Dr. Martin Olbermann Math 185
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Midterm Exam 2 - Solutions

1. (4 points)

Compute the integrals $\int_{[1,2+i\pi]} e^z dz$ and $\int_\gamma \sin(z^2)dz$, where γ is the counterclockwise oriented boundary of the rectangle with vertices $e^{\frac{2\pi i}{7}} - 2, e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} + 5i, e^{\frac{2\pi i}{7}} - 2 + 5i$.

Solution:
Either you really compute the first integral as $\int_{t=0}^1 e^{1+t(1+i\pi)}(1+i\pi)dt$ or you just remark that e^z is a primitive for e^z. In both cases you get $e^{2+i\pi} - e^1 = -e^2 - e$.
For the second part you should cite Cauchy's theorem for a rectangle, which implies that the integral is 0.

2. (5 points)

Prove that there is no branch of $log(z)$ on $B_1(0) \setminus \{0\}$.

Solution:
Again, you should have seen two proofs: In both consider a curve γ which is a circle with center 0 and say radius $\frac{1}{2}$. Either you assume a branch l of log exists and show that $l \circ \gamma(t) = c + 2\pi it$ (using the continuity of l and maybe the intermediate value theorem), which gives a contradiction. Or you explain that a branch l of log would be a primitive for $\frac{1}{z}$ in the given region, and that this is a contradiction to the fact that the integral over γ of $\frac{1}{z}$ is $2\pi i \neq 0$.

3. (5 points)

Let $G \subseteq \mathbb{C}$ be open and connected, and let D be a disk which is contained with its boundary in G. Let $f : G \to \mathbb{C}$ be a holomorphic function without zeros such that $|f(z)| = 1$ for all z in the boundary of D. Show that f is a constant function.

Solution:
By the maximum principle and the minimum principle, $|f(z)| = 1$ for all z in the disk D. By e.g. the maximum principle f is constant on D. By the identity theorem f is constant on G.

4. (2 + 4 + 2 points)

Let

$$f(z) = \frac{z^2 - 1}{\cos(\pi z) + 1}$$

have the power series expansion $\sum_{n=0}^{\infty} a_n z^n$ near $z = 0$.

a) Compute a_0 and explain how to compute a_2.

b) Prove that the isolated singularities of f are exactly the odd integers and classify each as removable / pole of order k / essential singularity.

c) What is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$? (Justify your answer.)

Solution:
$a_0 = f(0) = -\frac{1}{2}$, and $a_2 = \frac{1}{2!} f''(0)$. (One can compute a_2 also by an integral.)
We compute the zeros of $\cos(\pi z) + 1$:
$-1 = \cos(\pi z) = \frac{1}{2}(exp(i\pi z) + exp(-i\pi z)) \Leftrightarrow (exp(i\pi z))^2 + 2exp(i\pi z) + 1 = 0 \Leftrightarrow exp(i\pi z) = -1$ which is true if and only if z is an odd integer. So these values of z are exactly the isolated singularities of f. The derivative of $\cos(\pi z) + 1$ is $-\pi \sin(\pi z)$, which has the odd integers as zeros. The second derivative is $-\pi^2 \cos(\pi z)$, which has no odd integer as a zero. It follows that the denominator $\cos(\pi z) + 1$ is a function which has at each odd integer a zero of order 2. The numerator has ± 1 as a zero of order 1. Hence f has poles at all odd integers, of order 1 at ± 1, and of order 2 at all other odd integers.
By Cauchy's formula respectively the theorem that holomorphic functions are analytic, the radius of convergence is at least the biggest R such that f is defined and holomorphic in $B_R(0)$. This is $R = 1$. It also is not bigger since the function agrees with the series on $B_R(1)$, which means that the value of the series tends to ∞ as z tends to 1.

5. (5 points)

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Let ω_1, ω_2 be two complex numbers which are linearly independent over $\mathbb{R}$. This means that for $k_1, k_2 \in \mathbb{Z}$ the four complex numbers $k_1\omega_1 + k_2\omega_2, (k_1 + 1)\omega_1 + k_2\omega_2, (k_1 + 1)\omega_1 + (k_2 + 1)\omega_2, k_1\omega_1 + (k_2 + 1)\omega_2$ form a non-degenerated parallelogram in the complex plane. Suppose that f is double-periodic, i.e. $f(z) = f(z + \omega_1)$ and $f(z) = f(z + \omega_2)$ for all $z \in \mathbb{C}$.

Show that f is a constant function.

Solution:
Let P_{k_1,k_2} be the above parallelogram (boundary and interior). It suffices to observe that the compactness of $P_{0,0}$ implies that $|f|$ has a finite supremum=maximum on $P_{0,0}$ and that by double-periodicity this maximum is the maximum of $|f|$ on $\mathbb{C}$. So f is a bounded entire function, hence constant by Liouville's theorem.