curve. Then we found that the integral over an arc of the circle with radius ϵ tends to 0, as ϵ tends to 0, and that the integral over an arc of the circle with radius R tends to 0, as R tends to ∞. Finally we found that the symmetry of the function $\frac{1}{1+x^a}$ under replacing x by $xe^{\frac{2\pi i}{a}}$ allowed us to use the segment $[\epsilon e^{\frac{2\pi i}{a}}, Re^{\frac{2\pi i}{a}}]$ in order to get a closed integration curve. The first arguments work also for the function in this problem. But the function does not have this symmetry. We need a new idea to get the missing piece of the curve, which has to fulfill the condition that the integral over the missing piece must either tend to 0 or must be related to the integral we want to compute.
The idea is that we use the fact that the function x^{a-1} has several branches for our purpose. If we integrate one branch over $[\epsilon, R]$, and a different branch over the segment with the other orientation, then the two terms will not cancel.

Conclusion: We let $G = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \geq 0\}$. First show that there is a branch of $\frac{z^a}{1+z^2}$ on G. Denote this branch simply by $\frac{z^a}{1+z^2}$. For $r \in \mathbb{R}$, $r \geq 0$ compute $\lim_{z\to r, Im(z)>0} \frac{z^{a-1}}{1+z^2}$ and $\lim_{z\to r, Im(z)<0} \frac{z^{a-1}}{1+z^2}$. Then integrate the branch over the curve in the following picture.



Finally we let the inner radius tend to 0, the outer radius tend to ∞ and the angle between the two segments tend to 0.

Solution:
G is star-shaped, so there exists a branch l of log on G with imaginary part between 0 and 2π, and so also a branch of x^{a-1} on G given by $exp((a-1)l(z))$. We have

$$\lim_{z\to r, Im(z)>0} \frac{z^{a-1}}{1+z^2} = \frac{r^{a-1}}{1+r^2}$$

(where we use the usual real $(a-1)$st power of a real number) and

$$\lim_{z\to r, Im(z)<0} \frac{z^{a-1}}{1+z^2} = \frac{exp(a-1)(ln(r)+2\pi i)}{1+r^2} = e^{(a-1)2\pi i}\frac{r^{a-1}}{1+r^2}.$$

The function $\frac{z^{a-1}}{1+z^2}$ has singularities at $\pm i$, with residues $\frac{exp(\frac{(a-1)i\pi}{2})}{2i}$ and $\frac{exp(\frac{(a-1)3i\pi}{2})}{-2i}$. So the integral over the curve is equal to $\pi exp(\frac{(a-1)i\pi}{2}) - \pi exp(\frac{(a-1)3i\pi}{2})$ by the residue theorem.

To be precise, we have to take first the limit as the angle between the segments tends to 0. Then the integral over the upper segment tends to $\int_\epsilon^R \frac{r^{a-1}}{1+r^2}dr$, and the integral over the lower segment tends to $-e^{(a-1)2\pi i}\int_\epsilon^R \frac{r^{a-1}}{1+r^2}dr$. Use compactness of the interval $[\epsilon, R]$ to show that the integral of the limit function is the limit of the integrals: Let β be the angle between the two segments. Define (for the upper segment)

$$F : [0, \delta] \times [\epsilon, R] \to \mathbb{C}, \quad F(\beta, r) = \frac{(re^{i\beta})^{a-1}}{1 + (re^{i\beta})^2}.$$

We want to show that $\int_\epsilon^R F(0, r)dr = \lim_{\beta\to 0} \int_\epsilon^R F(\beta, r)dr$. Now since $[0, \delta] \times [\epsilon, R]$ is compact, F is uniformly continuous. This implies that for any sequence $(\beta_n)_n$ converging to zero the sequence of functions $F(\beta_n, 0)$ is uniformly convergent, and so the integral of the limit is the limit of the integral. The same argument applies to the lower segment.

With the standard estimate we show that the integrals over the curved parts tend to 0 as the inner radius tends to 0 and the outer radius tends to ∞. We obtain as result that

$$(1 - e^{(a-1)2\pi i}) \int_0^\infty \frac{x^{a-1}}{1 + x^2}dx = \pi(exp(\frac{(a-1)i\pi}{2}) - exp(\frac{(a-1)3i\pi}{2})),$$

and with a little work:

$$\int_0^\infty \frac{x^{a-1}}{1 + x^2}dx = \frac{\pi}{2\sin(\frac{\pi a}{2})}.$$

2. Sarason, X.10, exercise 6.
Hint: In general, integrals from 0 to 2π involving periodic functions (e.g. sine and cosine) can be transformed into integrals over the unit circle. If $w = e^{i\theta}$ is on the unit circle, then $\cos\theta = \frac{1}{2}(w + \frac{1}{w})$ (and a similar formula holds for $\sin\theta$).

Solution:
The integral is equal to

$$\int_{|z|=1} \frac{\frac{1}{2}(z + \frac{1}{z})}{a - \frac{1}{2}(z + \frac{1}{z})} \cdot \frac{1}{iz}dz = \int_{|z|=1} \frac{z^2 + 1}{(2az - z^2 - 1)iz}dz.$$

Singularities: at $z = 0$ the residue is i, the other singularity inside the unit circle is $a - \sqrt{a^2 - 1}$ with residue $\frac{a}{i\sqrt{a^2-1}}$. By the residue theorem, the value of the integral is $2\pi(\frac{a}{\sqrt{a^2-1}} - 1)$.

3. Sarason, X.11, exercise 1.

Solution:
The integral is $\frac{1}{3n}\frac{1}{2\pi i}\int_C \frac{nz^{n-1}}{z^n - \frac{1}{3}}dz$, and now this integral is of the form we need for the argument principle. The function $z^n - \frac{1}{3}$ has n simple zeros inside C, and no poles, so the value of the integral is $\frac{1}{3n} \cdot n = \frac{1}{3}$.

4. a) Let $f(z) = \pi\cot(\pi z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$. Show that $f(z)$ is holomorphic on $\mathbb{C}$ except for simple poles with residue 1 at each integer.

b) Let g be a function which is holomorphic on $\mathbb{C}$ except for finitely many isolated singularities. For $N \in \mathbb{N}$ let C_N be the boundary of the square with vertices $\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2})$. Suppose that $\int_{C_N} f(z)g(z)dz \to 0$ as $N \to \infty$. (For large N, the curve contains none of the singularities of fg, so the integral is well-defined.) Show that

$$\lim_{N\to\infty} \sum_{\substack{n=-N \\ n \text{ not a singularity of } g}}^{N} g(n) = -\sum_{z \text{ a singularity of } g} res_z(fg).$$

c) Show that the supremum of $|f|$ on the curve C_N is $\pi\frac{e^{2\pi(N+\frac{1}{2})}+1}{e^{2\pi(N+\frac{1}{2})}-1} \leq 2\pi$.

d) Apply the preceding parts to $g(z) = \frac{1}{z^2}$ and prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution:
f has singularities exactly at the zeros of $\sin(\pi z)$, which are the integers. All these zeros are simple, so the residues are equal to $\frac{\pi\cos(\pi z)}{\pi\cos(\pi z)} = 1$ by the formula in the preceding homework.
For the second part we just apply the residue theorem to fg:

$$0 = \lim_{N\to\infty} \int_{C_N} f(z)g(z)dz = \lim_{N\to\infty} 2\pi i \sum_{z \text{ a singularity of } fg \text{ inside } C_N} res_z(fg).$$

The set of singularities of fg is the union of the set of singularities of f and g. Split into the singularities of g and the singularities of f which are not singularities of g. If $n \in \mathbb{Z}$ is not a singularity of g, then the residue of fg is $res_n f \cdot g(n) = g(n)$. The formula follows.
The third part should be straightforward.
For the last part the standard estimate for the integral shows that $\int_{C_N} f(z)g(z)dz \to 0$ as $N \to \infty$. So $2\sum_{n=1}^{\infty} \frac{1}{n^2} = -res_{z=0}(fg)$. We have to determine the first few terms of a Laurent series development for $\cot(z)$ at 0, which has the form $a_{-1}z^{-1} + a_0 + a_1 z + \ldots$. We get this by multiplying this series with the series for the sine function; we must get the series of the cosine. So $(z - \frac{z^3}{6} + \ldots)(a_{-1}z^{-1} + a_0 + a_1 z + \ldots) = 1 - \frac{z^2}{2} + \ldots$. It follows that $a_{-1} = 1, a_0 = 0, a_1 = -\frac{1}{3}$. Hence $\frac{\pi\cot(\pi z)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3}\cdot\frac{1}{z} + \ldots$. We get $2\sum_{n=1}^{\infty}\frac{1}{n^2} = -(-\frac{\pi^2}{3})$ and $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

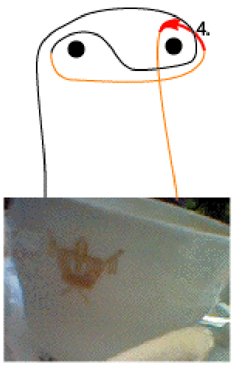
5. a) One can apply the residue theorem to all closed (smooth) curves γ in an open set G which are **null-homologous in** G, i.e. which satisfy that γ is contained with its interior in G. A closed curve γ is **null-homotopic in** G, if it is homotopic in G (as closed curve) to a constant curve.
Show that every (C^1-)curve that is (C^1-)null-homotopic in G is null-homologous in G. (In particular, it follows that one can apply the residue theorem to all (smooth) null-homotopic curves.)

b) Your aunt gave you a painting. You don't particularly like that painting. But when she comes to visit you'll have to put it up to not disappoint her. She's also a very stingy aunt and you know that when she sees a painting being fixed with two nails, she'll pull one out to use it for something else. Wouldn't it be nice if you could hang the painting in such a way using two nails that it falls down and breaks when your aunt pulls out either one of them? That way you'd be rid of the painting and your aunt couldn't complain.

(No cheating allowed. There is a solution without tricks.)

c) Find (without proof) an open set $G \subseteq \mathbb{C}$ and a curve γ which is null-homologous in G, but not null-homotopic in G.

Solution:
Let γ be a null-homotopic closed curve in G. Again use compactness to show that if $H : [0,1] \times [a,b] \to G$, $\gamma_s(t) = H(s,t)$ is a homotopy from γ to a constant curve, and $z_0 \notin G$, then $\frac{1}{2\pi i}\int_{\gamma_s} \frac{1}{z-z_0}dz$ is a continuous function of s. (Same argument as in the first problem.) But the value of the integral is an integer, and for $s = 1$, i.e. the constant curve, the value of the integral is 0. So the function of s is the constant function 0, which means that z_0 is in the exterior of γ. So γ is contained in G with its interior, i.e. γ is null-homologous in G.
A solution to the second part is (In class I drew the same solution in a more symmetric way.)



This also gives a solution for the last part. Use $G = \mathbb{C} \setminus \{0,1\}$ $(0,1$ are the nails) and a curve which is the union of the thread and the middle part of the upper edge of the picture. Then the index of 0 respectively 1 w.r.t. the curve is 0. (The picture falls if there is only 1 nail at 0 or 1 since the winding number is 0.) So the curve is null-homologous in G. Showing that the curve is not null-homotopic in G is more difficult.