Dr. Martin Olbermann | Math 114
Due: Wednesday, Apr. 23, in class | Spring 2008

Homework 11

Proofs and explanations should always be written using complete English sentences. You should always explain and justify each of the steps in your solution, unless otherwise noted. Write your name and "Math 114" on the top right of the first page.

1. Let $L : K$ be a finite field extension, $a \in L$. We consider the K-linear map

$$m_a : L \to L, \qquad x \mapsto ax.$$

As an endomorphism of a finite-dimensional K-vector space it has a well-defined trace $T(a) = tr(m_a)$ and determinant $N(a) = det(m_a)$, which are trace and determinant of any matrix representation using a basis for K over L.
Prove that if $L : K$ is normal and separable, then $N(a) = \prod_{\gamma \in \Gamma(L:K)} \gamma(a)$ and $T(a) = \sum_{\gamma \in \Gamma(L:K)} \gamma(a)$.

Instructions:
1. Consider the intermediate field $K(a)$. Let the minimal polynomial of a over K be $f = t^n + c_1 t^{n-1} + \cdots + c_n$. Choose a basis $b_1, \ldots b_m$ for L over $K(a)$ and the usual basis $1, a, a^2, \ldots a^{n-1}$ of $K(a)$ over K. Get from these two bases a basis for L over K. Now compute explicitly the matrix of the map m_a corresponding to this basis (it should have a nice block form), and find $det(m_a)$ and $tr(m_a)$.
2. f splits over L since $L : K$ is normal, and has n distinct zeros in L since the extension is separable. Show that for each zero b of f in L, there are exactly m elements in $\Gamma(L : K)$, which send a to b. Express the c_i in terms of the zeros, and conclude.

2. a) Determine N and T explicitly in the cases $\mathbb{C} : \mathbb{R}$ and $\mathbb{Q}(\sqrt{2}) : \mathbb{Q}$.

b) Let $L : K$ be a finite extension and $a, b \in L$. Show that $N(ab) = N(a)N(b)$ and that $N(a) = 0 \Leftrightarrow a = 0$, so that by restriction N defines a group homomorphism $L^* \to K^*$ between multiplicative groups. Also show that $T(a + b) = T(a) + T(b)$.

c) Let $L : K$, $M : L$ and $M : K$ be finite, normal, separable extensions. Show that $N_{M:K}(a) = N_{L:K}(N_{M:L}(a))$ and $T_{M:K}(a) = T_{L:K}(T_{M:L}(a))$ for all $a \in M$.
Hint: Use the description via the Galois group. A look at the proof of the fundamental theorem (in particular 11.1, part 5) is helpful.

3. Stewart, exercise 15.8. Use that K is a field of characteristic 0.

4. Stewart, exercise 15.4.

5. Stewart, exercise 15.5.

6. Stewart, exercise 15.9 (b) and (c).

7. Stewart, exercises 14.10 and 14.11.
Change the assumption "the Galois group of L:K is $\mathbb{Z}_2$" to "$[L : K] = 2$"! Use the result of the first for the second exercise.