Solutions to Homework 4

1. Stewart, exercise 3.9.

Solution:
$\mathbb{Q}(\alpha) \cong \mathbb{Q}[t]/\langle t^2 - 2\rangle$ and $\mathbb{Q}(\beta) \cong \mathbb{Q}[t]/\langle t^2 - 4t + 2\rangle$. Consider "evaluation at $t + 2$", i.e. the ring homomorphism $\phi : \mathbb{Q}[t] \to \mathbb{Q}[t]$ sending $p \mapsto p(t+2)$, and the composition ψ of ϕ with the quotient map $\mathbb{Q}[t] \to \mathbb{Q}[t]/\langle t^2 - 2\rangle$ sending $p \mapsto p + \langle t^2 - 2\rangle$.

Since ϕ is an isomorphism, ψ is surjective. The kernel of ψ consists of those p such that $p(t+2)$ is divisible by t^2-2, which are exactly those p such that p is divisible by $(t-2)^2-2 = t^2-4t+2$. Hence we get an isomorphism $\mathbb{Q}[t]/\langle t^2-2\rangle \cong \mathbb{Q}[t]/\langle t^2-4t+2\rangle$, i.e. an isomorphism $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$. Since this isomorphism sends each element of $\mathbb{Q}$ to itself, it is an isomorphism of the field extensions $\mathbb{Q}(\alpha) : \mathbb{Q}$ and $\mathbb{Q}(\beta) : \mathbb{Q}$.

2. Find the degree and a basis for the field extension $\mathbb{Q}(i, \sqrt{2} + \sqrt{5}) : \mathbb{Q}$.

Solution:
We consider the intermediate field $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}+\sqrt{5}) \subseteq \mathbb{Q}(i, \sqrt{2}+\sqrt{5})$. The minimal polynomial of $\alpha = \sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$ has degree 4: $\alpha^2 = 7 + 2\sqrt{10}$, so $(\alpha^2 - 7)^2 = 40$, so $\alpha^4 - 14\alpha^2 + 9 = 0$. Now check that $t^4 - 14t^2 + 9$ is irreducible over $\mathbb{Z}$. It is easy to see that there can be no linear factor, and a decomposition into quadratic polynomials would have to be of the form $(t^2 + at + b)(t^2 - at + b)$ (since the first and third coefficients have to be zero). But this also has no integral solutions a and b. Hence $t^4 - 14t^2 + 9$ is irreducible over $\mathbb{Q}$, i.e. it is the minimal polynomial of α. Since the minimal polynomial of i over $\mathbb{Q}(\sqrt{2} + \sqrt{5})$ has degree 2, we have $[\mathbb{Q}(i, \sqrt{2} + \sqrt{5}) : \mathbb{Q}] = 2 \cdot 4 = 8$, and a basis is $1, i, \sqrt{2} + \sqrt{5}, i(\sqrt{2} + \sqrt{5}), (\sqrt{2} + \sqrt{5})^2, i(\sqrt{2} + \sqrt{5})^2, (\sqrt{2} + \sqrt{5})^3, i(\sqrt{2} + \sqrt{5})^3$.

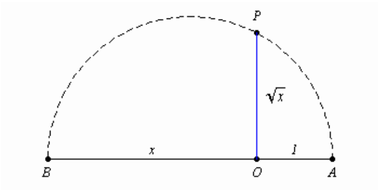
3. Stewart, exercise 4.8.

Solution:
Let p be a prime number. For each n, the polynomial $t^n - p$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. Now suppose $[A : \mathbb{Q}] = m$ is finite. For all n, since $\sqrt[n]{p}$ is algebraic with minimal polynomial $t^n - p$, we have $m = [A : \mathbb{Q}] = [A : \mathbb{Q}(\sqrt[n]{p})] : [\mathbb{Q}(\sqrt[n]{p})]$, so n divides m. Contradiction.

4. Stewart, exercise 4.12. You have to add the assumption that $deg(p) > 1$.

Solution:
The same argument as in the second half of the proof of the preceding problem.

5. Stewart, exercise 5.7.
Instructions:
a) Find a formula which gives $\cos(3\phi)$ as a function of $\cos(\phi)$.
b) Show that if $(0, 0), (1, 0) \in P_0$, then every point with coordinates in K_0 is constructible. (Show that the set of coordinates of constructible points forms a field. Use the intercept theorem to prove that it is closed under multiplication and taking multiplicative inverses.)
c) Show that if $(0, 0), (1, 0) \in P_0$, then every point with coordinates x, y which satisfy quadratic equations with coefficients in K_0 is constructible.



d) Apply this to $P_0 = \{(0,0),(1,0),(\cos\theta,\sin\theta)\}$.

Solution:
$\cos(3\phi) = 4\cos^3(3\phi) - 3\cos(\phi)$, so $\cos(\theta/3)$ is a zero of the polynomial $4t^3 - 3t - \cos(\theta)$.
Given distinct points p, q, r, one can construct the line through r which is orthogonal through the line through p and q, and the line through r which is parallel to the line through p and q. It follows that (x, y) is constructible if and only if we can construct two points whose distance is $|x|$ and two points whose distance is $|y|$.
So for part b) it remains to be proven that given line segments of length a, b, we can construct line segments of length $a + b$, ab, and $1/a$ if $a \neq 0$. Addition is easy, multiplication and inverse can be constructed using the intercept theorem
(see http://en.wikipedia.org/wiki/Intercept_theorem).
Part c) follows from the fact that the solution to a quadratic equation is given by a formula which (besides addition, subtraction, multiplication, division) only involves a square root, hence it can be constructed using the $h^2 = pq$ theorem for a right triangle (see the figure on the assignment sheet).
Conclusion: We start with an angle θ, say given by three points, i.e. $P_0 = \{(0,0),(1,0),(\cos\theta,\sin\theta)\}$. If the polynomial $4t^3 - 3t - \cos(\theta)$ is irreducible over $\mathbb{Q}(\cos\theta)$, then $[K_0(\cos(\theta/3)) : K_0] = 3$, from which it follows that $(\cos(\theta/3), \sin(\theta/3))$ is not constructible. If the polynomial is reducible, then $\cos(\theta/3)$ satisfies a quadratic equation with coefficients in K_0, hence $(\cos(\theta/3), 0)$ is constructible, hence $(\cos(\theta/3), \sin(\theta/3))$ is constructible.