Nonabelian Hodge theory in Characteristic $p > 0$

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Hodge theory in characteristic zero

Let $X/\mathbf{C}$ be a smooth projective algebraic variety and let X_{an} be its associated analytic space. Then there are canonical isomorphisms:

$$H^n(X_{an}, \mathbf{C}) \cong H^n(X, \Omega^{\cdot}_{X/\mathbf{C}}) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbf{C}}).$$

The grading can be though of as an action of the algebraic group $\mathbf{C}^*$ on $H^n(X_{an}, \mathbf{C})$. Simpson's nonabelian analog attempts to construct such an action on the fundamental group of X_{an}, or at least on the category of its linear representations.

The Riemann-Hilbert correspondence

Equivalences of $\otimes$-categories, compatible with cohomology:

$$Rep(\pi_1(X_{an})) \equiv LC(X_{an}, \mathbf{C}) \equiv MIC(X/\mathbf{C})$$

$$V \mapsto (V \otimes_{\mathbf{C}} \mathcal{O}_{X_{an}}, \mathrm{id} \otimes d), \quad (E, \nabla) \mapsto E_{an}^{\nabla}$$
$$H^n(X_{an}, V) \cong H^n(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}})$$

Hodge theory: If V is *constant*

$$H^n(X_{an}, V) \cong \bigoplus_{i+j=n} H^i(X, E \otimes \Omega^j_{X/\mathbf{C}}).$$

- This isomorphism is not algebraic and involves complex conjugation.

- *Fails* if V is not constant.

Simpson's' nonabelian Hodge theory

A *Higgs bundle* (or *sheaf*) on $X/\mathbf{C}$ is any of the following equivalent sets of data

- $\theta\colon E' \to E' \otimes \Omega^1_{X/\mathbf{C}}$, $\mathcal{O}_X$-linear, such that $\theta \wedge \theta = 0$.
- Action of $S^{\cdot}T_{X/\mathbf{C}}$ on E'
- Sheaf $\tilde{E}'$ on cotangent space $\mathbf{T}^*_{X/\mathbf{C}}$ of X.

$\mathbf{C}^*$ acts on $\mathbf{T}^*_{X/\mathbf{C}}$, hence on $HIG(X/\mathbf{C})$.

Theorem. [Simpson et al] *There is a natural equivalence of $\otimes$-categories (modulo a stability condition), compatible with cohomology:*

$$Rep(\pi_1(X_{an})) \equiv MIC(X/\mathbf{C}) \equiv HIG(X/\mathbf{C})$$

$H^n(X_{an}, V) \cong H^n(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}) \cong H^n(X, E' \otimes \Omega^{\cdot}_{X/\mathbf{C}})$ *if* (E', θ') *corresponds to* (E, ∇).

Proof: Uses PDE, harmonic metrics. $\square$

Warnings:

- The correspondence $(E, \nabla) \mapsto (E', \theta)$ is not algebraic or even holomorphic.

- Complex conjugation enters, as in the Hodge decomposition.

Characteristic $p > 0$: background and introduction

Fix: X/S smooth morphism of noetherian schemes in characteristic $p > 0$, and its relative Frobenius diagram.

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr] & X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\
 & S \arrow[r] & S
\end{tikzcd}$$

$MIC(X/S)$ makes sense and is important. What about Riemann-Hilbert?

Classical Cartier descent

Theorem. [Cartier] $E \mapsto E^{\nabla}$ *gives an equivalence of* $\otimes$*-categories:*

$$MIC_{\psi=0}(X/S) \equiv MOD(X'/S)$$

$_{\psi=0}$ means, for those objects for which the p-curvature vanishes.
p*-curvature:*

$$\psi\colon T_{X'/S} \to F_* \operatorname{End}_{\mathcal{O}_X}(E, \nabla) \quad D \mapsto \nabla_D^p - \nabla_{D^{(p)}}$$

$$\psi\colon E \to E \otimes F^*\Omega^1_{X'/S} \quad \text{“F-Higgs field”}$$

But: ψ is usually *not* zero. (Grothendieck, Katz)

Emerging Theme:

- Keep ψ in the picture.
- Untwist by F to somehow get Higgs field from ψ.

p-curvature and Differential Operators

$D_{X/S}$:= the ring of *PD-differential operators* on X/S

$T_{X/S} \subset D_{X/S}$ generates $D_{X/S}$ over $\mathcal{O}_X$

Center $\mathcal{Z}_{X/S}$ contains $\mathcal{O}_{X'}$ and in fact all of $S^{\cdot}T_{X'/S}$, via the p-curvature mapping:

$$c\colon S^{\cdot}T_{X'/S} \xrightarrow{\cong} \mathcal{Z}_{X/S} \quad D \mapsto D^p - D^{(p)}.$$

Theorem. [BMR] *$D_{X/S}$ is an Azumaya algebra over its center (fppf locally a matrix algebra) of rank $(p^d)^2$.*

Consequently, étale locally on $\mathbf{T}^*_{X'/S}$, $D_{X/S}$ is $\mathcal{E}nd_{\mathcal{O}_{\mathbf{T}^*}}(\mathcal{B})$ for some $\mathcal{B} \in MIC(X/S)$, and $\mathcal{H}om_D(\mathcal{B},)$ gives an equivalence $MIC(X/S) \equiv HIG(X'/S)$, with quasi-inverse $\mathcal{B}\otimes$.

Explicit étale splitting of $D_{X/S}$

Cartier operator: $C\colon F_* Z^1_{X/S} \twoheadrightarrow \Omega^1_{X'/S}$.
Choose a local section

$$\begin{aligned}
\zeta\colon \Omega^1_{X'/S} &\to F_* Z^1_{X/S} \subseteq F_* \Omega^1_{X/S} \\
&\in T_{X'/S} \otimes F_* Z^1_{X/S} \\
\tilde{\zeta}\colon F^* \Omega^1_{X'/S} &\to \Omega^1_{X/S} \\
\pi^* \tilde{\zeta}\colon F^*_{X'} \Omega^1_{X'/S} &\to \Omega^1_{X'/S} \\
\phi\colon T_{X'/S} &\to F^*_{X'} T_{X'/S} \\
\alpha &:= \mathrm{Spec}(\phi) \circ F - \mathrm{id}
\end{aligned}$$

(Here F is the relative Frobenius for $\mathbf{T}^*_{X'/S}$ over X'.)
α is a surjective étale morphism of group schemes over X'.

Theorem. *The Azumaya algebra $D_{X/S}$ splits when pulled back by α.*

Proof: Suffices to construct a splitting module.

$$(\mathcal{B}, \theta) := F^* S^{\cdot} T_{X'/S} \in HIG(X'/S)$$

$\exists! \nabla$ on $\mathcal{B}$: $\nabla(1) = \zeta \in T_{X'/S} \otimes F_* Z^1_{X/S}$ and θ is horizontal. $(\mathcal{B}, \theta)$ is a $D_{X/S}$-module, projective of rank p^d over $\mathbf{T}^*_{X'/S}$.

$$\text{Compute:} \qquad (\mathcal{B}, \psi) = \alpha_*(\mathcal{B}, \theta).$$

So the action of $D_{X/S}$ on $\mathcal{B}$ extends to an action of $\alpha^* D_{X/S} := D_{X/S} \otimes_{\mathcal{Z}_{X/S}, \alpha} S^{\cdot} T_{X'/S}$. $\square$

Local Consequences

p-curvature functor: $\quad MIC(X/S) \xrightarrow{\Psi} F\text{-}HIG(X/S)$

Not fully faithful or essentially surjective.

Theorem. *1. If $\Psi(E_1, \nabla_1) \cong \Psi(E_2, \nabla_2)$, and E_i are coherent, then $(E_1, \nabla_1) \cong (E_2, \nabla_2)$ locally on X.*

2. Étale locally on X, a coherent object of $F\text{-}HIG(X/S)$ is in the essential image of Ψ if and only if it descends to an object of $HIG(X'/S)$.

Abelian analog: $E = \mathcal{O}_X$, $\omega := \nabla(1) \in Z^1_{X/S}$.
Classical exact sequence: (0th step in p-adic Hodge theory)

$$0 \longrightarrow \mathcal{O}^*_{X'} \xrightarrow{F^*} \mathcal{O}^*_X \xrightarrow{dlog} Z^1_{X/S} \xrightarrow[\Psi]{\pi^* - C} \Omega^1_{X'/S} \longrightarrow 0$$

Idea of the proof:

(1) is elementary, (2) more subtle.

There is a 2-commutative diagram:

$$\begin{tikzcd}
HIG(X'/S) \arrow[r, "\alpha_{\zeta *}"] \arrow[d, "\mathcal{B}\otimes"'] & HIG(X'/S) \arrow[d, "F^*_{X/S}"] \\
MIC(X/S) \arrow[r, "\Psi"] & F\text{-}HIG(X/S)
\end{tikzcd}$$

For modules whose support in $\mathbf{T}^*_{X'/S}$ is finite over X', can split α_ζ étale locally over X'. (2) follows from this. $\square$

Remark: α has a canonical section in the formal neighborhood of the zero section, and is $-\mathrm{id}$ in its Frobenius neighborhood.

Riemann-Hilbert and the global Cartier transform

X/S smooth, relative dimension d.
$\tilde{X}'/\tilde{S}$ a lifting of X'/S mod p^2.
Construct: Analog of Riemann-Hilbert and Simpson's theory, a la Fontaine-Galois.

$$D^{\gamma}_{X/S} := D_{X/S} \otimes_{S^{\cdot}T_{X'/S}} \hat{\Gamma}^{\cdot}T_{X'/S}.$$

An action of $D^{\gamma}_{X/S}$ on E amounts to connection ∇ and a

$$\theta\colon \hat{\Gamma}^{\cdot}T_{X'/S} \to \mathcal{E}nd(E, \nabla),$$

extending the p-curvature. $\otimes$-Categories:

$$MIC_{\gamma}(X/S) \quad (D^{\gamma}_{X/S} - \text{modules}, \otimes)$$

$$HIG_{\gamma}(X'/S) \quad (\hat{\Gamma}^{\cdot}T_{X'/S} - \text{modules}, \circledast)$$

Theorem. *Given $\mathcal{X}/\mathcal{S} := (X/S, \tilde{X}'/\tilde{S}')$, there exists a canonical*

$$(\mathcal{A}_{\mathcal{X}/\mathcal{S}}, \nabla_{\mathcal{A}}, \theta_{\mathcal{A}}) \in MIC_{\gamma}(X/S)$$

and an equivalence of categories:

$$MIC_{\gamma}(X/S) \equiv HIG_{\gamma}(X'/S)$$

$$C_{\mathcal{X}/\mathcal{S}} : (E, \nabla, \theta) \mapsto (E \circledast \mathcal{A}_{\mathcal{X}/\mathcal{S}})^{\nabla,\theta}, \mathrm{id} \otimes \theta_{\mathcal{A}})$$
$$C^{-1}_{\mathcal{X}/\mathcal{S}} : (E', \theta') \mapsto (E' \circledast \mathcal{A}_{\mathcal{X}/\mathcal{S}})^{\theta}, \mathrm{id} \otimes \nabla_{\mathcal{A}}, \mathrm{id} \otimes \theta_{\mathcal{A}})$$

Variant: $MIC_p(X/S) \equiv HIG_p(X'/S)$: ($_p$ means where action of pth power of $S^+T_{X'/S}$ is zero.)

The dual of $\mathcal{A}_{\mathcal{X}/\mathcal{S}}$ is a splitting module for $D^{\gamma}_{X/S}$.

$\mathcal{A}_{\mathcal{X}/\mathcal{S}}$ has a ring structure, which makes $C_{\mathcal{X}/\mathcal{S}}$ an equivalence of $\otimes$-categories.

Construction and geometric interpretation of $\mathcal{A}_{\mathcal{X}/\mathcal{S}}$

Say $\tilde{T} \in Cris(X/\tilde{S})$ lifts $T \in Cris(X/S)$, thickening of X (or an open subset).

$$\begin{tikzcd} T \arrow[r, dashed, "f_{T/S}"] \arrow[dr, "F_{T/S}"'] & X' \arrow[d, "inc"] \\ & T' \end{tikzcd}$$

- $\mathcal{L}_{\mathcal{X}/\mathcal{S}}(\tilde{T}) := \{\tilde{T} \to \tilde{X}'$ lifting $f_{T/S}\}$.
- $\mathcal{L}_{\mathcal{X}/\mathcal{S}}(\tilde{T})$ depends only on T, forms a crystal on X/S.
- $\mathrm{Spec}(\mathcal{A}_{\mathcal{X}/\mathcal{S}})$ represents $\mathcal{L}_{\mathcal{X}/\mathcal{S}}$.

Explicitly:

$\mathcal{E}_{\mathcal{X}/\mathcal{S},\tilde{T}} :=$ conormal sheaf of

$$T \xrightarrow{\Gamma_{f_{T/S}}} T \times X' \xrightarrow{inc} \tilde{T} \times \tilde{X}'$$

- Depends only on T, forms a crystal on X/S, fitting into an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}_{\mathcal{X}/\mathcal{S}} \to F^*\Omega^1_{X'/S} \to 0.$$

- $\mathcal{A}_{\mathcal{X}/\mathcal{S}} = \varinjlim S^n \mathcal{E}_{\mathcal{X}/\mathcal{S}}$

- $\operatorname{Spec} \mathcal{A}_{\mathcal{X}/\mathcal{S}} \cong$ splittings of $\mathcal{E}_{\mathcal{X}/\mathcal{S}} \cong$ liftings of $f_{T/S}$

The derived Cartier transform

Theorem. *Suppose X/S is smooth of relative dimension d, and $(E, \nabla) \in MIC_{p-d}(X/S)$ corresponds to $(E', \theta') \in HIG_{p-d}(X'/S)$. Then there is an isomorphism in the derived category:*

$$(E \otimes \Omega^{\cdot}_{X/S}, d) \cong (E' \otimes \Omega^{\cdot}_{X'/S}, \psi')$$

- This generalize the Hodge decomposition of Deligne-Illusie, and also of Faltings, Illusie, Ogus.

- If (E, ∇) underlies a Fontaine-module on a lifting of $X/\tilde{S}$, then

$$C_{\mathcal{X}/\mathcal{S}}(E, \nabla) \cong (\mathrm{Gr}_A E'_{X'}, \mathrm{Gr}_A \nabla')$$

- Uses the action of G_m on $\mathbf{T}^*_{X'/S}$ and can be realized by an explicit and canonical double complex.

$$\mathcal{A}_n^{i,j} := \mathcal{A}_n \otimes \Omega^i_{X/S} \otimes F^*\Omega^j_{X'/S}$$

$$\begin{tikzcd}
E' \otimes \Omega^2_{X'/S} \arrow[r] & E \otimes \mathcal{A}^{0,2}_{n-2} \arrow[r, "\nabla"] & E \otimes \mathcal{A}^{1,2}_{n-2} \arrow[r, "\nabla"] & E \otimes \mathcal{A}^{2,2}_{n-2} \arrow[r] & \cdots \\
E' \otimes \Omega^1_{X'/S} \arrow[r] \arrow[u, "\psi'"] & E \otimes \mathcal{A}^{0,1}_{n-1} \arrow[r, "\nabla"] \arrow[u, "\mathrm{id} \otimes \psi"] & E \otimes \mathcal{A}^{1,1}_{n-1} \arrow[r, "\nabla"] \arrow[u, "\mathrm{id} \otimes \psi"] & E \otimes \mathcal{A}^{2,1}_{n-1} \arrow[r] \arrow[u, "\mathrm{id} \otimes \psi"] & \cdots \\
E' \arrow[r] \arrow[u, "\psi'"] & E \otimes \mathcal{A}^{0,0}_{n} \arrow[r, "\nabla"] \arrow[u, "\mathrm{id} \otimes \psi"] & E \otimes \mathcal{A}^{1,0}_{n} \arrow[r, "\nabla"] \arrow[u, "\mathrm{id} \otimes \psi"] & E \otimes \mathcal{A}^{2,0}_{n} \arrow[r] \arrow[u, "\mathrm{id} \otimes \psi"] & \cdots \\
 & E \arrow[r, "\nabla"] \arrow[u] & E \otimes \Omega^1_{X/S} \arrow[r, "\nabla"] \arrow[u] & E \otimes \Omega^2_{X/S} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$