Partial Solutions to Homework 11

1. a) Let K be a compact subset of a metric space X. Consider a cover of K with open subsets of K, i.e. a family $U_i \mid i \in I$ of subsets $U_i \subseteq K$, such that for every $x \in U_i$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \cap K \subseteq U_i$.
Show that the cover admits a Lebesgue number δ, i.e. show that there is $\delta > 0$ such that every subset $Y \subseteq K$ with diameter smaller than δ is contained in one of the U_i.
Instructions: Choose for each $x \in K$ an $\epsilon_x > 0$ such that $B_{2\epsilon_x}(x) \cap K \subseteq U_i$ for some i. Consider the cover of K by the open subsets of K of the form $B_{\epsilon_x}(x) \cap K$. By compactness, extract a finite subcover $B_{\epsilon_1}(x_1) \cap K, \ldots, B_{\epsilon_n}(x_n) \cap K$. Then show that $\delta = \min\{\epsilon_i\}$ works.

b) Now show that in IX.1 there exists a partition of $[a, b]$ into subintervals $[t_i, t_{i+1}]$ such that on each of these subintervals the image of ϕ is contained in a half-plane bounded by a line through the origin.

Solution:
With the instructions, the first part should be clear. If $Y \subseteq K$ has diameter smaller than δ, and $y \in Y$ is contained in $B_{\epsilon_j}(x_j)$, then Y is contained in $B_{2\epsilon_j}(x_j)$ and so in some U_i.
For each point $x \in [a, b]$, we have that $\phi(x) \neq 0$, and by continuity of ϕ, there is an open subset U_x of $[a, b]$ containing x such that the image of ϕ on U_x is contained in a half-plane bounded by a line through the origin. U_x can be chosen to be an open subinterval of $[a, b]$ if x is not a boundary point of the interval. For $x = a$, we may choose $U_x = [a, a + \epsilon)$, for $x = b$, we may choose $U_x = (b - \epsilon, b]$. In any case U_x is an open subset of $[a, b]$ in the above sense: for every $x \in U_x$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \cap K \subseteq U_x$. Now the family $\{U_x \mid x \in [a, b]\}$ forms a cover, which by the preceding part has a Lebesgue number δ. Now just take a subdivision of $[a, b]$ into subintervals of length smaller than δ. This has the required property.

2. Sarason, VIII.12, exercises 1 and 2.

Solution:
$h(z) = (z - z_0)f(z)$, where f is holomorphic in a neighborhood of z_0 and $f(z_0) = h'(z_0)$. Then $\frac{g(z)}{h(z)} = \frac{1}{z-z_0}\frac{g(z)}{f(z)}$. Now $\frac{g(z)}{f(z)}$ has a power series representation in $z - z_0$ near z_0 with constant term $\frac{g(z_0)}{f(z_0)}$, and so $\frac{g(z)}{h(z)}$ has a Laurent series representation in $z - z_0$ near z_0 with (-1)-th term $\frac{g(z_0)}{f(z_0)}(z - z_0)^{-1}$. The formula follows.
The residues in the second part are:
(a) $-\frac{1}{q}\zeta^{p+1}$ at each ζ which is one of the q-th roots of unity, using the first part.
(b) The residuum at -1 is the first coefficient of the Taylor series at -1 of $\frac{z^5}{(z-1)^2}$, since one has to divide this function by $(z + 1)^2$. So we get the derivative of $\frac{z^5}{(z-1)^2}$ at -1, which is 1. With the same method, one gets the residuum 1 at 1.
(c) $\frac{\cos(\zeta)}{1+2\zeta}$ at each ζ which is one of the two non-trivial third roots of unity, using the first part.
(d) $\frac{1}{\cos(z_0)}$ at each $z_0 = \pi k$, $k \in \mathbb{Z}$, by the first part. This means 1 at $2\pi k$, $k \in \mathbb{Z}$, and -1 at $\pi + 2\pi k$, $k \in \mathbb{Z}$.

3. Sarason, IX.5, exercise 3.

Solution:

The principal branch of log is a primitive, so we get $\frac{1}{2\pi i}(log(R-z_0)-log(-R-z_0))$. The real part of $log(R-z_0)-log(-R-z_0)$ is, using the real logarithm ln equal to $ln\left(\frac{|R-z_0|}{|-R-z_0|}\right)$, and tends to 0 as R tends to infinity. The imaginary part of $log(R-z_0)-log(-R-z_0)$ corresponds to the difference of the principal part of the arguments, it tends to πi if $Im(z_0)>0$ and to $-\pi i$ if $Im(z_0)<0$. The statement follows.

4. Sarason, IX.10, exercise 2.

Solution:
One has to show that if G is a convex subset, and Γ a contour in G, then $ind_\Gamma z_0 = 0$ for all z_0 in $\mathbb{C}\setminus G$. (Then IX.10 gives VII.2.) So let z_0 in $\mathbb{C}\setminus G$, and let $w\in G$. Consider the line through w and z_0. By convexity of G, if one moves z_0 on this line, away from w, we still get points not contained in G. So the index w.r.t. Γ is defined on the whole half-line with inital point z_0 which does not pass through w. But the index is constant on each connected component of $\mathbb{C}\setminus\Gamma$ and 0 on the unique unbounded component. It follows that z_0 belongs to the unbounded component and $ind_\Gamma z_0 = 0$.

5. Sarason, X.8, exercises 1, 2, 3, 4.

Solution:
1. Let f be the function we integrate. The value of the integral is

$$2\pi i\cdot(res_{z=a}f+res_{z=b}f)=2\pi i\cdot(\frac{a^k}{(a-b)^2}+\frac{(b-a)kb^{k-1}-b^k}{(b-a)^2}).$$

2. From the previous homework we know that $\int_C \frac{p}{q}=0$, where C is a large circle containing all zeros of q. But the residue theorem shows that the value of the integral is $2\pi i$ times the sum of the residues.
3. One can compute the residues at $-\frac{1}{2}$ and $\frac{1}{3}$, but computing the residue at 2 is easier. So we use the preceding exercise and get $-2\pi i\cdot res_{z=2}=-2\pi i\cdot 5^{-5}$.
4. For the first integral the denominator has zeros at $z=2$ and $z=\frac{1}{2}$. The residue at $z=\frac{1}{2}$ is $\frac{sin(1/2)}{-3}$, by VIII.12, exercise 1. The value of the integral is $2\pi i$ times this residue. For the second integral the denominator has only one zero inside the circle, $z=0$, and the residue is 2, again by the other homework problem, so the integral is $4\pi i$.

6. Sarason, X.10, exercises 2, 3, 4.

Solution: 2. The denominator has one zero inside the rectangle $z=i\frac{\pi}{2}$. The function has residue $-ie^{-\pi/2}$ there, so the integral over the boundary of the rectangle equals $2\pi e^{-\pi/2}$. Use the standard estimate for the integral to see that the integrals over the vertical parts tend to zero. Since on the real axis, the real part of the function is even, and the imaginary part is odd, the integral over the imaginary part vanishes. We get the integral $\int_{-\infty}^{\infty}\frac{\cos(x)}{\cosh(x)}dx$ as limit for the integral over the side which lies on the real axis, and $e^{-\pi}$ times this integral as the limit for the integral over the other horizontal side. Altogether:

$$(1+e^{-\pi})\int_{-\infty}^{\infty}\frac{\cos(x)}{\cosh(x)}dx=2\pi e^{-\pi/2},$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{\cosh(x)} dx = \frac{\pi}{\cosh(\frac{\pi}{2})}.$$

3. Integrating $\frac{1-e^{2iz}}{z^2}$ around $\Gamma_{\epsilon,R}$ gives 0, by Cauchy's theorem. The two straight parts give in the limit $4\int_0^\infty \frac{\sin^2(x)}{x^2}$ (imaginary part vanishes since it is odd). The integral over S_R tends to 0 using the standard estimate (the e^{2iz} in the numerator stays small). For the integral over S_ϵ, use the same trick as in example 1: add $\frac{2iz}{z^2}$ to $\frac{1-e^{2iz}}{z^2}$ get a new function with a removable singularity at 0. The integral over the new function tends to 0 as the new function is bounded in a neighborhood of 0. So the integral of $\frac{1-e^{2iz}}{z^2}$ over S_ϵ is in the limit equal to the integral of $\frac{2iz}{z^2}$, which is -2π. It follows that $\int_0^\infty \frac{\sin^2(x)}{x^2} = \frac{\pi}{4}$.

4. One can integrate $\frac{e^{iaz}}{(z^2+b^2)^2}$ over the curve Γ_R from example 2. (Or over a rectangle, ...) For $R >> 0$, the integral over Γ_R equals $2\pi i \cdot res_{z=ib}\frac{e^{iaz}}{(z^2+b^2)^2}$. The residue is the derivative at ib of $\frac{e^{iaz}}{(z+ib)^2}$, which is equal to $\frac{-2(ab+1)e^{-ab}}{-8ib^3}$. The integral over the curved part tends to 0 by the standard estimate for the integral. The integral over the real line is $2\int_0^\infty \frac{cos(ax)}{(x^2+b^2)^2} dx$. We get

$$\int_0^\infty \frac{cos(ax)}{(x^2+b^2)^2} dx = \frac{1}{2} \cdot 2\pi i \cdot \frac{-2(ab+1)e^{-ab}}{-8ib^3} = \frac{\pi}{4b^3}(1+ab)e^{-ab}.$$