Musings on microlocal analysis in characteristic p

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This sketch is intended as a summary of my thoughts during the fall of 2005, attempting to understand a conversation with Kontsevich that summer as well as a little bit of his preprint [1]. The theme is to explore further the geometric meaning of the p-curvature, which can provide an analogy to the characteristic variety in microlocal analysis.

Let S be a scheme and let X/S be a smooth S-scheme. We shall be interested in integrable systems of linear partial differential equations on X/S. These can be described in several ways. Let $\Omega^1_{X/S}$ be the sheaf of Kahler differentials on X/S and let $T_{X/S}$ be its dual, which can be identified with the sheaf of derivation $\mathcal{O}_X \to \mathcal{O}_X$ relative to S. Recall that a *connection* on a sheaf E of $\mathcal{O}_X$-modules is an $\mathcal{O}_S$-linear map

$$\nabla\colon E \to \Omega^1_{X/S} \otimes E$$

satisfying the Leibnitz rule: $\nabla(fe) = df \otimes e + f(\nabla e)$ for $f \in \mathcal{O}_X$, $e \in E$. Equivalently, ∇ can be viewed as an $\mathcal{O}_X$-linear map

$$T_{X/S} \to \mathrm{End}_{\mathcal{O}_S}(E)$$

such that $\nabla_D(fe) = D(f)e + f\nabla_D(e)$ for $f \in \mathcal{O}_X$, $e \in E$. The *curvature* of such a connection is the ($\mathcal{O}_X$-linear) map $\kappa\colon E \to \Omega^2_{X/S} \otimes E$ defined by composing ∇ with itself and projecting. If this map is zero, the connection is said to be *integrable*. Finding sections of E annihilated by ∇ amounts to solving a system of linear partial differential equations. In the complex analytic context, the integrability of ∇ guarantees the local existence of a basis of E annihilated by ∇.

Modules with integrable connection can also be viewed as modules over a suitable ring D of differential operators. Beware that there are many such rings, all of which coincide if $\mathcal{O}_X$ is a sheaf of $\mathbf{Q}$-algebras, but in general, and especially in characteristic p, more care is required.

Suppose first that Y/T is smooth and that T is flat over $\mathbf{Z}$. Let $D_{Y/T}$ denote the subsheaf of the sheaf of $\mathcal{O}_T$-linear endomorphisms of $\mathcal{O}_Y$ generated by the sheaf of derivations $T_{Y/T}$. Thus, if $T = \operatorname{Spec} R$ and $Y = \operatorname{Spec} R[t_1, \ldots t_n]$, $D_{Y/T}$ is generated by $D_1, \ldots D_n$, where $D_i := \partial/\partial t_i$. Note that the operators $D_i^n/n!$, allowed in [EGA IV], are not included in this ring in general. Now suppose that $\mathcal{O}_S$ is annihilated by a power of p. Then one finds in [2] a geometric construction of the ring of "PD-differential operators." This is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, endowed with injective maps $\mathcal{O}_X \to D_{X/S}$ and $T_{X/S}$, as well as an action of $D_{X/S}$ on $\mathcal{O}_X$ compatible with the standard action of $T_{X/S}$. Beware, however, that the action is not faithful in general. For example, if $X := \operatorname{Spec} \mathbf{F}_p[t]$, then $(\partial/\partial t)^p$ is a nonzero element of $D_{X/\mathbf{F}_p}$ whose action on $\mathcal{O}_X$ is identically zero. In general the sheaf $D_{X/S}$ of PD-differential operators of X/S is generated by $T_{X/S}$, and if Y/T is a lift of X/S with T flat over $\mathbf{Z}$, then $D_{X/S} \cong D_{Y/T} \otimes \mathcal{O}_S$. (This should follow from the explicit formulas in [2], but I haven't checked it carefully. Is there a better way, involving a geometric description of $D_{Y/T}$ itself?). If $X = \operatorname{Spec}_S \mathcal{O}_S[t_1, \ldots t_n]$, and $D_i := \partial/\partial t_i$, then $D_{X/S}$ is freely generated as an $\mathcal{O}_X$-module by the monomials $D^I := D_1^{I_1} \cdots D_n^{I_n}$. As an $\mathcal{O}_S$-algebra, it is the quotient of the free noncommutative polynomial algebra $\mathcal{O}_X < D_1, \ldots D_n >$ by the ideal generated by the elements $D_i t_j - t_j D_i - \delta_{i,j}$, $1 \leq i, j \leq n$.

Now let E be a sheaf of $\mathcal{O}_X$-modules with a connection ∇. For each section D of $T_{X/S}$, ∇_D is $\mathcal{O}_S$-linear endomorphism ∇_D of E. If ∇ is integrable then the mapping $D \mapsto \nabla_D$ extends uniquely to an action of the sheaf of rings $D_{X/S}$ on E. In this way we get an equivalence between the category of left $D_{X/S}$-modules and the category of $\mathcal{O}_X$-modules with integrable connection.

The equivalence between D-modules and connections has a simple and useful linear analog. A *Higgs* field on a sheaf of $\mathcal{O}_X$-modules E is an $\mathcal{O}_X$-linear map

$$\theta \colon E \to \Omega^1_{X/S} \otimes E$$

such that the composite $E \to E \otimes \Omega^2_{X/S}$ vanishes. For each section ξ of $T_{X/S}$ one gets an $\mathcal{O}_X$-linear endomorphism θ_ξ of E, and for any other $\xi' \in T_{X/S}$, θ_ξ and $\theta_{\xi'}$ commute. Thus θ extends uniquely to an action of the symmetric algebra $S^{\cdot}T_{X/S}$ on E, and we obtain an equivalence between the category of $S^{\cdot}T_{X/S}$-modules and the category of $\mathcal{O}_X$-modules equipped with a Higgs field. If E is quasi-coherent on X one gets from the action of $S^{\cdot}T_{X/S}$ a quasi-coherent sheaf on $\mathrm{Spec}_X\, S^{\cdot}T_{X/S}$, which is non other than the cotangent bundle $\mathbf{T}^*_{X/S}$.

When X has characteristic p there are two related special phenomena that we want to exploit. The first is the existence of the Frobenius map $F_X\colon X \to X$. This is just the identity on the space X, but F_X^* takes a function f to its pth power. Since $p = 0$ in $\mathcal{O}_X$, F_X^* is actually a ring homomorphism. Note also that $dF_X^*(f) = pf^{p-1}df = 0$ for all $f \in \mathcal{O}_X$. The second phenomenon is the fact that the pth iterate of a derivation $\mathcal{O}_X \to \mathcal{O}_X$ in characteristic p is again a derivation. For example, when X is affine n-space, the pth derivative of any function with respect to any coordinate is zero. Thus the pth iterative of D_i as a derivation is zero, although D_i^p is not zero in the ring $D_{X/S}$. This phenomenon underlies the notion of the *p-curvature* of an integrable connection ∇. This is the map ψ which sends a derivation D of X/S to the endomorphism $(\nabla_D)^p - \nabla_{D^{(p)}}$ of E, which turns out to be $\mathcal{O}_X$-linear. Alternatively, we can write

$$\psi_D := \nabla_{D^p - D^{(p)}}, \tag{1}$$

where here $D^p - D^{(p)}$ is computed in the ring $D_{X/S}$.

To express this in a convenient and coordinate-free way, consider the relative Frobenius diagram:

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[dr] & X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

Here πF is the absolute Frobenius endomorphism F_X of X and the square is Cartesian, so that locally each element of $\mathcal{O}_{X'}$ is a sum of elements of the form gf^p, where g is pulled back from S and $f \in \mathcal{O}_X$.

Theorem 1 *The map $D \mapsto D^p - D^{(p)}$ above induces an injective homomorphism*

$$c\colon S^{\cdot}T_{X'/S} \to F_{X/S*}D_{X/S}$$

whose image is the center of $F_{X/S*}D_{X/S}$, and $F_{X/S*}D_{X/S}$ is an Azumaya algebra over its center.

We should remark that the proof of the $\mathcal{O}_X$-linearity of this map is non trivial. Now suppose that E is a quasi-coherent sheaf of $\mathcal{O}_X$-modules with connection, or equivalently, a $D_{X/S}$-module structure. Then c_*E is a quasi-coherent sheaf of $S^{\cdot}T_{X'/S}$-modules, and hence defines a quasi-coherent sheaf $\Psi'(E)$ on $\operatorname{Spec}_{X'} S^{\cdot}T_{X'/S}$, *i.e.*, the cotangent space $\mathbf{T}^*_{X'/S}$ of X'. Equivalently, one can note that the p-curvature mapping defines an $\mathcal{O}_{X'}$-linear map

$$T_{X'/S} \to F_{X/S*} \operatorname{End}_{\mathcal{O}_X} E.$$

Just as in the case of a Higgs field, this map induces an $S^{\cdot}T_{X'/S}$ module structure on E, and $\Psi'(E)$ is just sheaf corresponding to the induced sheaf of $S^{\cdot}F^*\Omega^1_{X'/S}$-modules. Thus in this way we have associated to the D-module E some linear data on the cotangent space of X'/S. Let $I_E \subseteq S^{\cdot}T_{X'/S}$ denote the annihilator of $\Psi'(E)$ and let $\sqrt{I_E}$ be its radical. This is the ideal defining the (support) of $\Psi'(E)$, and can perhaps be viewed as an analog of the characteristic variety of E used in microlocal analysis. Let us note that the actions of $\mathcal{O}_{X'}$ on E via the action of $S^{\cdot}T_{X'/S}$ and via the map $F^*_{X/S}\mathcal{O}_{X'} \to \mathcal{O}_X$ agree. Thus in fact E has a natural structure of a module over $\mathcal{O}_X \otimes_{\mathcal{O}'_X} S^{\cdot}T_{X'/S}$. This is the sheaf of functions on the pullback $\mathbf{T}^{*\prime}_{X/S}$ of $\mathbf{T}^*X'/S$ to X via $F_{X/S}$. If $\Psi(E)$ is the corresponding quasi-coherent sheaf on $\mathbf{T}{*\prime}_{X/S}$, then

$$\Psi'(E) \cong F_{\mathbf{T}^*/X}\Psi(E).$$

Example 2 Let us consider the case of a connection ∇ on $E := \mathcal{O}_X$. In this case ∇ is determined by $\nabla(1)$, which just an arbitrary one-form ω. The curvature of the connection is $d\nabla$, so the connection is integrable if and only if ω is closed. It follows from some tricky formulas due to Jacobson [3] that the p-curvature of such a connection corresponds to the F-Higgs field sending 1 to $F^*(C(\omega) - \pi^*(\omega))$, where $C\colon Z^1_{X/S} \to \Omega^1_{X'/S}$ is the Cartier operator [4].

In characteristic zero, a key role in microlocal analysis is played by the Poisson bracket structure $\{\ ,\ \}$ on the ring $S^{\cdot}T_{X/S} = \mathcal{O}_{\mathbf{T}^*_{X/S}}$. This can be expressed in many ways; probably the one most relevant for us involves commutators in $D_{X/S}$. Let $V_nD_{X/S}$ denote the sheaf of differential operators of order less than or equal to n. Then there is a natural isomorphism (the symbol map)

$$\sigma\colon \operatorname{Gr}^V_n D_{X/S} \cong S^nT_{X/S},$$

and if $\alpha \in V_m D_{X/S}$, $\beta \in V_n D_{X/S}$, then $[\alpha, \beta] \in V_{n+m-1} D_{X/S}$ and

$$\sigma[\alpha, \beta] = \{\sigma(\alpha), \sigma(\beta)\} \tag{2}$$

Belov-Kanel and Kontsevich have noted a p-adic expression for the Poisson bracket [1].

Proposition 3 *Let $\tilde{D}$ be a ring, flat over $\mathbf{Z}/p^2\mathbf{Z}$, let D be its reduction modulo p, and let $\mathcal{Z}$ denote the center of D. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be elements of D whose images α and β in D lie in $\mathcal{Z}$. Then there is a unique element γ of $\mathcal{Z}$ such that $[p]\gamma = [\tilde{\alpha}, \tilde{\beta}]$. Furthermore, γ depends only on α and β, and the paring $\{\alpha, \beta\} \mapsto \gamma$, defines a Poisson bracket structure on $\mathcal{Z}$.*

Proof: Since α and β lie in the center of D, $[\tilde{\alpha}, \tilde{\beta}]$ is divisible by p, so there is a $\tilde{\gamma} \in \tilde{D}$ such that $p\tilde{\gamma} = [\tilde{\alpha}, \tilde{\beta}]$. Since D is flat over $\mathbf{Z}/p^2\mathbf{Z}$, the image γ of $\tilde{\gamma}$ in D is independent of the choice of $\tilde{\gamma}$. By definition, $[p]\gamma = p\tilde{\gamma}$.

To see that γ is central, note that the Jacobi identity says that for any $\tilde{\delta} \in \tilde{D}$,

$$\tilde{[}\delta, \tilde{[}\alpha, \tilde{\beta}]] = [\tilde{\delta}, \tilde{\alpha}], \tilde{\beta}] + [\tilde{\alpha}, [\tilde{\delta}, \tilde{\beta}]].$$

Since $\tilde{\alpha}$ and $\tilde{\beta}$ are central mod p, $[\tilde{\delta}, \tilde{\alpha}] = p\tilde{\eta}$ for some η and $[\tilde{\delta}, \tilde{\beta}] = p\tilde{\zeta}$. Thus

$$[\tilde{\delta}, p\tilde{\gamma}] = [p\tilde{\eta}, \tilde{\beta}] + [\tilde{\gamma}, p\tilde{\zeta}].$$

Since α and β are central, the right side is divisible by p^2, and so $\delta, \gamma] = 0$.

For the independence of γ on the choice of the lifting, note that if $\tilde{\delta} \in D$, $[\tilde{\alpha} + p\tilde{\delta}, \tilde{\beta}] = [\tilde{\alpha}, \tilde{\beta}] + p[\tilde{\delta}, \tilde{\beta}]$. Now $[\tilde{\delta}, \tilde{\beta}]$ is divisible by p since $\tilde{\beta}$ belongs to the center of D modulo p, so the last term vanishes.

It is clear from the definition that the expression $\{\xi_1, \xi_2\}$ is antisymmetric and that it satisfies the Jacobi identity. Let us also check that it is a derivation, *i.e.*, that

$$\{\xi_1, \xi_2\xi_3\} = \{\xi_1, \xi_2\}\xi_3 + \xi_2\{\xi_1, \xi_3\}.$$

Choosing appropriate lifts, we have

$$\begin{aligned}
p\{\xi_1, \xi_2\xi_3\} &= [D_1, D_2D_3] \\
&= D_1D_2D_3 - D_2D_3D_1 \\
&= D_2D_1D_3 + [D_1, D_2]D_3 - D_2D_3D_1 \\
&= D_2[D_1, D_3] + [D_1, D_2]D_3 \\
&= pD_2\{\xi_1, \xi_3\} + p\{\xi_1, \xi_2\}D_3 \\
&= p\xi_2\{\xi_1, \xi_3\} + p\{\xi_1, \xi_2\}\xi_3.
\end{aligned}$$

$\square$

Proposition 4 *The Poisson structure defined above is the negative of the standard one defined in 2.*

Proof: It suffices to check this locally on X, so we may assume that we are given a set of coordinates, *i.e.*, an étale map $X \to \mathbf{A}^n/S$. Since the Poisson bracket defined above and the standard one both satisfy the derivation rule, it suffices to check the proposition on affine space itself. Thus we may assume that $X = \mathbf{A}^n/S$, with standard coordinates $(x_1, \dots x_n)$. Let $D_i := \frac{\partial}{\partial x_i}$. Then the above formula follows from the following calculation.

Lemma 5 *In the ring of differential operators of $\mathbf{A}^1/\mathbf{Z}$, one has the relation*

$$[D^p, x^p] \equiv -p \pmod{p)^2},$$

where $D := \frac{d}{dx}$.

Proof: Use the formula:

$$D^p(fg) = \sum_{i+j=p} \binom{p}{i} D^j f D^i g.$$

Apply this with $f = x^p$ to see that

$$D^p(x^p g) = \sum_{i+j=p} \binom{p}{i} \frac{p!}{(p-j)!} x^{p-j} D^i g = \sum_{i+j=p} \binom{p}{i} \frac{p!}{i!} x^i D^i g$$

Hence as endomorphisms of the ring of polynomials,

$$[D^p, x^p] = p! \sum_{i=0}^{p-1} \binom{p}{i} \frac{x^i}{i!} D^i.$$

This shows that in the ring of differential operators over $\mathbf{Z}$, the commutator $[D^p, x^p]$ is divisible by p, and

$$p^{-1}[D^p, x^p] = (p-1)! \sum_{i=0}^{p-1} \binom{p}{i} \frac{x^i}{i!} D^i = (p-1)! + \sum_{i=1}^{p-1} \binom{p}{i} \frac{x^i}{i!} D^i.$$

$\square$

Reducing modulo p and using Wilson's theorem, we see the desired formula. $\square$

A key property of the characteristic variety of a D-module in characteristic zero is that it is involutive, that is, that the ideal defining it is closed under Poisson bracket. This is not true of the annihilator ideal I (or of its radical) in general, as the following example shows.

Example 6 Let $X/S := \operatorname{Spec} \mathbf{F}_p[x_1, x_2]$, let $\omega := x_1^p x_2^{p-1} dx_2$. and let ∇ be the unique connection on $\mathcal{O}_X$ sending 1 to ω. Since ω is closed, this connection is integrable. Now as we saw in 2, the p-curvature of ∇ sends 1 to $F^*(C(\omega) - \pi^*(\omega)) = F^*(x_1' dx_2')$. Since the form $x_1 dx_2$ is not closed, the corresponding ideal is not closed under Poisson bracket. Explicitly, the $F^* S^{\cdot} T_{X'/S}$-module $\Psi(E)$ is given by the section of $F^* \mathbf{T}^*_{X'/S}$ corresponding to the one-form $x_1' dx_2'$. In terms of coordinates, this is given by the ideal $(\xi_2' - x_1', \xi_1')$, which is evidently not closed under Poisson bracket.

In this example, ω is closed in characteristic p but cannot be lifted to a closed form in characteristic zero.

Proposition 7 *Let X/S be smooth, where S has characteristic $p > 0$, and let E be a $D_{X/S}$-module on X/S. Suppose that $\tilde{X}/\tilde{S}$ is a lifting of X/S, where $\tilde{S}$ is flat over $\mathbf{Z}/p^2\mathbf{Z}$, and that $\tilde{E}$ is a lifting of E to a $D_{\tilde{X}/\tilde{S}}$-module, also flat over $\mathbf{Z}/p^2\mathbf{Z}$. Then the annihilator I_E of $\Psi'(E)$ in $S^{\cdot} T_{X'/S}$ is closed under Poisson bracket.*

Proof: Since $\tilde{\nabla}$ is integrable, it extends uniquely to a ring homomorphism $D_{\tilde{X}/\tilde{S}} \to \operatorname{End}_{\mathcal{O}_{\tilde{S}}} \tilde{E}$, which we also denote by $\tilde{\nabla}$. Let ξ_2 and ξ_2 be elements of $S^{\cdot} T_{X'/S}$ which annihilate E. We view them as central differential operators, so that ∇_{ξ_i} acts as zero on E. Hence if we choose lifting $\tilde{D}_i$ of ξ_i to $D_{\tilde{X}/\tilde{S}}$, $\tilde{\nabla}_{\tilde{D}_i}$ is divisible by p, say $\tilde{\nabla}_{\tilde{D}_i} = p\eta_i$, where η_i is an endomorphism of $\tilde{E}$. Now by 3, $[\tilde{D}_1, \tilde{D}_2] = p\xi$, where

$$\xi = \{\xi_i, \xi_2\} \in S^{\cdot} T_{X'/S} \subset F_{X/S*} D_{X/S}.$$

Hence we can write

$$\tilde{\nabla}_{p\xi} = [\nabla_{\tilde{D}_1}, \nabla_{\tilde{D}_2}] = [p\eta_1, p\eta_2] = p^2[\eta_1, \eta_2].$$

Since this is zero modulo p^2, it follows that ∇_ξ is zero modulo p. $\square$

Question 8 *With the previous hypotheses, is it also true that the radical of I_E is closed under Poisson bracket?*

In the case of connections on $\mathcal{O}_X$ we can give a partial converse to 7. If $\omega := \nabla(1)$ is closed, then $\psi(1) = F^*(C(\omega) - \pi^*\omega))$. This Higgs field is involutive if and only if $C(\omega) - \pi^*\omega$ is closed, *i.e.*, if and only if $C(\omega)$ is also closed. The following results are special cases of results about indefinitely closed one-forms in the de Rham Witt complex; see [].

Proposition 9 *Let ω be a closed i-form on X/S. If lifts to a closed i-form on $\tilde{X}/\tilde{S}$, then $C_{X/S}(\omega)$ is also closed. Conversely, if $C_{X/S}(\omega)$ is closed, then locally on X, ω is homologus to a closed form which lifts to $Z^i_{\tilde{X}/\tilde{S}}$.*

Here is a more precise statement

Proposition 10 *Let $\tilde{\omega}$ be an i-form on $\tilde{X}/\tilde{S}$ whose reduction modulo p ω is closed, and write $d\tilde{\omega} = [p]\gamma$, where $\gamma \in \Omega^{i+1}_{X/S}$. Then in fact γ is closed, and*

$$dC_{X/S}(\omega) = C_{X/S}(\gamma) \in \Omega^{i+1}_{X'/S}.$$

Proof: This statement is local on X/S, so we may assume that X is affine and choose a lifting $\tilde{F}: \tilde{X} \to \tilde{X}'$ of $F: X \to S$. Let $\omega' := C_{X'/S}(\omega)$. Recall that the inverse Cartier isomorphism is an isomorphism

$$C^{-1}_{X/S}: \Omega^i_{X'/S} \to F_*\mathcal{H}^i(\Omega^{\cdot}_{X/S})$$

and that $C^{-1}_{X/S} \circ C_{X/S}$ is the natural projection $F_*Z^i_{X/S} \to F_*\mathcal{H}^i_{X/S}$. According to Mazur's formula, if $\omega' \in \Omega^i_{X'/S}$ and $\tilde{\omega}' \in \Omega^i_{\tilde{X}'/\tilde{S}}$ lifts ω', then $C^{-1}_{X/S}(\omega')$ is the reduction of $p^{-i}\tilde{F}^*\tilde{\omega}'$ mod p. Hence there exist α and β such that

$$\tilde{\omega} = p^{-i}\tilde{F}^*(\tilde{\omega}') + p\alpha + d\beta.$$

Since $d\tilde{\omega} = p\gamma$,

$$\gamma = p^{-i-i}\tilde{F}^*(d\tilde{\omega}') + d\alpha,$$

so $C(\gamma) = d\omega'$, as claimed. □

In particular, if $\tilde{\omega}$ is exact, $C_{X/S}(\omega) = 0$. Conversely, suppose that $\omega \in F_*Z^i_{X/S}$ and $dC_{X/S}(\omega) = 0$. The formula implies that $C_{X/S}(\gamma) = 0$, and hence that the class of γ in $\mathcal{H}^i(\Omega^{\cdot}_{X/S})$ vanishes. Hence locally on X, $\gamma = d\delta$. Replacing $\tilde{\omega}$ by $\tilde{\omega} - pd\delta$, we see that ω is homologous to a liftably exact form.

Question 11 *Is there a generalization of this result to the case of general modules with connection? When can a module with integrable connection in characteristic p be lifted to a module with integrable connection modulo p^2?*

Remark 12 If S is anything and X/S is smooth, I expect that $D_{X/S}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules I also expect that $D_{X/S}$ is coherent, probably even left noetherian, as a sheaf of rings, If X has characteristic p and E is a coherent $D_{X/S}$-module, I hope E has a good filtration F, and then $\mathrm{Gr}_F E$ will become an $S^{\cdot}T_{X/S}$-module. I do not expect the annihilator of this module, or its radical, to be closed under Poisson bracket. But I do expect the dimension of the support of this module to be the same as the dimension of the p-curvature module $\Psi'(E)$. All these should be fairly straightforward to verify.

Let me now turn to one of the conjectures of [1] that has especially caught my interest. The conjecture relates connections in characteristic zero to their reductions modulo almost all primes p. To make sense of this, let us introduce the following notation. Let R be an integral domain which is finitely generated and flat as an algebra over $\mathbf{Z}$—for example, $\mathbf{Z}[n^{-1}]$ for some n. Let K be the fraction field of R, a field of characteristic zero—for example just the field $\mathbf{Q}$. Let $S := \operatorname{Spec} R$, let $\sigma := \operatorname{Spec} K$, and let X/S be a smooth morphism, of relative dimension d. Its generic fiber X_σ is thus a smooth K-scheme of dimension d. Let E be a sheaf of $D_{X/S}$-modules on X which is coherent as a sheaf of $D_{X/S}$-modules. For each closed point s of S, the residue field $k(s)$ is a finite field. In particular, if E is a $D_{X/S}$-modules, its restriction E_s to the fiber X_s of X over $k(s)$ has an associated $S^{\cdot}T_{X'_s}$-module $\Psi'(E_s)$. In general, the dependence of $\Psi'(E_s)$ on s is quite complicated. However, Belov-Kanel and Kontsevich have made the following conjecture [1]. (I have changed the statement slightly.)

Conjecture 13 *Let $Y \subseteq \mathbf{T}^*_{X_\sigma}$ be a smooth and Lagrangian subvariety. Assume that the de Rham cohomology $H^1_{DR}(Y/K)$ vanishes. Then (after replacing S by a Zariski open subset) there exists a coherent $D_{X/S}$-module E with the following properties:*

1. *E_σ is holonomic*

2. *For all closed points s of S, the action of $S^{\cdot}T_{X'_s}$ on E_s factors through $\mathcal{O}_{Y'_s}$, and in fact $\Psi'(E_s)$ is locally free over $\mathcal{O}_{Y'_s}$ of rank p^d.*

Furthermore, E_σ is uniquely characterized (up to isomorphism) by the above properties.

Remark 14 The characterization "up to isomorphism" is annoyingly vague, in particular I don't see why E can't have automorphisms. Can this be made more precise by saying something more rigid about the action of $\mathcal{O}_{Y'_s}$ on E_s? It might be tempting to fix an $\mathcal{O}_Y$-structure on E ahead of time which is somehow used in (2) above. However, see the example 16 below.

Let me also remark that condition (2) above says that E_s is a splitting module for the Azumaya algebra $D_{X_s} \otimes \mathcal{O}_{Y'}$. (Here the tensor product is taken over the center of D_{X_s}.) It follows from results of [5] that it is equivalent to the statement that $\Psi(E)$ is becomes an invertible sheaf over $X_s \times_{X'_s} Y'_s$.

Example 15 Let us consider the case in which Y is given by a section of $\mathbf{T}^*_{X_\sigma}$. Such a section is in turn given by a global section θ of $\pi^*\Omega^1_{X_\sigma/\sigma}$ on Y, and the condition that Y be Lagrangian says that the image of θ in $\Omega^1_{Y_\sigma/\sigma}$ should be closed. The hypothesis on the de Rham cohomology of $Y = X$ then implies θ is exact. After shrinking S, this can be achieved on X/S. Consider the connection on $\mathcal{O}_X$ sending 1 to θ. It follows from the formula 2 for the p-curvature of this connection (in characteristic p sends 1 to $F^*\theta$, as desired.

Example 16 Let $X := \operatorname{Spec} \mathbf{Z}[x]$, and let ξ be the coordinate of $\mathbf{T}^*_X$ corresponding to d/dx, and let Y be the closed subscheme of $\mathbf{T}^*_X$ defined by $\xi^2 - x$. In this case, the D_X-module $D_X/(D_X(D^2 - x)$ works in 13. This module corresponds to the module with connection E with basis (e_0, e_1) and $\nabla(e_0) = e_1$, $\nabla(e_1) = xdx \otimes e_0$.

To see that this works, one can use the Fourier transform (Kontsevich) or compute directly, using Jacobson's identity. Let us note that the restriction of the universal one-form ξdx to Y is $2\xi^2 d\xi = (2/3)d\xi^3$, which is exact as soon as we invert 3. Note also that things look suspicious at 2.

It is instructive to look at some explicit formulas for small primes p. Here we list the p-curvature matrix (in the above basis) as well as its square. (These were calculated using Macintosh Common Lisp.)

p	Ψ	Ψ^2

2	$\begin{pmatrix} X^1 & 1 \\ 0 & X^1 \end{pmatrix}$	$\begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix}$
3	$\begin{pmatrix} 1 & X^2 \\ X^1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1+X^3 & 0 \\ 0 & 1+X^3 \end{pmatrix}$
5	$\begin{pmatrix} 4X^1 & 4+X^3 \\ X^2 & X^1 \end{pmatrix}$	$\begin{pmatrix} X^5 & 0 \\ 0 & X^5 \end{pmatrix}$
7	$\begin{pmatrix} 2X^2 & X^4 \\ 3+X^3 & 5X^2 \end{pmatrix}$	$\begin{pmatrix} X^7 & 0 \\ 0 & X^7 \end{pmatrix}$
11	$\begin{pmatrix} 5X^1+3X^4 & 5+7X^3+X^6 \\ 6X^2+X^5 & 6X^1+8X^4 \end{pmatrix}$	$\begin{pmatrix} X^{11} & 0 \\ 0 & X^{11} \end{pmatrix}$
13	$\begin{pmatrix} 2X^2+10X^5 & X^4+X^7 \\ 9+3X^3+X^6 & 11X^2+3X^5 \end{pmatrix}$	$\begin{pmatrix} X^{13} & 0 \\ 0 & X^{13} \end{pmatrix}$
17	$\begin{pmatrix} 15X^1+2X^4+13X^7 & 15+10X^3+12X^6+X^9 \\ 2X^2+6X^5+X^8 & 2X^1+15X^4+4X^7 \end{pmatrix}$	$\begin{pmatrix} X^{17} & 0 \\ 0 & X^{17} \end{pmatrix}$

When $p = 3$, the answer is incorrect, as we predicted, and if $p > 3$, it is correct. However for $p = 2$, the matrix is not generically semisimple, and again gives the wrong answer.

Question 17 *With the notation of Conjecture 13, let us replace the hypothesis on $H^1_{DR}(Y/K)$ by the condition that the universal one-form Θ on $\mathbf{T}^*_{X_\sigma/K}$ be exact when restricted to Y. (Note that this implies that Y/K is Lagrangian.)*

1. *Is this hypothesis sufficient to insure the existence of a module E with connection as in 13.1?*
2. *If so, can we classify all such (E, ∇)?*

Example 18 The answer to question 17.1 is yes if Y/X is étale. Namely, consider the connection on $\mathcal{O}_Y$ (viewed as an $\mathcal{O}_Y$-module) sending 1 to the restriction Θ_Y of the universal one-form to Y. Since $\Theta_{|Y}$ is exact, the p-curvature of this connection is the F-Higgs field sending 1 to $F_Y^*\Theta_{|_Y}$. Now since Y/S is étale, $\pi_*\Omega^1_{Y/S} \cong \pi_*(\mathcal{O}_Y) \otimes \Omega^1_{X/S}$, and our connection on the $\mathcal{O}_Y$-module $\mathcal{O}_Y$ gives us a connection on the $\mathcal{O}_X$-module $\pi_*\mathcal{O}_X$. The p-curvature of this connection is still the $\mathcal{O}_Y$-linear map $\pi_*\mathcal{O}_Y \to F_X^*\Omega^1_{X/S} \otimes \pi_*\mathcal{O}_Y$ sending 1 to $\Theta_{|_Y}$ and hence the action of $F_X^* S^{\cdot} T_{X/S}$ it induces is exactly the action of $\mathcal{O}_Y$ on itself.

However, it seems difficult to classify all such E. For example, suppose that $X := \mathbf{G}_m$, with coordinate x, and let $Y \subseteq \mathbf{T}^*_X$ be the closed subscheme defined by $\xi^2 - x$. The above construction gives one such connection on $\pi_* \mathcal{O}_Y$:

$$\begin{array}{rcl} 1 & \mapsto & \xi dx \\ \xi & \mapsto & d\xi 1 + \xi\xi dx = \dfrac{\xi}{2x} dx + x dx \end{array}$$

Note that this connection has a regular singularity at the origin. On the other hand, we saw above a connection on a free $\mathcal{O}_X$-module of rank two with no such singularity and which also has the right p-curvature. In the basis (e_0, e_1) discussed above, we have

$$\begin{array}{rcl} e_0 & \mapsto & e_1 dx \\ e_1 & \mapsto & x e_0 dx \end{array}$$

Is there some "standard" way to see a relationship between these (nonisomorphic) connections?

Let us remark that although the category of coherent sheaves with integrable connections over a formal power series ring over $\mathbf{C}$ is trivial, this is not the case over $\mathbf{Z}$. Thus it seems reasonable to ask question 17 with $X := \operatorname{Spec} R[[x_1, \ldots x_n]]$, for example. The uniqueness in this case seems especially problematic, however, since the Katz Grothendieck conjecture [4] fails for such X, as the following example shows.

Example 19 The Katz Grothendieck conjecture asserts that if X/S is as in 13 and (E, ∇) is a coherent sheaf on X with integrable connection such that for the p-curvature of each E_s vanishes, then E_σ has a full set of horizontal sections, after replacing X by a finite étale cover. This seems to be false with X replaced by $\mathbf{Z}[[x]]$. Let

$$\omega := \sum a_n x^n \operatorname{dlog} x, \qquad \text{where } a_n := \sum_p \{p : p^2 | n\}.$$

Evidently $\omega \in \mathbf{Z}[[x]]dx$, and in fact

$$\omega := \sum_p \omega_p, \qquad \text{where } \omega_p := \sum p(x^{p^2} + x^{2p^2} + x^{3p^2} + \cdots) \operatorname{dlog} x.$$

In fact,

$$\omega_p = \operatorname{dlog} g_p, \qquad \text{where } g_p := (1 - x^{p^2})^{-1/p}.$$

Note that $g_p \in \mathbf{Z}[p^{-1}][[x]]$, and $g := \prod g_p \in \mathbf{Q}[[x]]$ satisfies $\operatorname{dlog} g = \omega$.

Remark 20 It might be useful to investigate some higher p-curvature operators. Let us suppose that S is flat over $\mathbf{Z}$ and p-adically complete and that (E, ∇) is a module with integrable connection on X/S. Suppose that the p-curvature of the reduction modulo p of E vanishes, and suppose we are given a coordinate system for X/S. Then each $\nabla^p_{D_i}$ is divisible by p; write $\nabla^p_{D_i} = p\eta_i$ Note that $D_i^{p^2}$ is divisible by $p^2!$ and hence by p^{p+1}, whereas $\nabla^{p^2}_{D_i} = p^p \eta_i^p$ is a priori only divisble by p^p. In fact, $\eta_i^p/p^p \bmod p$ is an $\mathcal{O}_X$-linear and horizontal endomorphism of E/pE, which is an obstruction to solving the differential equations of (E, ∇). Some obvious things to investigate:

1. a coordinate free treatment of this map
2. its relationship to Frobenius descent, in particular to the p-curvature of the F-descent of (E, ∇).
3. its relationship to Berthelot's higher level differential operators.

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