Dr. Martin Olbermann
Due: Wednesday, Feb. 13, in class

Math 114
Spring 2008

Homework 3

Proofs and explanations should always be written using complete English sentences. You should always explain and justify each of the steps in your solution, unless otherwise noted. Write your name and "Math 114" on the top right of the first page.

1. a) Prove the following lemma using the quotient map $r : \mathbb{Z}[t] \to \mathbb{Z}_p[t]$.

Lemma. *Let p be a prime number and $f, g, h \in \mathbb{Z}[t]$ such that $f = gh$. If p divides f, then p divides g or p divides h.*

b) Prove the following proposition:

Proposition. *If $f \in \mathbb{Z}[t]$ is irreducible, then f is also irreducible as an element of $\mathbb{Q}[t]$.*

(If $f = gh$ with $g, h \in \mathbb{Q}[t]$, show that there exists $n \in \mathbb{Z}$ and $g', h' \in \mathbb{Z}[t]$ such that $nf = g'h'$. Then use the above lemma to prove that one can in fact use $n = 1$.)

c) Prove the following proposition using the quotient map $r : \mathbb{Z}[t] \to \mathbb{Z}_p[t]$.

Proposition. *(Eisenstein Criterion) Let $f = a_n t^n + \ldots + a_0 \in \mathbb{Z}[t]$, and let p be a prime number such that p does not divide a_n, such that p divides a_i for $i = 0, 1, \ldots n-1$ and such that p^2 does not divide a_0. Then f is irreducible as an element of $\mathbb{Q}[t]$.*

(Use also the preceding proposition and the fact that $K[t]$ is a UFD if K is a field.)

2. Stewart, exercise 2.1 (with proof!)

3. Stewart, exercise 2.13.
(A slight correction to his instructions: you should find, for each monomial in p of maximal height, a polynomial of the form $ks_1^{b_1} \ldots s_n^{b_n}$ whose highest term is equal to the given monomial.)

4. Stewart, exercise 3.8.